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Editors
M. J. Klein   G. J. Toomer

O. Neugebauer

A History of Ancient Mathematical Astronomy

In Three Parts
with 9 Plates and 619 Figures

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Otto Neugebauer
Brown University, Providence, Rhode Island 02912, USA

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To
the Owl and the rabbit

The opposite of an Introduction is a Contradiction

Owl (The House at Pooh Corner)

Pendant que j'étudie l'astronomie, je ne pense
ni à Balkis, ni à quoi que ce soit au monde.
Les sciences sont bienfaisantes: elles empêchent
les hommes de penser.

A. France, Balthasar (Œuvres IV, p. 141)

Preface

This work could properly go under the title which Petrarch, in 1367, gave to one of his latest writings: "*De sui ipsius et multorum ignorantia.*" By ignorantia I do not mean the obvious fact that only a small fraction of ancient astronomical theory can be restored from the scattered fragments that have survived. What I mean is the ignorantia auctoris in comparison with the scholarship of the 18th and 19th centuries and the ignorantia multorum to whom such a work might be addressed.

In many years of study I have tried to become familiar with the ancient methods of mathematical astronomy, to realize their problems, and to understand their interconnections and development. Perhaps I may say that my approach is nearest to Delambre's in his Histoire de l'astronomie (ancienne: 1817, Moyen age: 1819, moderne: 1821) though I fully realize that I do not have by far the professional competence of Delambre. Yet I have tried to come as close as possible to the astronomical problems themselves without hiding my ignorantia behind the smoke-screen of sociological, biographical and bibliographical irrelevancies.

The general plan of the following is simple enough. I begin with the discussion of the Almagest since it is fully preserved and constitutes the keystone to the understanding of all ancient and mediaeval astronomy. Then we go back somewhat in time to the investigation of earlier periods, in particular to Babylonian astronomy, for which we have a fair amount of contemporary original sources. Next comes the most fragmentary and most complex section: the investigation of early Greek astronomy and its relation to Babylonian methods. Finally Book V brings us back to safer ground, i.e. to material for which original sources are again extant: Hellenistic astronomy as known from papyri, Ptolemy's minor works and the "Handy Tables". The appendices (Book VI) contain details concerning technical terminology and descriptions of chronological, astronomical, and mathematical tools.

The present work covers only about the first half of a much more ambitious plan (laid out in the early 1950s). I had hoped to be able to carry the discussion down to the latest aspects of "ancient" astronomy, i.e. the astronomy of Copernicus, Brahe, and Kepler. I did not feel it was necessary to eliminate all traces of this overly optimistic plan.

As for all books on a complex scientific subject there exists only one ideal reader, namely the author. Topics are selected, viewpoints taken, and answers formulated as they appeal to his taste and prejudices. In the course of more than twenty years, students, friends and collaborators have been exposed to my way of looking at the history of astronomy, and in turn they have influenced my views while adopting and developing some of mine in their own work. This gives me some hope that also in the future sympathetic readers might exist who are willing to penetrate the jungle of technical details and become fascinated by the kaleidoscopic picture which I have tried to unfold here of the history of the first and oldest natural science.

It is with feelings of sincere gratitude that I acknowledge my indebtedness to the generous support of my work that I have enjoyed for many years from Brown University and from the Institute for Advanced Study in Princeton. At the Institute I also had the help of Mrs. E. S. Gorman and of Miss Betty Horton whose patience and accuracy greatly facilitated the preparation of the manuscript.

My thanks are due in no small measure to the Springer Verlag whose initiative made the publication of this work possible, just as it did that of my first book, fifty years ago, and repeatedly thereafter.

Finally I want to acknowledge with gratitude the work of my good friends and associates Janet Sachs and Gerald Toomer for their persistent efforts to improve my English usage and to modify untenable positions in some topics. What remains uncorrected is entirely my responsibility.

By deciding to put this manuscript into print, the moment has come when these pages themselves turn into a part of the past. I can only ask for the indulgence of my younger colleagues and friends, and of their pupils, when they see that I have overlooked or misinterpreted what a new generation now can see more clearly in this never-ending process. *Καὶ οὕτως ἀπέρχομαι ... ὡς μηδὲ ἀρξάμενος.*[1]

Providence, June 1975 O. N.

[1] *'Αββᾶς Παμβώ* (Migne, PG 65, 369 η').

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Part One

Introduction

Book I
The Almagest and its Direct Predecessors

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Introduction

§ 1. Limitations

> "... *excellens philosophus, cosmographus, mathematicus, historicus, stultus, linguarum non ignarus, sed nullius ad unguem peritus.*"
>
> J. J. Scaliger

Many things are omitted here. The reader who wants to hear about Archimedes taking a bath or about the silver nose of Tycho Brahe can find innumerable books which dwell on these important biographical matters. Nor do I enumerate the pros and cons concerning the place or movement of the earth and the substance of the spheres. The history of these discussions has filled volumes which succeeded, by their sheer mass, almost completely to remove from sight the crucial mathematical arguments which are needed for the prediction of planetary positions, of eclipses and of secular changes, or for the determination of sizes and distances of the celestial bodies.

In short: I shall deal here neither with early cosmogony nor with philosophy, but exclusively with mathematical astronomy, i.e. with the numerical, geometrical, and graphical methods devised to control the mechanism of the planetary system. It was my goal to convey to the reader some insight into the different aspects these problems have taken during their wanderings from culture to culture, beginning in the latest period of Mesopotamian civilization and ending with the discovery of the elliptic orbit of Mars, thus preparing the way for Newton's dynamics which brought to its conclusion all ancient mathematical astronomy.

But even in this restricted area of mathematical astronomy much is omitted which would have deserved fuller treatment. Besides the personal impossibility to cover evenly such an enormous variety of topics, the lack of easily accessible sources shows its influence. What is badly needed is a systematic corpus of mediaeval sources, Indian, Islamic, and Western alike, a task which would require a great cooperative effort not likely to be made in the foreseeable future. Thus it remains accidental, to a large measure, what can be said in any discussion of mediaeval astronomy.

An omission for which I alone am responsible is the theory of instruments. The reason is simply lack of competence. The theory of sun dials, of astrolabes, clocks, planetary computers, etc. constitutes a large field of its own with which I am not sufficiently familiar to bring it into significant relation to the topics discussed here. This is regrettable but I cannot do better than to admit the fact.

Again by reason of incompetence I have omitted all discussion of the history of astronomy in China. Its influence upon the Islamic and Western development is probably not visible earlier than the creation of Mongol states in Western Asia. Thus the damage done by omitting China is perhaps not too great and at any rate alleviated by ignorantia.

No relation whatsoever exists between our study and Maya astronomy. Consequently no reference to this field of study will be found in the following pages.

I have also refrained from more than casual comparisons of ancient with modern numerical data. Not only can such data be easily obtained from any good modern textbook and from modern tables (in particular Tuckerman's planetary tables for the period from −600 to +1650), but such comparisons are usually rather misleading. For example, the often found references to gross "errors" in ancient solstitial or equinoctial dates are of no value without a careful discussion of the influence on the solar theory as a whole. The reader of the subsequent pages will not need constant reminders of these rather obvious facts. I shall not explain how "good" or how "bad" ancient astronomy was but I will try to describe what seemed to be the essential problems and the methods developed toward their solution.

§ 2. The Major Historical Periods, An Outline

The history of astronomy falls into three sharply distinct periods: (a) prehistory until about 700 B.C. when (probably) Mesopotamian astronomy begins, (b) the ancient and mediaeval period (in the customary sense of political history) to the middle of the 17th century, and (c) modern astronomy beginning with the time of Newton. These three periods are as sharply distinct from one another as a stone-age settlement, mediaeval Paris, and modern New York. Figuratively speaking we are dealing here exclusively with the history of mediaeval Paris.

The main chronological fixed-points for our discussion can be given in a few sentences. Not much before 300 or 400 B.C. there originated in Mesopotamia arithmetical methods for very accurate predictions of lunar and planetary phenomena. Perhaps inspired by these successes, but only in a very small measure depending upon them, cinematic models were developed by Greeks, notably by Apollonius, around 200 B.C. Careful systematic observations by Hipparchus (about 150 B.C.) made it clear, however, that the actual motions were more complicated and that further progress would not be easy. Indeed, only after much groping in the dark, about two centuries later (about A.D. 100) the important tool of spherical astronomy was put on a sound mathematical basis by Menelaos, while a satisfactory planetary theory and an improved theory for the lunar motion had to wait until Ptolemy (about 150 A.D.). His monumental work remained the foundation for all mathematical astronomy until Kepler (around 1600). Late antiquity, Byzantium, and Islam upheld theoretical astronomy on an intelligent level, even improving on important mathematical and some astronomical details. But it is only in the late Renaissance that a real renewal of astronomy

took place, most clearly expressed in Tycho Brahe's observational program (about 1560 to 1600) which provided the foundation for Kepler's "*Astronomia Nova*".

Since the subsequent chapters are not following a strictly chronological order, it will be useful to provide the reader for his orientation with a schematic summary of the main lines of development and their interrelations. Such a summary is of necessity not only very incomplete but also much too dogmatic in formulation. It is only meant as a kind of preliminary scaffolding that has to be removed as the building itself progresses.

A. The Hellenistic Period

The "Babylonian" astronomy of the Seleucid and Arsacid period is of primary importance for our investigations. Here we have authentic source material, representing a substantial amount of astronomical records, exactly as they were written. Only for papyri could the same be said, but their number and deplorable state of preservation places them far below the cuneiform tablets. It is particularly fortunate that these Babylonian documents were preserved because they allow us to study a type of mathematical astronomy the existence of which we would never have deduced from our Greek sources. From these we knew of systematic observational records in Mesopotamia and of the use of some of the resulting basic parameters by Greek astronomers. This was not very surprising in itself and found confirmation in the official reports on astronomical phenomena (i.e. omina) in the Assyrian royal archives. But it was only from the tablets discovered in Babylon and Uruk that it became clear that a theoretical astronomy existed from the Persian time to the first century A.D., operating with methods entirely different from the Greek ones which were based on the combination of circular motions. From the cuneiform texts we learned that ephemerides had been computed exclusively by means of intricate difference sequences which, often by the superposition of several numerical columns, gave step by step the desired coordinates of the celestial bodies – all this with no attempt of a geometrical representation, which seems to us so necessary for the development of any theory of natural phenomena. It is a historical insight of great significance that the earliest existing mathematical astronomy was governed by numerical techniques, not by geometrical considerations, and, on the other hand, that the development of geometrical explanations is by no means such a "natural" step as it might seem to us who grew up in the tradition founded by the Greek astronomers of the Hellensitic and Roman period. Beyond this, our knowledge of Babylonian methods has become a valuable tool for the discovery of historical connections between the Mediterranean world and India and the Islamic centers because remnants of undoubtedly Babylonian methods were discovered in Sanscrit and in Arabic sources.

We know only very little about the prehistory of this Babylonian astronomy. In the extant texts from the Hellenistic period almost all methods appear fully developed. On the other hand it is virtually certain that they did not exist at the

end of the Assyrian period. Thus one must assume a rather rapid development during the fourth or fifth century B.C.

The same two centuries witness also the first steps in Greek astronomy. The beginning is made by the "school" of Meton and Euctemon (around 430 B.C.) with observations concerning the length of the solar year and with the formulation of the "Metonic" 19-year cycle which may or may not be independent of the contemporary Mesopotamian discovery of the same cycle. In general the whole astronomy of the century from Meton to Eudoxus is dominated by calendaric problems. The variation of the length of daylight or the variation of the length of the shadow during the day are represented by the simplest arithmetical schemes. The rising and setting of stars during the year is related to the seasons and used for weather prognostication. Obviously the mathematical tools did not yet exist which are indispensible for a numerical description of the motion of the celestial bodies. On the other hand we know that Eudoxus had gained profound insight in the theory of irrational quantities (in geometrical formulation); his demonstration that the planetary retrogradations can in principle be explained as the result of superimposed rotations of inclined concentric spheres opened the road to the search for geometric representations of the planetary system. How little equipped Greek mathematics was at that time to handle actual astronomical problems is, however, drastically shown by Aristarchus' clumsy procedure (about 300 B.C.) to solve one right triangle in order to determine (by a most inaccurate method) the relative distances of sun and moon. Similarly in Euclid's "Phaenomena" a first attempt is made to give a mathematical formulation for the most elementary consequences of the introduction of a "celestial sphere" as reference system – obviously a novel and by no means trivial hypothesis.

While spherical astronomy got slowly on its way a mathematical theory of great elegance and sophistication for the motion of the planets was developed by Apollonius (around 200 B.C.) and determined the direction of all subsequent efforts of cinematic planetary theories. All basic ideas for the description of planetary motion by means of eccentric or epicyclic circular motions were fully developed by Apollonius. Here we see for the first time the full force of highly developed mathematical methods applied to the solution of specific astronomical problems; the theory of stationary points is perhaps the most outstanding example. Unfortunately our information is so fragmentary that we have no idea about Apollonius' attitude toward the numerical part of astronomical problems.

For us the influence of Babylonian data, accompanied of course by the sexagesimal number system, is first clearly visible with Hipparchus (around 150 B.C.). Now astronomy becomes a real science in which observable numerical data are made the decisive criterium for the correctness of whatever theory is suggested for the description of astronomical phenomena. It is not surprising that as a result of this attitude no new model of planetary or lunar motion can be ascribed to Hipparchus. His importance lies clearly in the new methodology of exactness, both in definitions and in observational techniques which he introduced. Hipparchus' role is best compared with the role of Tycho Brahe, just as Ptolemy parallels Kepler in the profound theoretical evaluation of his predecessors' material.

Babylonian influence in the second century B.C. is not restricted to the methodology of Hipparchus. In the same period astrology begins to flourish in Hellenistic Egypt. The roots of astrology are undoubtedly to be found in Mesopotamia, emerging from the general omen literature. Yet, we know much less about the history of Babylonian astrology than is generally assumed. Only that much seems clear that it was a far less developed doctrine than we find in Greek astrological literature. The real center of ancient astrology, from which it eventually spread over the whole world, is undoubtedly Hellenistic Alexandria. The claim of astrology that it is based on age-old wisdom or on revelation by a deity could only reduce the incentive for a search for better astronomical methods. Consequently we see that astrological practice is often based on antiquated methods, on a conglomerate of diverse, even contradictory, parameters – facts most welcome for the analysis by the historian but very detrimental for the progress of astronomy.

B. The Roman Period

More than the hellenistic monarchies the Roman empire contributed to the spread of hellenistic science and hellenistic astrology far beyond its political frontiers, though Rome itself seems not to have made any significant contribution. Seneca tells us[1] that the explanation of the planetary phases had reached Rome "only recently", i.e. around the beginning of our era, a century after Apollonius or Hipparchus. Recent discoveries by D. Pingree have shown that hellenistic astrology, and through it the Babylonian arithmetical methods, reached India as early as the second century A.D.[2] This early date is of fundamental importance for the understanding of the fact that pre-Ptolemaic strata become visible centuries later in Indian astronomy, and consequently also in Islamic and finally in Western astronomy.

Alexandria in the second century A.D. saw the publication of Ptolemy's remarkable works, the Almagest and the Handy Tables, the Geography, the Tetrabiblos, the Optics, the Harmonics, treatises on logic, on sundials, on stereographic projection, all masterfully written, products of one of the greatest scientific minds of all times. The eminence of these works, in particular the Almagest, had been evident already to Ptolemy's contemporaries. This caused an almost total obliteration of the prehistory of the Ptolemaic astronomy.

Ptolemy had no successor. What is extant from the later time of Roman Egypt is rather sad: huge commentaries on the Almagest or the Handy Tables by Pappus (about 320), by Theon (about 370) and by his daughter Hypatia.[3] The style of these commentaries is the style later customary in mediaeval schools; as A. Rome has formulated it, they were written "non pour apprendre aux élèves à raisonner, mais pour les empécher de réfléchir".[4] But it is probably through

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1 Seneca, Nat. Quaest. VII 25, 5 (Loeb II, p. 279/281).

2 Cf., e.g., Pingree [1963, 1].

3 Nothing of her writings is preserved but it is unlikely to have been essentially different from her father's.

4 Rome [1948], p. 518.

the editions of these schoolmasters that much of the ancient scientific classics remained in circulation, thus providing the basis for the Arabic translations as well as for the studies of the Byzantine scholars.

C. Indian Astronomy

I think it is fair to say that practically all fundamental concepts and methods of ancient astronomy, for the better or the worse, can be traced back either to Babylonian or to Greek astronomy. In other words, none of the other civilizations of antiquity, which have otherwise contributed so much to the material and artistic culture of the world, have ever reached an independent level of scientific thought. Only into astrology were incorporated two remnants of prescientific astronomical lore from other than Mesopotamian or Greek background: the 36 Egyptian "Decans" and the 28 Indian "Lunar Mansions" (nakshatra). Both are the result of crude qualitative descriptions of the most immediate astronomical observations: the decans reflect the steady shift of the risings and settings of the stars during the course of the year, the nakshatras represent the lunar motion in the sidereal month. Both concepts reached accurate numerical definition only after being assimilated with (and in part transformed by) the ecliptic coordinate system that was developed in Babylonian astronomy.

But while the nakshatras were known in India since the first millennium B.C., the contact with western astronomy dates only from the Roman imperial period. That the preceding Greek occupation of the Punjab should have brought astronomical knowledge to India is not very likely, simply because the Greeks themselves at that period seem not to have had any knowledge of a mathematical astronomy. However, we know so little about the early phase of Greek and of Mesopotamian astronomy at the beginning of the hellenistic era that the argument from negative evidence could be deceptive.

So much, however, can be considered as fairly certain, namely that the first and lasting impact of western astronomy on India came via Greek astrological texts, operating with Babylonian arithmetical methods.[5] These methods were well known in Alexandria at least at the beginning of our era, as is attested through Greek and demotic papyri and from the astrological literature. This agrees well with the appearance of similar works in India by the middle of the second century A.D. when a branch of the Śaka dynasty ruled over a large area of western India with Ujjain as its capital,[6] a locality which since then defined the zero meridian for Hindu astronomy. The Śaka era, however, (epoch: A.D. 78 March 15) is probably the era starting with the regnal years of the Kushana ruler Kanishka.[7]

The first Sanskrit astronomical treatises still preserved were written in the late fifth and sixth centuries A.D., notably the Āryabhaṭīya of Āryabhaṭa and works by Varāhamihira. It is of great interest to see that at that time there existed

[5] Cf. for the details of transmission the masterful paper by D. Pingree [1963, 1].

[6] Mentioned in Ptolemy's Geography (II, 1 § 63) under the name Ὀζηνή with $\varphi = 20$. Actually Ujjain is located at $\varphi = 23;11$ N and 75;50 east of Greenwich.

[7] Cf. Nilakanta-Sastri, CHI II, p. 233 ff.

already the need for a historical survey of Hindu astronomy, thus causing Varāhamihira to explain in his Pañcasiddhāntikā the systems followed by five different siddhāntas (thus the name), among others the early version of the Sūrya-siddhānta, a famous work which has come down to us only in a much later version.

Of particular influence on Islamic astronomy became the Khaṇḍakhādyaka of Brahmagupta, written in A.D. 665, known as the al-Arkand to the Muslims. Parameters from this work and characteristic Hindu modifications of the Greek planetary theories found their way, through Islamic intermediaries, in particular through al-Khwārizmī, into Western European astronomy. The use of the zero meridian of "Arin" (= Ujjain) is the most obvious example of this influence.

In spite of the pioneering work done by H. T. Colebrooke (1765–1837), G. Thibaut (1848–1914) and others the study of Hindu astronomy is still at its beginning. The mass of uninvestigated manuscript material in India as well as in Western collections is enormous. May it suffice to remark that many hundreds of planetary tables are easily accessible in American libraries. So far only a preliminary study of this material has been made revealing a great number of parameters for lunar and planetary tables.[8] The planetary tables themselves are of great extent and based on methods so far not encountered in western material, the basic idea being that the planetary positions are computed for a whole year as function of the initial conditions at the entry of the sun into Aries. When these methods were developed we do not know – the extant texts suggest dates of the 14th century A.D.

D. The Islamic Period

The reception of western methods by the Indians proceeded in two steps, an earlier one mainly based on ultimately Babylonian arithmetical methods, soon to be followed by the Greek cinematic procedures. The first impact on Islamic science was made by Indian astronomy (by that time, the ninth century, already a mixture of peculiarly modified Babylonian and Greek methods), shortly followed by the direct reception through Greek sources of the Ptolemaic system. This complex origin of Islamic astronomy left its traces wherever its influence was felt, from Persia to Spain and from Byzantium to Italy, France, and England.

The details of the sudden beginning of Islamic astronomy at the Abbasid court in Baghdad are no longer discernable. It is customary to assume Syrian works as intermediary, besides the well attested Hindu-Iranian contacts. Although some pre-islamic Syriac astronomical works are known to have existed proof seems lacking that they exercised any influence on the Islamic treatises and tables. Of course, Syrians, knowing Greek as well as Arabic, were instrumental as translators of Greek originals but this is very different from postulating the existence of independent Syrian (or, in particular, Nestorian) works which would have been translated into Arabic. As long as Syria and Egypt were still under Byzantine rule there was very little need for the translation of Greek theoretical

[8] Cf. Poleman, A Census of Indic Manuscripts in the United States and Canada. Am. Oriental Series 12, 1938 (e.g. p. 236); Pingree, SATUS; also Neugebauer-Pingree [1967].

works into Syriac. Only with the gradual diminution of the Greek ruling class after the Islamic conquest does a translation into the language of the new rulers make some sense. I think one may safely assume that the Greek as well as the Indian astronomical treatises were transmitted directly to the scholars of Baghdad and Damascus.[9]

Quite another matter is the problem of Persian influences. There can be no doubt about the existence of a substantial body of astronomical and astrological literature in pre-islamic Persia. We know of Pahlavī translations of such first and second century astrological writings as Teucer and Vettius Valens[10] and of the presence of "Indian books" as well as of the "Roman *μεγίστη*" around A.D. 250 under Shapur I.[11] Under Khosro I Anōsharwān (i.e. "of immortal soul") was revised, around A.D. 550, the famous Zīj ash-Shāh, which has been shown[12] to be greatly dependent on Hindu sources. Its zero meridian was Babylon, probably because of the proximity of Ktesiphon, the Sasanian capital.

When Baghdad was founded by al-Manṣūr in 762 the propitious moment[13] was determined by the Persian astrologer Naubakht and the converted Jew Māshā'allāh, the latter of international mediaeval fame as attested, e.g., by Chaucer's version of his treatise on the astrolabe (1391). Māshā'allāh died around 815/820. Of even greater fame and influence was Abū Ma'shar (787[14] to 886) whose writings were spread in many Greek and Latin versions widely over Europe. That Bīrūnī had, rightly, a low opinion about Abū Ma'shar's astronomical competence[15] did not do any damage to his fame.

To the same period, however, also belongs a group of competent astronomers, e.g., al-Khwārizmī (who died probably before 850), the authors of the Mumtaḥan zīj (the "*Tabulae probatae*" of the West) prepared under al-Ma'mūn (813–833),[16] Ḥabash "the computor" (died 862), Thābit b. Qurra (died 901), and others. During the ninth century both the Almagest and the Handy Tables became available in Arabic translations.

The greatest astronomers of Islam lived in the next two centuries: al-Battānī (858 to 929), aṣ-Ṣūfī (903 to 986), Abū'l Wafā (940 to 997/8), Ibn Yūnis (died 1009), and the universal scholar al-Bīrūnī (973 to 1048). The disintegration of the Muslim world into a great number of independent states is reflected in the geographical dispersion of these men. Battānī worked in ar-Raqqa on the upper Euphrates,[17] Sūfī and Abū-l Wafā in Baghdad, then under Būyid domination, Ibn Yūnis in Cairo under the Fāṭimid ruler al-Ḥākim, while Bīrūnī, born in Khwārazm, had to follow the conqueror of his country, Maḥmūd of Ghazna, to Afghanistan[18]

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[9] The situation seems to be different in the case of Greek and Syriac medicine but there is no necessity to assume that different fields follow identical patterns.

[10] Cf. e.g., Nallino, Scritti VI, p. 295.

[11] Cf. e.g., Bailey, Zor. Probl., p. 86.

[12] Kennedy [1958] and Bīrūnī, Transits; also Pingree [1963, 1], p. 242.

[13] The date is fixed by the horoscope, as given in Bīrūnī's Chronology (trsl. Sachau p. 263 with the omission of ♄ in ♈ 26;40) as 762 July 31, not July 24 as usually stated.

[14] His horoscope was dated by D. Pingree (1962], p. 487 n. 6 to 787 Aug. 10.

[15] Cf. Kennedy, Survey, p. 133, No. 63.

[16] Cf. Vernet [1956].

[17] $\varphi = 35;57$ (36;0 according to Battānī), 39;3 East of Greenwich.

[18] Ghazna is located southwest of Kabul, at $\varphi = 33;33$ (Bīrūnī found 33;35) and 68;28 east of Greenwich.

and then to India. Bīrūnī's greatest astronomical work is dedicated to the son and successor of Maḥmūd, the Sultan Masʿūd, therefore called Qānūn al-Masʿūdī. Shortly before the Qānūn Bīrūnī had completed his work on India (1030), a real mine of information on Indian astronomy and culture. Bīrūnī's earliest great work, the "Chronology of Ancient Nations" (completed about A.D. 1000) belongs still to his life in Khwārazm. The total of Bīrūnī's writings on a great variety of subjects in all branches of exact and natural sciences, as well as literature and philosophy, comes to some 180 titles.[19]

Bīrūnī's work is in all its aspects of outstanding individuality and belongs to a class all its own, both from a purely literary point of view and with respect to the wealth of information it contains for the modern historian. This concerns not only the contemporary oriental civilizations and their history but also the history of Greek mathematics and astronomy, much of which was still accessible to Bīrūnī but is now lost.

Compared with Bīrūnī, Battānī's work appears very pedestrian. But by consistently following the procedures of the Almagest and giving a clear account of the underlying empirical data, Battānī's tables became one of the most important works for the astronomy of the Middle Ages in the Orient and later in the Renaissance.

Aṣ-Ṣūfī became a great historical influence in a different way. In his "Book on the Fixed Stars" he collected whatever he could find of Arabic names of constellations and tried to establish accurate boundaries for each of them, in many cases arbitrarily fixing a rather fluid tradition.[20] In this way a definitive terminology was established which remained the norm for the iconography of the constellations in Islamic astronomy. On the basis of Ṣūfī's catalogue of stars in combination with Ptolemy's catalogue (Almagest VII, 5 and VIII, 1) Bayer in his "*Uranometria*" (1603) established the modern terminology, again with some arbitrary modification of the historical data.[21]

Abū'l Wafā's contributions belong mainly to the field of mathematics in so far as he modernized spherical trigonometry, e.g. by replacing the Menelaos theorem by the sine theorem for spherical triangles. The replacement of the Greek chord-function by the half-chords, i.e. by the sine-function, took place centuries earlier in India.[22]

Parallel to the progress made by the Islamic scholars in spherical geometry numerical techniques were advanced. Improvements in the arrangement and refinement of tables constitute a definite step beyond the level of classical antiquity. In the concluding phase of Islamic astronomy the construction of colossal observational instruments was undertaken. Thus we see a development in the same direction which became of decisive importance in Europe in the 16th century.

We are only very insufficiently informed about the influence of Islamic on Byzantine astronomy. This is because of the almost complete neglect by modern

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[19] Boilot [1955] and Enc. of Islam(2) I, p. 1236–1238.

[20] Cf. Kunitzsch, Sternnomenklatur.

[21] Cf., e.g., Boll, Sphaera, p. 450.

[22] On the other hand Copernicus still speaks only about "half-chord of the double arc" (cf., e.g., De Revol. I, 12).

scholarship of Byzantine astronomical treatises though they are still extant in a quite respectable number. But even on the basis of the available fragmentary information it seems evident that from the 10th century on Islamic material had been studied and, in part, translated into Greek.

The Byzantines were in a position to observe the scientific activities of their neighbors closely. Under Masʿūd began the steady progress of Turkish tribes under the Seljuks. He was defeated by the Turks in 1040 (and murdered the year following); the Seljuks succeeded in occupying Persia, eliminated the Būyids from Baghdad, and reached the Mediterranean on the south coast of Asia Minor. Under the able rule of Jalāl-ad-Dīn Malikshāh (1072–1092) a well staffed observatory came into being probably in Isfahan[23] (1074). It seems to have reached almost the canonical age of 30 years of existence, deemed desirable for a planetary observational program-obviously because of Saturn's 30-year period.

With Malikshāh's name is also connected a famous calendar reform which introduced (instead of the inconvenient Arabic lunar years and instead of the very convenient Persian calendar years) accurate tropical years as the basis of the so-called Jalālī calendar. References to this calendar and its patron, *Μελιξά*, appear repeatedly in Byzantine manuscripts.

To a son of Malikshāh, the Sultan Sanjar (1118 to 1157) was dedicated the famous Sanjarī zīj, translated into Greek and often mentioned in Byzantine astronomical writings. Its author, al-Khāzinī, was himself a Greek freedman of a judge in Marv (Turkestan).

In 1255 the Mongol prince Hulāgū Khān, grandson of Genghis Khān, succeeded in taking the mountain stronghold of the Assassins, al-Alamūt, which had been a major obstacle for a Mongol domination of Persia and Iraq. Among the prisoners taken by Hulāgū in al-Alamūt was the astronomer Naṣīr ad-Dīn aṭ-Ṭūsī, who had been held as honored prisoner and astrological adviser by the "Old Man of the Mountain," the Grand Master of the Assassins. Aṭ-Ṭūsī once more became the trusted adviser of his master, now Hulāgū, and seems to have played in this capacity a somewhat sinister role in the end of the last Abbasid Caliph when Baghdad was taken by the Mongols in 1258.[24]

In the following year the construction of an observatory was begun[25] at Marāgha,[26] south of Tabriz, under the directorship of aṭ-Ṭūsī. Its main product was the Persian Īlkhānī tables (known as *ἐλκανῆ* in Byzantium), completed in 1271, six years after Hulāgū's death. This observatory functioned for an unusual length of time, until the end of the Ilkhānī dynasty itself, i.e. until about 1336.[27]

Aṭ-Ṭūsī's work ranges wide over all branches of astronomy and astrology, optics, mathematics, and the construction of instruments. One small incident may be mentioned here, namely Ṭūsī's discovery that a simple harmonic motion can be produced by the rolling of a circle of diameter 1 inside of a circle of radius 1.

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[23] Sayili, OI, p. 164.
[24] Cf. Boyle [1961].
[25] Cf. Sayili OI, p. 190.
[26] $\varphi = 37;25$ (37;20 in the Khāqānī zīj) and 46;13° east of Greenwich.
[27] Sayili OI, p. 211.

This device was used in the theory of the motion of Mercury in order to replace a crank mechanism used by Ptolemy, intended to move Mercury's epicycle periodically nearer to, or away from, the observer. Aṭ-Ṭūsī's methods were further developed by his pupils, e.g. by Ibn ash-Shātir (1306–1375), and appear finally in Copernicus' De Revolutionibus.[28] Aṭ-Ṭūsī died in 1274, his name being known from Byzantium to China.

The last important Islamic observatory was founded in 1420 by Ulugh Beg in Samarkand, the city which his grandfather Timur (who died in 1405) had made the capital of his empire. A group of prominent scholars, among them al-Kashī (died 1429), were entrusted with the planning and the development of the observatory; its colossal ruins are still extant. The tables which were produced there became very famous, in particular the catalogue of stars which was, e.g., incorporated into Flamsteed's "*Historia coelestis Brittanica*" (1725). Gauss, in 1799, applied his new method of least squares to the equation of time as given in Ulugh Beg's tables.[29] Tycho Brahe, however, seems not to have known these tables.[30]

Modern scholars became aware of one particular case of transmission of oriental astronomy, from Tabriz to Constantinople, through a Greek version of "Persian Tables", probably the Sanjari zīj and the Zīj al-ʿAlaʾī, and canons written by the physician Gregory Chioniades in Constantinople about 1300.[31] Manuscripts of the version by Georgios Chrysokokkes, which reached European collections in comparatively large numbers, and excerpts concerning planetary periods, were made known by Boulliau (1605 to 1694) in his "*Astronomia Philolaica*" (1645).

I do not think, however, that we are dealing here with a very significant incident. The direct tradition from Ptolemy and Theon had never been broken in the Eastern Roman Empire. The interest in astronomy of the emperor Heraclius (610 to 641) is well attested. One of the most famous manuscripts of the "Handy Tables" (Vat. gr. 1291) was written under Leon V (813 to 820). References to Islamic works which were composed from the 9th to the 14th centuries are frequently found in Byzantine treatises. There is no reason to assume that there is any period in which Islamic astronomy was not known in Constantinople. Nevertheless there are certain periods where we know that astronomy became fashionable in the highest circles of Byzantine society, e.g. under the Emperor Manuel Comnenus (1143 to 1180). At the end of the 13th century Theodoros Metochites (died 1332) wrote a commentary to the Almagest; from the 14th century we have treatises on the theory of the astrolabe by Nicephoros Gregoras and by Isaac Argyros. The "Tribiblos" of Theodoros Meliteniotes belongs to the end of the 14th century.

Our chronological summary has thus reached the time of the final collapse of the Byzantine empire. The story has often been told how this affected the West and contributed to the revival of Greek literature and science. As we slowly begin

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[28] Roberts [1957]; Kennedy [1966]; Hartner [1969, 2], [1971]. Cf. also below p. 1035.

[29] Gauss, Werke 12, p. 64–68.

[30] The first (incomplete) western publication seems to be Joh. Gravius, *Epochae celebriores* ..., London 1650. For discussion, cf. Ideler [1832].

[31] Pingree [1964].

to obtain some knowledge of the contents of the astronomical material that reached in this way the European collections of Greek manuscripts, we can say that Byzantine and Islamic astronomy from the Near East was of a higher level of competence and inner consistency than the material that originated from Spain which is reflected in such famous works as the Toledan or Alfonsine Tables.

Nevertheless there can be little doubt that Spain exercised a far greater influence on the revival of astronomy than Byzantium. The earliest work that became widely known in the Latin West is a version of al-Khwārizmī's tables, prepared by Maslama ibn Aḥmad al-Majrīṭī (who died in 1008), translated in the 12th century into Latin, presumably by Adelard of Bath. Thus England and France became influenced by a work reflecting by its mixture of Hindu and Greek elements the earliest phase of Islamic astronomy at a time when this period had long been outdated by eastern Islamic works.

Majrīṭī also revised an Arabic translation of Ptolemy's "Planisphaerium" and thus preserved for us, in a Latin translation,[32] this treatise which represents the basis of the theory of the astrolabe. This is not the only case in which the Spanish tradition connects us directly with Greek astronomy. We know from papyri that there existed a category of astronomical tables in Graeco-Roman Egypt that can be described as "Almanacs," i.e. tables computed for specific calendar years, giving the positions and phenomena of the celestial bodies. It is this approach which appears again in mediaeval Spain, e.g. in the "Perpetual Almanac" of Zarqālī (epoch year 1089), a competent astronomer from whom we have observations for the years around 1060. He also wrote on instruments and is closely related to the preparation of the "Toledan Tables" of which translations and adaptations, e.g. by Gerard of Cremona (1114 to 1187), were widely distributed in Europe. We even have a translation of these tables into Greek, preserved in a Vatican codex of the 14th century.[33] These continuous adaptations certainly did not contribute to the reliability of the tables, which from the very beginning were of a composite character.[34]

Several of the important Spanish scholars were Jews. Ibn Ezra (born about 1090, died 1167) wrote on the astrolabe, on astrology, translated into Hebrew a commentary by Ibn al-Muthannā to al-Khwārizmī, etc. He had travelled extensively, lived some years in Italy and France, visited Cyprus, was in London 1158/9 and thus contributed much to the spread of Islamic astronomy in Europe. Highly competent in astronomy (and strictly opposed to astrology) was Maimonides (1135 to 1204) who was expelled in his youth from Spain, during one of the comparatively rare waves of Muslim intolerance, and became famous as physician and scholar at the Egyptian court under Saladin and his son. In Book III, 8 of his "Code" Maimonides instructed his co-religionaries in the Ptolemaic theory of the lunar motion in order to compute in advance the date of the visibility of the first crescent. Philosophically, however, he objected to the Ptolemaic methods as not based exclusively on circular motions. In the "Guide of the Perplexed" he therefore advocated in principle the system of al-Biṭrūjī (died 1204)

[32] Cf. Heiberg in Ptolemaeus, Opera minora, Prolegomena, p. CLXXXVII; also Suter, MAA, p. 77. Below V B 3.
[33] Vat. gr. 212 (Mercati-Cavalieri CVG I, p. 270).
[34] Cf. Toomer [1968, 2].

which was based on concentric spheres rotating about inclined axes.[35] Michael Scot's translation of Biṭrūjī's "*De motibus celorum*," completed 1217 in Toledo, made this astronomically hopeless theory known in Europe. Even Regiomontanus studied it seriously and wrote a refutation.[36] Nevertheless it found in Fracastoro[37] (1478–1553) again a convinced advocate.

The Toledan Tables were superseded by the Alfonsine Tables whose epoch, 1252 June 1, is the day of coronation of Alfonso X; actually the tables were completed about 20 years later. These tables reached a very wide distribution, eventually through many printed editions, but astronomically they are not essentially superior to the Toledan Tables. Alfonso, who ruled from 1252 to 1284, instigated, or wrote himself, a great number of works. The famous *Libro del Saber* (1276/9) is a compilation on instruments; the *Libros de las Cruces* with its endless enumeration of trivial combinations of astrological influences reveals an unusual dullness of mind.

At the end of Spanish astronomy we find again a Jewish scholar, Abraham Zacut, born about 1452 in Salamanca, who wrote, among others, a "perpetual almanac" and published tables used by Columbus and Vasco da Gama. Zacut was expelled from Spain in 1492 and died about 1522 in Damascus.

The rapid spread of astronomical interest in the late Middle Ages is illustrated by the fact that a Jewish scholar from Tarascon, Immanuel Bonfils composed, about 1370, an astronomical treatise in Hebrew, commonly called "Six Wings" from its six chapters.[38] This work was translated to Greek and a commentary written in 1435 by Michael Chrysokokkes[39] under the title Hexapterygon.[40] The work has a rather restricted scope since it concerns only the computation of lunar and solar eclipses. The mean conjunctions are found by means of the Jewish lunar calendar. The true conjunctions and the corrections for parallax can be found from very extensive tables with a minimum of computation. It is probably for this reason that this work became so popular.

Already at the Council of Basle (1431 to 1448) the question of the "reform" of the julian calendar was put on the agenda. The bishop (later cardinal) Nicholas of Cusa (1401 to 1464) was one of the main supporters of the plan, having made detailed suggestions of his own for the procedures for the shift to a new calendar. He himself was an ardent, if not very successful, student of astronomy. At the Council he met in 1434/5 the metropolitan, also later cardinal, Isidore of Kiev, who led the embassy sent by Andronicus III Palaiologus which should negotiate the "Union" of the Eastern and the Western churches. In Basle Isidore showed Cusanus "Persian Tables" in Greek, a fact proudly recorded by the latter, who translated certain sections into Latin.[41] The result proved as ineffective as the calendar reform,[42] the Union, and the Council.

[35] It has been shown by B. Goldstein that Biṭrūjī's model is not a simple modification of the Eudoxan-Callippic model but is based on an independent idea (cf. Biṭrûjî, Astron., ed. Goldstein Vol. 1).

[36] Cf. Rosen [1961].

[37] The physician, often mentioned for his poem on syphilis.

[38] The original title was Eagle Wings.

[39] He is not to be confused with Georgios Chrysokokkes, mentioned on p.

[40] Cf. the summary of the contents in Solon [1970].

[41] Cf. Stegemann, Cusanus, p. 54/55 and p. 117, note 11.

[42] It had still to wait until 1582.

E. Epilogue

Under the influence of the tradition of the 18th and 19th centuries, astronomy seems to us almost synonymous with continuous systematic observation of the celestial objects. The historical development, however, shows a quite different pattern. Both Babylonian and Greek astronomy are based on a set of relatively few data, like period relations, orbital inclinations, nodes and apogees, etc. The selection of these data undoubtedly required a great number of observations and much experience to know what to look for. Nevertheless, a mathematical system constructed at the earliest possible stage of the game was generally no longer systematically tested under modified conditions.

This attitude can be well defended. Ancient observers were aware of the many sources of inaccuracies which made individual data very insecure, e.g. the finite diameters of slits and pins at instruments, and of the human eye, the inaccuracy of shadow- and water-clocks, etc. On the other hand period relations, e.g. time intervals between planetary oppositions and sidereal periods, can be established within a few decades with comparatively high accuracy because the error of individual observations is distributed over the whole interval of time. If a theory was capable of guaranteeing correct periodic recurrence of the characteristic phenomena then intermediate deviations would matter little, in particular when one had no means to distinguish clearly between theoretical and observational causes of the errors. Thus we notice an outspoken tendency in the development of astronomy to take refuge behind mathematical schemes, rather than to embark on systematic observational programs which are so characteristic for astronomy since the invention of the telescope.

Yet we know of countless observations recorded between the 9th and the 16th centuries and practically every astronomer claimed, probably correctly, to have made (some) observations of his own. Nevertheless the results of these observations turned out to be insignificant. They were not designed to test over and over again the validity of the accepted mathematical idealizations, or to follow up all the ramifications of a change in any one of the basic parameters. For example, Islamic astronomers persisted in the measurement of the inclination of the ecliptic and of other elements of the solar orbit, like eccentricity and apogee. Yet the effects of an improved determination of the solar orbit on the elements of the planetary orbits were not investigated simultaneously. Thus it became possible that more and more inconsistent elements were contributing to mediaeval tables until it no longer could be doubted that a revision of all its foundations had become necessary. Rightly Tycho Brahe called his observational program the foundation of an *Astronomia Nova*. *Ancient Astronomy* is overwhelmingly *mathematical* astronomy.

A reader who ventures beyond this point must be warned that he should not except to find in the following a "History of Ideas". All I intended to do is to illustrate some of the enormous technical difficulties which had to be overcome before astronomy could become fully associated with the progress of mathematics after both fields had laid stagnant for some 1300 years. How the self-appointed custodians of "ideas" reacted to astronomical and mathematical developments

seems to me of little interest. Nevertheless it might be useful, if not amusing, to see how the crowning achievement in classical astronomy is reflected in a famous philosopher's mind:

"Es wird von den Mathematikern selber zugestanden, daß die Newtonischen Formeln sich aus den Kepplerischen Gesetzen ableiten lassen. Die ganz unmittelbare Ableitung aber ist einfach diese: Im dritten Kepplerischen Gesetz ist $\frac{A^3}{T^2}$ das Constante. Dies als $\frac{A \cdot A^2}{T^2}$ gesetzt und mit Newton $\frac{A}{T^2}$ die allgemeine Schwere genannt, so ist dessen Ausdruck von der Wirkung dieser sogenannten Schwere im umgekehrten Verhältnisse des Quadrats der Entfernungen vorhanden".(Hegel, System der Philosophie. II. Die Naturphilosophie. Sämtliche Werke IX (Stuttgart 1929), p. 124.)[43]

§ 3. General Bibliography

A list of all bibliographical abbreviations used throughout in the present work will be found at the end of Book VI (cf. p. 1165ff.). Several sections in the following text are supplemented by bibliographical notes; cf. the list given below p. 18. Also the subject index (p. 1133ff.) may be consulted for the literature on specific topics.

A. Source Material

The following references are meant as a first guide to the primary sources, i.e. to the editions of texts or to works where such references can be found; completeness is not attempted.

For the pre-Greek period the situation is simple: the sources for Babylonian mathematical astronomy are collected in Neugebauer, ACT (1955); for Egypt cf. p. 566f.

For classical antiquity and the Middle Ages no systematic collection of mathematical or astronomical treatises exists. No attempt has ever been made to compile basic collections comparable to the Loeb Classical Library or the Budé collection, or Migne's Patrologia, the Monumenta Germaniae Historica, the Bonn Corpus of Byzantine historians, etc. Except for the Teubner editions of Greek and Latin authors no set of texts and translations exists[1] and one must find one's way from author to author through handbooks and bibliographies. This fact alone suffices to show that the so-called "History of Science" is still operating on an exceedingly primitive level.

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[43] Similarly Schopenhauer, in his "Farbenlehre", about Newton's optics (Sämtliche Werke 6, Wiesbaden 1947, p. 211).

[1] In the year A.D. 1964 I was informed that the editors of the Loeb Classical Library "felt that they had discharged their duties toward the Loeb Library as well as toward Greek Mathematics" (and by inference astronomy) by publishing a two-volume anthology (for which cf. my review in AJP 64, 1943, p. 452–457).

In this situation the five huge tomes of Sarton's "Introduction" (1927 to 1948), ending at 1400, are a useful tool, although being hardly more than a compilation from all available encyclopaedias. Unfortunately this method resulted in an overburdening with irrelevant material, not to speak about the absurdly rigid chronological arrangement and the historical introductions which are reminiscent of the mentality of Isidore of Seville.

For Greek and Latin authors one will naturally consult first the "Real-Encyclopädie" of Pauly-Wissowa which, however, does not include the Byzantine period. For this period one has now a bibliography in Vol. IV, 2 of the second edition of the Cambr. Med. Hist. (1967).[2] Unpublished manuscripts, extant in vast quantities, can only be identified through the descriptions in library catalogues, few of which are sufficiently detailed, in particular when scientific subjects are involved.[3] For the location of Greek manuscripts and their catalogues one has an excellent guide in Richard, Répertoire (1958, 1964). The 12 volumes of the CCAG contain in many cases also information about purely astronomical sections in "astrological" manuscripts. For Greek papyri cf. the bibliographies Neugebauer [1962, 1] and Neugebauer-Van Hoesen [1964].

For the Islamic period one has, of course, the standard work of Brockelmann, GAL, and the articles in the Encyclopaedia of Islam (second edition, beginning 1960). For the exact sciences Suter, MAA (1900, Nachtr. 1902; also Renaud [1932]) is fundamental, for astronomy Kennedy, Survey (1956). A mine of information is Nallino, Battānī, and his Scritti, Vol. V and VI. For Indian astronomical texts cf. Pingree SATUS and Pingree [1963].

The worst conditions prevail for the mediaeval european scientific literature in Latin. The best guide to the sources and their interrelation is Haskins SMS. Thorndike's Magic (in 8 volumes, 1923 to 1958, reaching to the 17th century) is not fruitful for the astronomical literature which was beyond the author's competence.

Steinschneider's unsurpassed mastery of mediaeval Arabic and Hebrew sources has produced his great bibliographical works which are an indispensible tool for all studies in mediaeval sciences; for the Spanish area cf. the publications of Millás Vallicrosa (cf. VI D 2).

B. Modern Literature

At the end of the 18th century France was undoubtedly the center of mathematical and astronomical progress. At that period ancient or oriental astronomical data were still of current practical interest and it is therefore not surprising to see the modern historical study of astronomy originate in France at the turn of the century. Here we have the monumental works, in part strictly historical, in part introductory to astronomy itself, by Montucla (published between 1758 and 1802), by Bailly (between 1775 and 1782), by Lalande (between 1764 and 1803),

[2] By K. Vogel; the text, covering all sciences on 40 pages is of little use but the bibliography (p. 452–456, 463–469) is the best one has at present; omission: Pingree [1964].

[3] Descriptions like "astronomical tables" is usually all the information one can hope for.

by Delambre (between 1817 and 1827), and many historical excurses in the contemporary works, e.g. in Laplace's "*Exposition du système du monde*" (1796). Much more elementary but very useful for bibliographical matters are the publications of Joh. Fridr. Weidler "*Historia astronomiae*" (1741) and "*Bibliographia astronomica*" (1755), both published in Wittenberg.

Work on the history of astronomy became stagnant during the main part of the 19th century. Its modern revival may perhaps be reckoned from P. Tannery who published as fruits of his researches a "Histoire de l'astronomie ancienne" in 1893. More or less in the same style of a historical outline is Dreyer's "History of the Planetary Systems from Thales to Kepler" (1906) whereas Heath, "Aristarchus" (1913) deals mainly with earliest Greek astronomy or cosmogony. These works suffer from the overemphasis on philosophical speculations during the prescientific period and (unavoidable at that time) from the ignorance of oriental sources. The same tendency still prevails in Duhem's "Système du monde" (1913 to 1917) which provides very little insight into the history of technical astronomy. An outstanding summary of the history of Islamic astronomy was given by Nallino in his lectures in Cairo 1909/10, published in an Italian version in Vol. V (p. 88–329) of his Scritti. The professional attitude of Delambre was resumed by N. Hertz in his "Geschichte der Bahnbestimmung" (1887, 1894) which concerns mathematical astronomy from Eudoxus to the time of Halley and Cassini.

The impact of Kugler's discoveries in Babylonian astronomy (1900 to 1914) is first felt in Pannekoek's "History of Astronomy" (1951), to become the dominant theme in my "Exact Sciences" (1951, 1957) and in van der Waerden's "Anfänge der Astronomie" (1966).

The works which we have mentioned in connection with the primary sources (above p. 15f.) deal, of course, frequently with the general historical background. Beyond this exist special reference works of which shall be mentioned Lalande, Bibliographie (1803) and his Astronomie (1764 to 1792), the bibliographical references in Wolf's "Handbuch" Vol. II (1872) and in his "Geschichte der Astronomie" (1877); finally Houzeau "Vademecum" (1882) which is particularly useful for technical details, parameters and empirical constants, and Houzeau-Lancaster, "Bibliographie générale" (1887 to 1892).

For tables, designed for historical purposes, should be mentioned:

for chronology: Schram, Tafeln;

for the computation of all types of astronomical phenomena: P. V. Neugebauer, Tafeln, and Astron. Chron.; also Baehr, Tafeln, and Ahnert, Tafeln;

for the positions of sun, moon, and planets: Tuckerman, Tables (from -600 to $+1649$); Goldstine, New and Full Moons (from -1000 to $+1651$).

Chronological Summaries are found below in the following sections:

C. Sectional Bibliographies

The following sections contain bibliographical information pertaining to their specific topic:

chronological tables: p. 1075 f.
chronology in general: p. 1074 f.
demotic and coptic texts: p. 567 f.
eclipses: p. 1093
Egyptian astronomy in general: p. 566 f.
fixed star phases: p. 1092, p. 589
Handy Tables: p. 976 f.
lunar inequalities: p. 1108 ff.
papyri: p. 787 f.
parapegmata: p. 589
planetary phases: p. 1091 f.
Ptolemy's "Geography" and geography in general: p. 940
stars, names and catal. of stars: p. 1087.

The Subject Index (p. 1133–1165) is of central importance for the use of this work. The arrangement of the references is intended to furnish access to interconnections and topics which are not expressly treated as main subjects in the text.

The Bibliographical Abbreviations (p. 1165–1203), extensive as they are, are not meant to be taken as a complete bibliography of ancient astronomy.

Book I

The Almagest and its Direct Predecessors

Addidit Ptolomeus in astronomia plus quam esset id totum quod ante se scriptum invenit.

Anon., Ashmole 191. II fol. 138v marg.

A. Spherical Astronomy

La sphère inspire les méditations des géomètres par le nombre de ses propriétés; quand elle procède de la nature physique, elle en acquiert des qualités nouvelles.

Anatole France, L'Île des Pingouins (Œuvres, Vol. 18, p. 70)

Εἴ τις λέγει ὡς τὸ τοῦ κυρίου ἐξ ἀναστάσεως σῶμα αἰθέριόν τε καὶ σφαιροειδὲς τῷ σχήματι ... ἀνάθεμα ἔστω.

Concilium Constantinopolitanum (A.D. 543); Canones concilii V adversus Origenem. X. (Joannes Dominicus Mansi, Sacrorum Conciliorum ... collectio IX, ed. nov., Florentiae 1763, col. 400)

§ 1. Plane Trigonometry

1. Chords

The Chaps. 10 and 11 of Book I of the Almagest contain the ancient theory of plane trigonometry and the resulting tables. The basic function, however, is not the sine function but its equivalent, the length of the chord subtended by the given angle in the unit circle. We shall use the notation crd α if the radius of the unit circle is 1, but Crd α if the radius R of the circle is the sexagesimal unit $R=1{,}0=60$. Ptolemy uses Crd α exclusively but we shall frequently replace it by crd α which only implies a shift in the sexagesimal place value.

Since one trigonometric function (e.g. sin α) suffices to express all the other ones and since crd α and sin α are related by the simple rules (cf. Fig. 1)

$$\operatorname{crd}\alpha = 2\sin\frac{\alpha}{2} \qquad \sin\alpha = \frac{1}{2}\operatorname{crd} 2\alpha$$

it is obvious that there can be no essential difference between the methods for the solution of elementary trigonometric problems when using chords or modern trigonometric functions. The really important difference between ancient and modern procedures lies in a practical point, namely in the absence of tables for the equivalent of the function tan α, i.e. of the quotients crd α/crd (180-α). Consequently every problem which in modern terminology involves tan α required in antiquity two entries in the tables to find separately crd α and the value of the co-function crd (180-α) and finally the computation of the quotient. Obviously it is the smallness of the number of people who were interested and able to undertake productive work in theoretical astronomy that is responsible for the slow progress in the mechanisation of procedures. Whenever a large number of practitioners is involved, as, e.g., in calendaric or in astrological computations, we notice the tabulation of a variety of sometimes very complicated functions taking place. But trigonometric computations only occur in the derivation of new procedures and this does not happen between Ptolemy and the Abbasid period.

In our description of the ancient procedures we shall often use modern trigonometric functions but only so far as it does not influence the method itself. In other words we will never use trigonometric identities for which there does not exist a counterpart within ancient procedures.

2. The Table of Chords

Before giving some examples for the operation with chords we shall describe shortly the procedure which led to the computation of a table of chords in steps of 1/2°.

The basic idea is simple: several chords are easily determined since they are the sides of constructible regular polygons. Using an important theorem on quadrilaterals inscribed in a circle one then can find crd $(\alpha+\beta)$ and crd $(\alpha/2)$ and in this way reach the chords of all arguments which are multiples of 3/4°. Finally an interpolation procedure is employed to obtain steps of only 1/2°.

The method described in the Almagest is based on parameters which we modify only in so far as we use a circle of radius $r=1$ instead of Ptolemy's $R=1{,}0$.

Let s_n be the side of the regular n-gon, then (cf. Fig. 2)

$$s_n = \text{crd}\,(360/n) = \text{crd}\,\alpha_n$$
$$s_n' = \text{crd}\,(180\text{-}\alpha_n) = \sqrt{4r^2 - s_n^2}.$$

The following values are known

$$\begin{array}{lll} \alpha_3 = 120° & s_3 = \sqrt{3} = 1;43,55,23 & \\ \alpha_4 = 90 & s_4 = \sqrt{2} = 1;24,51,10 & \\ \alpha_5 = 72 & s_5^2 = s_6^2 + s_{10}^2 & \text{(Euclid XIII, 10; cf. Fig. 3)} \qquad (1) \\ \alpha_6 = 60 & s_6 = 1 & \\ \alpha_{10} = 36 & (s_{10}+s_6)/s_6 = s_6/s_{10} & \text{("golden section"; cf. Fig. 3).} \end{array}$$

Ptolemy does not say how the approximations for $\sqrt{2}$ and $\sqrt{3}$ were found but we know that at least the above given value for $\sqrt{2}$ was used as early as in Old Babylonian times. Obviously this whole set of relations was considered as well known.

The next step consists in the proof of the following theorem (cf. Fig. 4):

$$a_1 a_3 + a_2 a_4 = d_1 d_2. \tag{2}$$

One constructs on the diagonal d_1 a point Q such that the angle $P_3 P_2 P_4$ = angle $P_1 P_2 Q$. Then one has to notice that angles which are marked in our figure by equal symbols are equal, either because they subtend the same chord, or by construction. This shows that we have two pairs of similar triangles

$$P_1 Q P_2 \approx P_4 P_3 P_2 \quad \text{and} \quad P_2 Q P_3 \approx P_1 P_2 P_4.$$

Thus

$$a_1 a_3 = d_2 x \qquad \text{and} \qquad a_2 a_4 = d_2(d_1 - x),$$

two relations which combine to (2).

It is now easy to show that this theorem leads to the desired relations by specializing it to the case where one side of the quadrilateral coincides with the diameter d of the circle.

(A.) From given $s_1 = \text{crd}\,\alpha_1$ and $s_2 = \text{crd}\,\alpha_2$ (cf. Fig. 5) one can find $x = \text{crd}\,(\alpha_1 - \alpha_2)$ linearly from

$$x d + s_2 \sqrt{d^2 - s_1^2} = s_1 \sqrt{d^2 - s_2^2}.$$

Thus crd 12° = crd (72 − 60) can be computed, using (1).

(B.) From given $s = \text{crd}\,\alpha$ one can find linearly $x^2 = \text{crd}^2\,(\alpha/2)$ because (Fig. 6)

$$x d + x \sqrt{d^2 - s^2} = s \sqrt{d^2 - x^2}.$$

Hence, starting with crd 12° one can obtain the chords of 6°, 3°, 3/2°, and 3/4°.

(C.) From given s_1 and s_2 (Fig. 7) one can find $x = \text{crd}\,(\alpha_1 + \alpha_2)$ from the quadratic equation

$$s_1 \sqrt{d^2 - x^2} + s_2 d = x \sqrt{d^2 - s_1^2}.$$

For (B) and (C) Ptolemy does not use the above given procedures. In case (C) he avoids the determination of x from a quadratic equation by finding first the chord x' (cf. Fig. 8) from

$$x' d + s_1 s_2 = t_1 t_2 \qquad t_1 = \sqrt{d^2 - s_1^2} \qquad t_2 = \sqrt{d^2 - s_2^2}$$

and then x from $\sqrt{d^2 - x'^2}$.

In case (B), however, Ptolemy does not use the quadrilateral theorem at all but follows a different procedure which goes back to Archimedes, as we know from a section in Thābit ibn Qurra's translation of Archimedes' treatise on the heptagon[1]. Since BC = CD (cf. Fig. 9) we have also $\beta_1 = \beta_2$. If we now make AF = AB we have also FC = CB = CD = x. If one draws CE as altitude in the

[1] Schoy, Al-Bir. p. 81 (No. 14); cf. also below p. 776.

triangle FDC one can find $y = \mathrm{DE}$ from $y = 1/2\,\mathrm{DF} = 1/2\,(d - \sqrt{d^2 - s^2})$. But $x : y = d : x$ and therefore $x = \mathrm{crd}\,\alpha/2$ can be computed.

The fact that here an Archimedean construction replaces the quadrilateral theorem might be used as an argument for a more recent date of the latter theorem which, for us, is for the first time attested in the Almagest.

(D.) Ptolemy now states, without proof, that crd (1/2°) cannot be found "by geometric construction,"[2] as is indeed the case.[3] He therefore proves the inequality

$$4/3\ \mathrm{crd}\ 3°/4 > \mathrm{crd}\ 1° > 2/3\ \mathrm{crd}\ 3°/2.$$

Within an accuracy of two sexagesimal places the values obtainable from the parameters (1) for 4/3 Crd 3°/4 and 2/3 Crd 3°/2 are both found to be 1;2,50. Thus one has also Crd 1° = 1;2,50. This completes the list of fundamental parameters needed to compute a table of chords in steps of half degrees.

Almagest I, 11 gives this table, for arguments from 1°/2 to 180°. Tabulated are the chords to minutes and seconds of units in which the radius R of the circle is 60. There follows a column of coefficients of interpolation for minutes of arc. Repeatedly the last digit of these coefficients does not agree exactly with the 30th part of the difference in the preceding column. It seems as if the underlying table of chords was computed to one more sexagesimal place.

3. Examples

No. 1. Find the length s of the noon shadow of a gnomon of length 60 for the summer solstice at Rhodes (Alm. II, 5). We assume the geographical latitude to be $\varphi = 36°$, the obliquity of the ecliptic $\varepsilon = 23;51,20°$. Thus (Fig. 10)

$$\alpha = \varphi - \varepsilon = 12;8,40 \qquad 2\alpha = 24;17,20 \qquad 2\beta = 155;42,40.$$

From the table of chords

$$s = g\,\frac{\mathrm{Crd}\,2\alpha}{\mathrm{Crd}\,2\beta} = 1,0\,\frac{25;14,43}{1,57;18,51} \approx 12;54,42.$$

In modern terms one would have said $s = g \tan\alpha$ and $\tan\alpha = 0.21519 \approx 0;12,54,41$ which agrees with Ptolemy's result.

No. 2. We assume that the sun S moves on a circle of radius $R = 60$ with center M (cf. Fig. 11). We take as eccentricity $\mathrm{OM} = e = 2;30$ and as position of the apogee A = ♊ 5;30. The angle $\bar{\kappa}$, i.e. the distance of S from A, is called the *mean eccentric anomaly*. We wish to find the value of $\bar{\kappa}$ if S is seen from O in ♎ 0° (Alm. III, 7).

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[2] διὰ τῶν γραμμῶν, meaning "rigorous" methods (cf. below p. 771 n. 1).

[3] If one could construct crd(1/2°), one could find, by virtue of the preceding steps, the chord for 1°, thus for 2°, 4°, 4 + 6 = 10°, hence also for 20° and finally for 40° which is the side of the regular 9-gon. But Gauss has shown (Disquis. arithm. § 365, Werke I, p. 461) that the construction of a regular n-gon by ruler and compass is only possible when n is a prime number of the form $2^k + 1$ (k an integer).

We know the angle $\gamma=\lambda_A=65;30$, hence $2\gamma=131;0$. Furthermore

$$MK=\frac{e}{2,0}\operatorname{Crd} 2\gamma \quad \text{as well as} \quad MK=\frac{R}{2,0}\operatorname{Crd} 2c;$$

hence from the table of chords

$$\operatorname{Crd} 2c=\frac{e}{R}\operatorname{Crd} 2\gamma=\frac{2;30}{1,0}\cdot\operatorname{Crd} 131=2;30\cdot 1;49,11,44\approx 4;33.$$

Again from the tables $2c=4;20$ thus $c=2;10$ and

$$180-\bar{\kappa}=\gamma-c=63;20 \quad \text{hence} \quad \bar{\kappa}=116;40.$$

No. 3. Find the angle c, the so-called "equation," to given mean eccentric anomaly $\bar{\kappa}=30$ (Alm. III, 5). Apogee and eccentricity are the same as in the preceding example.

Make (cf. Fig. 12) OK perpendicular to SMK. Then

$$OK=\frac{e}{2,0}\operatorname{Crd} 2\bar{\kappa}=\frac{2;30}{2,0}\operatorname{Crd} 60=\frac{2;30}{2,0}\cdot 1,0=1;15.$$

From the table of chords

$$MK=\frac{e}{2,0}\operatorname{Crd}(180-2\bar{\kappa})=\frac{2;30}{2,0}\operatorname{Crd} 120=\frac{2;30}{2,0}\cdot 1,43;55\approx 2;10.$$

Consequently we have in the right triangle SOK with $SM=R=60$

$$SK=SM+MK=1,2;10 \quad \text{thus} \quad SO=\sqrt{SK^2+OK^2}\approx 1,2;11^4$$

and therefore, if $ON=NS=1/2\,SO$

$$OK=\frac{ON}{2,0}\operatorname{Crd} 2c \quad \text{or} \quad \operatorname{Crd} 2c=\frac{1,0\cdot OK}{SO}=\frac{2,30}{1,2;11}\approx 2;25.$$

From the table one finds for the corresponding angle $2c=2;18$, thus $c=1;9$. This value is also found tabulated for $\bar{\kappa}=30$ in the table for the solar anomaly (Alm. III, 6).

The examples Nos. 2 and 3 deal with complementary problems. In the first case the mean longitude $\bar{\kappa}$ had to be found from the (observable) true longitude κ; in the second case the mean longitude is assumed to be known and the true longitude is asked for, to be found from $\kappa=\bar{\kappa}-c$.

4. Summary

For the sake of convenience I write down the three basic formulae for the solution of right triangles in the form in which they appear in ancient trigonometry, assuming the norm $2R=120$ for the function $\operatorname{Crd}\alpha$. I also give the equivalent

[4] Using the procedure described below p. 1114 we get for $\sqrt{1,2;10^2+1;15^2}=\sqrt{1,4,26;15,25}$ as first approximation $\alpha_1=1,2;10$. Thus $\beta_1=1,4,26;15,25/1,2;10\approx 1,2;12$. Hence $\alpha_2=1,2;11$.

formulae for $R=1$ and the modern trigonometric functions (cf. Fig. 13):

$$\frac{a}{c}=\frac{1}{2{,}0}\,\mathrm{Crd}\,2\alpha=1/2\,\mathrm{crd}\,2\alpha=\sin\alpha$$

$$\frac{b}{c}=\frac{1}{2{,}0}\,\mathrm{Crd}\,(180-2\alpha)=1/2\,\mathrm{crd}\,(180-2\alpha)=\cos\alpha$$

$$\frac{a}{b}=\frac{\mathrm{Crd}\,2\alpha}{\mathrm{Crd}\,(180-2\alpha)}=\frac{\mathrm{crd}\,2\alpha}{\mathrm{crd}\,(180-2\alpha)}=\tan\alpha.$$

For the general triangle the equivalent of the sine theorem is a direct consequence of the fact that the angle in the center is twice the angle at the periphery over the same chord (cf. Fig. 14):

$$a\!:\!b\!:\!c=\mathrm{Crd}\,2\alpha\!:\!\mathrm{Crd}\,2\beta\!:\!\mathrm{Crd}\,2\gamma=\sin\alpha\!:\!\sin\beta\!:\!\sin\gamma.$$

This theorem is repeatedly used, e.g. in the determination of the eccentricity of the planets (cf. p. 174, Steps 1 and 2).

Fig. 13 also illustrates the reason for a terminology which is frequently encountered in the solution of right triangles. Where we would say $a=\sin\alpha$ ancient trigonometry has to express the fact that a is the chord to the angle 2α at the center of the circumscribed circle. This then is expressed in the form[5]

angle CAB $=\alpha$ as 4 right angles are 360

or $=2\alpha$ as 2 right angles are 360.

Now one enters the table of chords with the argument 2α and finds $a=\mathrm{crd}\,2\alpha$.

§ 2. Spherical Trigonometry

1. The Menelaos Theorem

The trigonometry of the right spherical triangle (cf. Fig. 15) rests on the following four formulae[1]

$$\sin\alpha=\frac{\sin a}{\sin c} \tag{1}$$

$$\cos\alpha=\frac{\tan b}{\tan c} \tag{2}$$

$$\tan\alpha=\frac{\tan a}{\sin b} \tag{3}$$

$$\cos c=\cos a\cos b. \tag{4}$$

[5] Cf., e.g., ed. Heiberg I, p. 317, 22f., et passim.

[1] Two additional formulae (or their counterparts for chords), namely $\cos a=\cos\alpha/\sin\beta$ and $\cos c=\cot\alpha\cot\beta$ which express the sides by means of the angles never occur in ancient spherical trigonometry, although it was known that a spherical triangle is determined by its angles (Menelaos I, 18; Krause, p. 138). The equivalent of (1), (4) and $\cos a=\cos\alpha/\sin\beta$ is proved by Copernicus (De revol. I, 14 Theorems 3 and 4) but he has still no formula in which a tangent occurs.

With decreasing sides the formulae (1) to (3) tend toward the defining relations for the trigonometric functions in the plane right triangle whereas (4) is the equivalent of the Pythagorean theorem $c^2=a^2+b^2$.

These relations were not discovered before the Islamic period. In Greek astronomy relations were utilized which involve six parts in a certain configuration of which the triangle under consideration is a part and therefore called by the Arabs "the rule of the six quantities." The discovery of this theorem is due to Menelaos whose date is known through a reference in the Almagest[2] to an observation of his made in the time of Trajan (in A.D. 98). The proof of this "*Menelaos Theorem*" is preserved not only in the Almagest (I, 13) but also in Menelaos' original work on "Spherics," extant in an Arabic version. It proceeds in two steps. First it is shown to hold for a plane configuration and then it is associated with a similar spherical arrangement. In order to present the idea of the proof as simply as possible we introduce the following terminology. We call a "Menelaos Configuration" a configuration as represented in Fig. 16. Two "outer parts" AV and BV meet in the "vertex" V, two "inner parts" BC and AD intersect in K. In this way six line segments are defined which we denote as follows

$$\begin{array}{lll} \text{outer parts: } m_1, m_2 & \text{and} & m_1+m_2=m \\ \phantom{\text{outer parts: }} n_1, n_2 & \text{and} & n_1+n_2=n \\ \text{inner parts: } r_1, r_2 & \text{and} & r_1+r_2=r \\ \phantom{\text{inner parts: }} s_1, s_2 & \text{and} & s_1+s_2=s. \end{array}$$

For these six parts two theorems are established. If c is parallel to r_1 (cf. Fig. 16) then

$$\frac{m}{m_1}=\frac{r}{c}=\frac{r}{r_1}\cdot\frac{r_1}{c} \quad \text{and} \quad \frac{r_1}{c}=\frac{s_2}{s};$$

hence

Theorem I.
$$\frac{m}{m_1}=\frac{r}{r_1}\cdot\frac{s_2}{s}$$

in which two outer elements are expressed by means of inner ones. Similarly, if we make d parallel to s_1 (Fig. 16) we have

$$\frac{m_2}{m_1}=\frac{r_2}{r_1+e}=\frac{r_2}{r_1}\cdot\frac{r_1}{r_1+e} \quad \text{and} \quad \frac{r_1}{r_1+e}=\frac{n_2}{n}$$

thus

$$\frac{m_2}{m_1}=\frac{r_2}{r_1}\cdot\frac{n_2}{n}$$

or

Theorem II.
$$\frac{r_2}{r_1}=\frac{m_2}{m_1}\cdot\frac{n}{n_2}.$$

Here two inner parts are expressed by means of outer ones.

We now prove that similar relations hold for a Menelaos configuration made up by great circle arcs on a sphere (cf. Fig. 17). Let A be the vertex, AB and $A\Gamma$

[2] VII, 3 (Heib. II, p. 30, 18).

the outer parts, BE and $\Gamma\Delta$ the inner parts, intersecting in Z. We consider the plane triangle $\mathrm{A}\Gamma\Delta$ and the intersections Λ, K, Θ of the radii of the sphere HE, HZ, HB, respectively with the plane of this triangle. Thus Λ lies on the side $\mathrm{A}\Gamma$, K on $\Gamma\Delta$, Θ on $\mathrm{A}\Delta$ (extended). But Λ, K, Θ do not only belong to the plane of the triangle $\mathrm{A}\Gamma\Delta$ but also to the plane (shaded in Fig. 17) of the great circle EZB with center H. Thus Λ, K, Θ lie on the straight line in which the plane of the triangle $\mathrm{A}\Gamma\Delta$ intersects the plane of the great circle EZB. This shows that associated with the spherical Menelaos configuration of vertex A is a plane Menelaos configuration with the outer elements $\mathrm{A}\Theta$, $\mathrm{A}\Gamma$ and with the inner elements $\Theta\Lambda$, $\Gamma\Delta$ which intersect in K.

The final transformation of the two above theorems from the plane configuration to a spherical configuration rests on the relation

$$\frac{\mathrm{EA}}{\mathrm{E}\Gamma}=\frac{\operatorname{crd}\alpha}{\operatorname{crd}\beta} \tag{1}$$

which can be easily verified for the two cases represented in Figs. 18 and 19.

This relation (1) permits us to replace the ratios which enter the Menelaos theorems for the plane configuration by ratios between corresponding arcs of the spherical configuration. If, e.g., in Fig. 17 $\mathrm{A}\Gamma$ represents the outer part m in the plane configuration, μ in the spherical configuration, then it follows from (1) that (cf. Fig. 20)

$$\frac{m_1}{m_2}=\frac{\operatorname{crd}(2\mu_1)}{\operatorname{crd}(2\mu_2)}. \tag{2}$$

If we now introduce, for the sake of simplicity, the notation

$$\operatorname{crd}(2\alpha)=2\sin\alpha=(\alpha)$$

then we can write, because of (2) and of similar identities for the other parts, the two Menelaos theorems in the form

Theorem I. $$\frac{(m)}{(m_1)}=\frac{(r)}{(r_1)}\cdot\frac{(s_2)}{(s)}$$

Theorem II. $$\frac{(r_2)}{(r_1)}=\frac{(m_2)}{(m_1)}\cdot\frac{(n)}{(n_2)}$$

where m, n, r, s, etc., now represent great circle arcs.

It is, of course, always possible in a Menelaos configuration to replace the outer part m by n by interchanging the inner parts accordingly. This leads to equivalent forms for both theorems

Theorem I. $$\frac{(n)}{(n_1)}=\frac{(r_2)}{(r)}\cdot\frac{(s)}{(s_1)}$$

Theorem II. $$\frac{(s_2)}{(s_1)}=\frac{(m)}{(m_2)}\cdot\frac{(n_2)}{(n_1)}.$$

In all the above formulae we may interpret the symbols (m), (n), etc., as sines instead of as chords, because

$$(\theta) = \text{crd}\, 2\theta = 2 \sin \theta$$

and therefore

$$\frac{(a)}{(b)} = \frac{\text{crd}\, 2a}{\text{crd}\, 2b} = \frac{\sin a}{\sin b}.$$

In this interpretation the Menelaos theorems appear as the "theorems of the six quantities" in Islamic spherical astronomy. We shall use the same device if we wish to compare Ptolemy's results with the corresponding solutions given by modern trigonometry.

2. Supplementary Remarks

Before turning to the astronomical applications of the Menelaos theorems we shall make two additional remarks. One is due to Theon of Alexandria who realized that Theorem I can be derived from Theorem II.[3] The other remark concerns the fact that the four basic formulae (1) to (4) (above p. 26) for the right spherical triangle are the exact equivalents of the Menelaos Theorems I and II.

To show that Theorem I can be derived from Theorem II we consider the Menelaos configuration AV_1C (Fig. 21). Then it follows from Theorem II that

$$\frac{(m_2)}{(m_1)} = \frac{(r_2)}{(r_1)} \cdot \frac{(n_2)}{(n)}. \tag{1}$$

We now extend the great circles AV_1 and AB until they intersect in the point $\bar{A}$ which is diametrically opposite to A. Then we have in $CV_2\bar{A}$ a new Menelaos configuration in which the previously considered inner parts s_1, s_2 now play the role of outer parts v_1, v_2 and conversely $n_1 = \sigma_1$, $n_2 = \sigma_2$. We have furthermore $m_1 = \mu_1$, $r_1 = \rho_1$, $m_2 = 180 - \mu$ (thus $2m_2 = 360 - 2\mu$ or $|(m_2)| = |(\mu)|$) and $r_2 = 180 - \rho$ (thus $|(r_2)| = |(\rho)|$). Substituting the new elements in (1) one thus obtains

$$\frac{(\mu)}{(\mu_1)} = \frac{(\rho)}{(\rho_1)} \cdot \frac{(\sigma_2)}{(\sigma)}$$

which is Theorem I for the new configuration. Since every Menelaos configuration can be completed such that it plays the role of the second configuration in the above proof, Theorem I will hold for it by virtue of the fact that Theorem II holds for all Menelaos configurations; q.e.d.

We now shall prove that the fundamental formulae (1) to (4) (p. 26) for right spherical triangles are a direct consequence of the Menelaos Theorems I and II.

We follow a procedure repeatedly applied by Ptolemy. If ABC is our right spherical triangle (cf. Fig. 22) we use A as pole for the great circle PM, B as pole of QN. It follows from this construction that each side of our right triangle is extended in each direction by an arc which is the complement to 90°.

[3] Cf. Rome, CA I, p. 569.

This configuration obviously represents the combination of two Menelaos configurations, one with vertex M, one with vertex N. For these figures one finds, replacing chords by sines,

$$\text{for M: Th. I:} \quad \frac{1}{\sin\alpha}=\frac{1}{\sin a}\cdot\frac{\sin c}{1} \quad \text{or:} \quad \sin\alpha=\frac{\sin a}{\sin c} \tag{1}$$

$$\text{Th. II:} \quad \frac{\cos\alpha}{\sin\alpha}=\frac{\cos a}{\sin a}\cdot\frac{\sin b}{1} \quad \text{or:} \quad \tan\alpha=\frac{\tan a}{\sin b} \tag{3}$$

$$\text{for N: Th. I:} \quad \frac{1}{\cos a}=\frac{1}{\cos c}\cdot\frac{\cos b}{1} \quad \text{or:} \quad \cos c=\cos a\cdot\cos b \tag{4}$$

$$\text{Th. II:} \quad \frac{\sin a}{\cos a}=\frac{\sin c}{\cos c}\cdot\frac{\cos\beta}{1} \quad \text{or:} \quad \cos\beta=\frac{\tan a}{\tan c}. \tag{2}$$

Thus every problem that can be solved by means of the relations (1) to (4) can also be solved by a proper Menelaos configuration.

A general triangle can always be broken up by altitudes into right triangles. The application of (1) then produces

$$\sin\alpha:\sin\beta:\sin\gamma=\sin a:\sin b:\sin c \tag{5}$$

i.e. the general sine theorem for spherical triangles[4].

§ 3. Equatorial and Ecliptic Coordinates

1. Solar Declinations

For many problems of spherical astronomy it is necessary to know the distance of the sun from the equator. For a given date one can find the longitude λ of the sun, either accurately from the theory of solar motion or approximately by assuming as many degrees for λ as days elapsed since the vernal equinox. Thus we take λ to be known and we wish to find the declination of the point of the ecliptic with longitude λ.

The solution of this problem given in Almagest I, 14 and 15 is based on the Menelaos theorem. Let VA in Fig. 23 be the equator, VB the ecliptic, N the north pole, V = ♈ 0° the vernal point, VA = 90°. Then we consider the Menelaos configuration with vertex A, outer parts AN and AV, and with inner parts intersecting at the point H of the ecliptic where the sun is located. We wish to find $r_1=\delta$ to given $s_2=\lambda$.

According to Theorem I we have

$$\frac{(m)}{(m_1)}=\frac{(r)}{(r_1)}\cdot\frac{(s_2)}{(s)}$$

[4] For plane trigonometry cf. above p. 26.

and the following parts are known:

$$m = r = s = 90°$$
$$m_1 = \varepsilon \text{ the obliquity of the ecliptic}$$
$$s_2 = \lambda \text{ given.}$$

Thus the sixth quantity $r_1 = \delta$ can be found from

$$(\delta) = 1/2(\varepsilon)(\lambda) \tag{1}$$

because crd $180° = 2$. This is the equivalent of the modern formula

$$\sin \delta = \sin \varepsilon \cdot \sin \lambda \tag{2}$$

derived from the right spherical triangle VΘH.

Ptolemy uses for ε the value 23;51,20 to which corresponds according to the table of chords (Alm. I, 11)

$$(\varepsilon) = \text{Crd } 47;42,40 = 48;31,55.$$

Substituting this value in the above equation (1) he can compute a table of declinations for single degrees of λ. The results are tabulated in Alm. I, 15.

2. Right Ascensions

The same configuration (Fig. 23) which we used for the computation of the declination $r_1 = \delta$ to given solar longitude λ can be utilized to find the right ascension $n_2 = \alpha$ to given λ and previously determined δ. Indeed Menelaos' Theorem II

$$\frac{(r_2)}{(r_1)} = \frac{(m_2)}{(m_1)} \cdot \frac{(n)}{(n_2)} \tag{3a}$$

shows that $n_2 = \alpha$ can be found because we know

$$\begin{array}{c} m_1 = \varepsilon \quad m_2 = 90 - \varepsilon \quad n = 90 \\ r_1 = \delta \text{ (tabulated in Alm. I, 15)} \quad r_2 = 90 - \delta. \end{array} \tag{3b}$$

The numerical results are tabulated twice in the Almagest, first at the end of I, 16, then in II, 8. Here the terminology requires some explanation. Let us assume that we are located on the terrestrial equator. Then (cf. Fig. 24) the equator ΘV goes through the zenith Z and the circle of declination ΘHN coincides with the horizon. This case is called the case of "*sphaera recta.*" In the situation represented in Fig. 24 the vernal point V has risen by an arc ΘV at the moment when the sun is in the horizon rising at H. Since the uniform rotation of the equator measures time the arc $\alpha = \text{V}\Theta$ represents the time it took the ecliptic arc $\lambda = \text{VH}$ to rise. Consequently α is called the "*rising time* (of λ) *at sphaera recta*" or in short "*ascensio recta*" i.e. "*right ascension.*"[1] Therefore the table for α as function of

[1] Contrast: "*sphaera obliqua*" denotes geographical latitudes different from zero; the corresponding rising times are called "*oblique ascensions.*"

λ is headed "sphaera recta" within a group of tables (Alm. II, 8) for "rising times" of ecliptic arcs for different geographical latitudes. The second column of the table for α as function of λ proceeds in steps of 10° from $\lambda = 10°$ to 360° while the first column contains the corresponding differences, i.e. the time it takes each individual 10°-section of the ecliptic (the so-called "*decans*") to cross the horizon at sphaera recta.

The Ptolemaic procedure of computing α as function of λ requires twice the use of a Menelaos theorem: first to find δ by means of what amounts to

$$\sin \delta = \sin \varepsilon \sin \lambda \tag{1}$$

and then, for the determination of α from (3a) and (3b), a formula which is the equivalent of

$$\sin \alpha = \frac{\cos \varepsilon}{\sin \varepsilon} \cdot \frac{\sin \delta}{\cos \delta}. \tag{4}$$

Here the modern formulae applied to the right triangle VΘH would have given α directly as function of λ in the form of

$$\tan \alpha = \cos \varepsilon \tan \lambda. \tag{5}$$

Again, it is the absence of the tangent function which causes the complication of the ancient procedure.[2]

3. Transformation from Ecliptic to Equatorial Coordinates

The important problem of finding the equatorial coordinates α and δ of a celestial object of given ecliptic coordinates λ and β appears in the Almagest only in Book VIII (Chap. 5) in connection with the Catalogue of Stars (VII, 5 and VIII, 1) in which the positions of all stars are given in ecliptic coordinates. Then the problem is discussed in VIII, 5 of finding the date for the simultaneous culmination of the sun and a given star. It is easy to see that this problem leads to the above-mentioned coordinate transformation.

Indeed, simultaneous culmination of a star S and the sun Σ (cf. Fig. 25) implies that at a certain moment S and Σ are simultaneously in the meridian. But the meridian intersects the equator at right angles; therefore S and Σ must be located at the same circle of declination in order to culminate simultaneously; thus S and Σ have the same right ascension α. The position of the star S is given by its ecliptic coordinates $\lambda = \text{VL}$ and $\beta = \text{LS}$ (cf. Fig. 26). Suppose we are able to find from λ and β the equator coordinates $\alpha = \text{VA}$ and $\delta = \text{AS}$; then we know that also the sun Σ has the right ascension α. But the table of rising times for sphaera recta in Alm. II, 8 which gives α as function of λ can also be used inversely to find λ to given α. Thus we can find the solar longitude VΣ (and hence the day) when the sun will culminate simultaneously with S *if* we are able to find α and δ from given λ and β.

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[2] The equivalence with (5) of (1) plus (4) requires the use of the relation $\cos \alpha \cos \delta = \cos \lambda$ for which see p. 26 (4).

In order to solve this problem we have to make some preliminary remarks which Ptolemy took for granted.[1] First: since VL, the longitude of S, is given we can find the arc VK in the table of sphaera recta (Alm. II, 8) by entering it with the value VL as "right ascension" and finding the corresponding argument in the column for "longitudes." In other words, a right triangle (VLK) with acute angle ε at V can always be used as if the hypotenuse represented longitudes and the two other legs right ascension and declination, respectively. By this same argument we can also find the arc KL from the table of declinations (Alm. I, 15) by entering it with the arc VK, which we have just found, as if it were an arc of longitude and find in the column of arguments the corresponding arc KL as if it were an arc of declination.

It follows from these preliminary remarks that we can consider to be given not only the coordinates $\lambda = \mathrm{VL}$ and $\beta = \mathrm{LS}$ of S but also the arcs VK and KL.

We now take the Menelaos configuration with vertex B, outer legs BA and BP, inner arcs AN and PK, intersecting at S. Then, in Theorem I

$$\frac{(m)}{(m_1)} = \frac{(r)}{(r_1)} \cdot \frac{(s_2)}{(s)}$$

the following arcs are known:

$m_1 = \mathrm{BN} = 90 \quad m_2 = \mathrm{NP} = \varepsilon$ as angle between north pole and pole of the ecliptic, thus $m = 90 + \varepsilon$

$r = \mathrm{KL} + \mathrm{LP} = \mathrm{KL} + 90 \quad r_1 = \mathrm{KL} + \mathrm{LS} = \mathrm{KL} + \beta$

$s = \mathrm{NA} = 90.$

Hence $s_2 = \mathrm{AS} = \delta$, the declination of S, can be found.

With this coordinate determined we can find also α from Theorem II

$$\frac{(s_2)}{(s_1)} = \frac{(n_2)}{(n_1)} \cdot \frac{(m)}{(m_2)}$$

because we know

$s_2 = \mathrm{AS} = \delta$ found in the first step

$s_1 = \mathrm{SN} = 90 - \delta$

$n_1 = \mathrm{KB} = \mathrm{VB} - \mathrm{VK} = 90 - \mathrm{VK}$

$m = \mathrm{BP} = 90 + \varepsilon \quad m_2 = \mathrm{NP} = \varepsilon.$

Thus $n_2 = \mathrm{AK}$ can be found which gives us $n = \mathrm{AB}$. But then also

$$\alpha = \mathrm{VA} = \mathrm{VB} - \mathrm{AB} = 90 - n$$

is known, i.e. the right ascension of the star.[2]

[1] This omission has misled Manitius in his translation of the Almagest (edition of 1912) and then Vogt (in his Griech. Kal. 4, p. 44) who blamed Ptolemy for an essential error, instead of Manitius.

[2] The modern formulae for the transformation of λ, β into α and δ are based on the consideration of the triangle SNP in which the sides $\mathrm{NP} = \varepsilon$ and $\mathrm{PS} = 90 - \beta$ are known and the enclosed angle $90 - \lambda$ at P. Thus $90 - \delta = \mathrm{NS}$ and $90 + \alpha$ at N can be found. This results in the formulae

$$\sin\delta = \sin\beta\cos\varepsilon + \cos\beta\sin\varepsilon\sin\lambda$$

$$\cos\alpha = \cos\beta\cos\lambda/\cos\delta.$$

For conversion tables see below p. 1080.

That declinations of fixed stars were recorded is told us by Ptolemy (Alm. VII, 3) who compares declinations observed by Timocharis (about 290/270 B.C.) and his pupil Aristyllos (about 250) with declinations found by Hipparchus and by himself in order to demonstrate that the slow motion of precession proceeds about the pole of the ecliptic and not about the pole of the equator.

§ 4. Geographical Latitude; Length of Daylight

1. Oblique Ascensions

Perhaps the most important problem in ancient spherical astronomy is the problem of determining the "*rising times*" or "*oblique ascensions*" of arcs of the ecliptic: find for a given arc of the ecliptic the arc of the equator simultaneously crossing the horizon at a given geographical location.

Obviously it suffices to solve this problem for the special case that one end-point of the arc of the ecliptic is the vernal point ($\lambda=0$). The rising time $\rho(\lambda)$ is defined as the equator arc VE which rises simultaneously with the ecliptic arc $\mathrm{VH}=\lambda$ (cf. Fig. 27). The rising time of an arbitrary ecliptic arc of length $\Delta\lambda=\lambda_2-\lambda_1$ is then known from

$$\rho(\Delta\lambda)=\rho(\lambda_2)-\rho(\lambda_1)$$

as soon as we are able to find $\rho(\lambda)$ at a given geographical latitude φ.

The solution of this problem is given in the Almagest and the numerical results are tabulated[1] for values of λ proceeding in steps of 10° and for 11 different geographical latitudes, beginning with $\varphi=0$ (sphaera recta) and ending with $\varphi=54;1$. For intermediate values linear interpolation is considered sufficiently accurate.

We have already described the solution of the problem for the case of "sphaera recta".[2] Then the north pole N is a point of the horizon and the rising time of the arc VH is given by the arc VA of the equator, i.e. by the right ascension of the point H of the ecliptic. The resulting values $\alpha(\lambda)$ are tabulated in the first column of Alm. II, 8.

Thus we may assume $\varphi>0$ and the equatorial coordinates α and δ of the ecliptic point H of longitude λ known. We then make use of the Menelaos Theorem II, taking T as vertex (cf. Fig. 27), ZNT being the meridian:

$$\frac{(m_2)}{(m_1)}=\frac{(r_2)}{(r_1)}\cdot\frac{(n_2)}{(n)} \tag{1a}$$

with the following parts known:

$$\begin{array}{lll} m_2=\varphi & m_1=90-\varphi & n=90 \\ r_1=\delta(\lambda) & r_2=90-\delta & \text{(tabulated in Alm. I, 15).} \end{array} \tag{1b}$$

[1] In Alm. II, 8.

[2] Cf. above p. 31.

Thus n_2 can be found and from it

$$\rho(\lambda) = \mathrm{VE} = \mathrm{VA} - n_2 = \alpha(\lambda) - n_2.$$

This solves our problem.

2. Symmetries

The practical computation of the rising times is simplified by symmetries which reduce the problem to the consideration of one quadrant only. In the Almagest these symmetries are mentioned and extensively used but the proofs are incomplete or not very clear. We therefore prove these theorems by means of stereographic projection of the celestial sphere from its south pole on the plane of the equator, a procedure well known long before Ptolemy.[3]

(A). The rising times of equal ecliptic arcs before and after an equinox are equal.

Proof (cf. Fig. 28): we obtain two equal ecliptic arcs AB and BC (B being an equinox) by constructing equal arcs of corresponding right ascensions $\mathrm{A'B} = \mathrm{BC'}$. Since A rises in D and sets in $\bar{\mathrm{E}}$ we have equal angles

$$\mathrm{DNB} = \bar{\mathrm{E}}\mathrm{N}\bar{\mathrm{B}}.$$

Now C and $\bar{\mathrm{C}}$ and therefore also E and $\bar{\mathrm{E}}$ are diametrically opposite points of the ecliptic; hence E and $\bar{\mathrm{E}}$ lie on a straight line which passes through N. Thus we also have

$$\mathrm{BNE} = \bar{\mathrm{E}}\mathrm{N}\bar{\mathrm{B}}$$

and hence the remaining angles which subtend the arcs DA and CE are equal, q.e.d.

(B). The sum of the rising times of two ecliptic arcs which are symmetrically located to a solstice is equal to the sum of the rising times of these two arcs at sphaera recta.

It obviously suffices to prove this statement under the assumption that the two ecliptic arcs join at the solstice, such that $\mathrm{AB} = \mathrm{BC}$ (Fig. 29). The rising time of AC at sphaera obliqua is represented by the equator arc ac, at sphaera recta by $\alpha\gamma$. But $a\alpha = c\gamma$, therefore $ac = \alpha\gamma$ q.e.d.

Examples taken from the tables in Almagest II, 8:

(A). For Lower Egypt: $\rho(♈0, ♈10) = 6;48 = \rho(♓20, ♓30)$.

(B). For sphaera recta:

$$\alpha(♊10, ♊20) + \alpha(♋10, ♋20) = 10;47 + 10;47 = 21;34.$$

For Lower Egypt:

$$\rho(♊10, ♊20) + \rho(♋10, ♋20) = 10;0 + 11;34 = 21;34.$$

For South Britain:

$$\rho(♊10, ♊20) + \rho(♋10, ♋20) = 8;49 + 12;45 = 21;34.$$

[3] Cf. below V B 3, 7.

3. Ascensional Differences

The method described in our first section (p. 34f.) for the determination of the rising times $\rho(\lambda)$ can also be rendered in the following fashion: the oblique ascensions $\rho(\lambda)$ can be computed from the right ascensions $\alpha(\lambda)$ by means of a correction $n_2(\lambda)$ in the form

$$\rho(\lambda)=\alpha(\lambda)-n_2(\lambda) \qquad (1)$$

where n_2 can be found from[4]

$$\sin n_2=\frac{\sin\varphi}{\cos\varphi}\cdot\frac{\sin\delta}{\cos\delta}=\tan\varphi\cdot\tan\delta. \qquad (2)$$

In the terminology of the Middle Ages these corrections n_2 are called "*ascensional differences*," in Islamic astronomy "*equation of daylight*". At least as early as in the Almagest methods have been derived to adapt the computation of these quantities, which transform right ascensions into oblique ascensions, to the data which determine the geographical latitude of a given locality.

Consequently Ptolemy describes a procedure which is aimed at the replacement of the geographical latitude φ by the length of the longest daylight. Let in Fig. 30 H be a point of the ecliptic of given longitude λ at the moment of rising, F the point of the horizon at which the summer solstitial point C of the ecliptic rises. Then $\mathrm{GF}=\varepsilon$ is the obliquity of the ecliptic, $\mathrm{AH}=\delta$ the declination of the point $\mathrm{H}=\lambda$ for which we wish to find rising time $\rho=\mathrm{VE}=\mathrm{VA}-\mathrm{EA}=\alpha-n_2$. The longest daylight is obviously measured by 180° + twice the arc EG. We consider this quantity, henceforth denoted by M, to be known for the given locality (expressed either in degrees or in hours). Then the Menelaos Theorem II

$$\frac{(n)}{(n_2)}=\frac{(m_1)}{(m_2)}\cdot\frac{(r_2)}{(r_1)}$$

with

$$m_1=\varepsilon \quad m_2=90-\varepsilon \quad r_1=\delta \quad r_2=90-\delta$$

shows that

$$\frac{(n)}{(n_2)}=\frac{(\varepsilon)}{(90-\varepsilon)}\cdot\frac{(90-\delta)}{(\delta)}$$

is independent of the geographical location and can be computed once and for all as function of λ. The arc n, however, is known for a given locality because

$$n=\mathrm{EG}=1/2\,M-90.$$

Thus the right ascensional differences n_2 can be found from

$$(n_2)=(n)\frac{(90-\varepsilon)}{(\varepsilon)}\cdot\frac{(\delta)}{(90-\delta)} \qquad (3\,\mathrm{a})$$

or, in modern notation, from the equivalent relation

$$\sin n_2=-\cos 1/2\,M\cot\varepsilon\tan\delta. \qquad (3\,\mathrm{b})$$

[4] Cf. p. 34 (1 a) and (1 b).

Comparison of (3) with (2) p. 36 shows that we have implicitly proved

$$-\cos 1/2\,M = \tan\varphi \tan\varepsilon. \tag{4}$$

Ptolemy's procedure of determining, by means of a Menelaos configuration, the longest daylight M for a place of geographical latitude φ is the exact equivalent of (4).

4. Ortive Amplitude

The inverse problem, of finding φ to given M, cannot be solved in ancient terms by means of (4) because the function tan φ is not available. Consequently Ptolemy has to proceed in two steps, the first of which utilizes the arc EF in Fig. 30, later, in mediaeval astronomy, known as the "*ortive amplitude*", i.e. "*rising amplitude*", of the solstitial point C. It should be noted, however, that the rising amplitudes of the solstitial points were already used by Aristotle for the definition of wind directions[4a].

If we make in Fig. 31 ET = 90 and use T as vertex we have Menelaos Theorem I

$$\frac{(m_1)}{(m)} = \frac{(s)}{(s_2)} \cdot \frac{(r_1)}{(r)} \tag{5a}$$

with

$$EG = m_2 = 1/2\,M - 90 \qquad m = 90 \qquad m_1 = 90 - m_2$$
$$FG = s_1 = \varepsilon \qquad s = 90 \qquad s_2 = 90 - s_1$$
$$EK = r = 90.$$

Hence we can find r_1 and therefore also the ortive amplitude $EF = r_2 = 90 - r_1$. With $EF = r_2$ known Menelaos Theorem II is now used in the same configuration (Fig. 31):

$$\frac{(m_2)}{(m_1)} = \frac{(r_2)}{(r_1)} \cdot \frac{(n_2)}{(n)} \tag{5b}$$

where $n_2 = \varphi$, $n = 90$. Thus $n_2 = \varphi$ can be found.

This is an excellent illustration for the fact that the absence of the tangent-function introduces serious complications into ancient trigonometry. Instead of using (4) in the form

$$\tan\varphi = -\cos 1/2\,M \cot\varepsilon \tag{4a}$$

Ptolemy has to use

$$\cos r_2 = \cos\varepsilon \sin 1/2\,M \tag{5a}$$

and then

$$\sin\varphi = -\frac{\cos 1/2\,M}{\sin 1/2\,M} \cdot \frac{\cos r_2}{\sin r_2}. \tag{5b}$$

That indeed (5 a) and (5 b) are the equivalent of (4 a) becomes clear if one uses the right triangle EFG for which

$$\sin(90 - \varphi) = \cos\varphi = \frac{\sin\varepsilon}{\sin r_2}. \tag{6}$$

[4a] Aristotle, Meteorol. II, 6 (Loeb, p. 189 f.).

Thus we obtain from (5b) and (5a)

$$\tan\varphi=\frac{\sin\varphi}{\cos\varphi}=-\frac{\cos 1/2M}{\sin 1/2M}\cdot\frac{\cos r_2}{\sin r_2}\cdot\frac{\sin r_2}{\sin\varepsilon}=-\frac{\cos 1/2M}{\sin 1/2M}\cdot\frac{\cos\varepsilon\sin 1/2M}{\sin\varepsilon}$$
$$=-\cos 1/2\,M\cot\varepsilon$$

which is (4a). - For the results cf. below Table 2 (p. 44).

The relation (6) which we used to transform (5a) and (5b) into (4a) appears explicitly in the Almagest[5] in connection with the problem of finding directly the ortive amplitude $r_2=\mathrm{EF}=\eta_0$ of the summer solstice for given φ, i.e. without making use of the longest daylight M.

The concept of ortive amplitude is not restricted to the solstitial points, i.e. to the extremal azimuthal distance from E of any point of the ecliptic. We shall describe later on[6] a curious classification of "inclinations" of solar and lunar eclipses for which it is of importance to know the distance in the horizon of the rising point H of the ecliptic from E. Ptolemy therefore gives a list of the rising (and setting) amplitudes for the endpoints of all zodiacal signs, computed for the latitudes of the seven climata.[7] These values are, however, not listed in the usual tabular form (as in our Table 1) but entered in a circular diagram.[8] A circle is divided into 16 equal sectors by 8 diameters, three of which on each side of the east-west and north-south diameter are used to show the corresponding rising or setting amplitude in one of the seven spaces bounded by concentric circles which represent the climata.[9]

Table 1

			ortive amplitude		
Cl.	M	φ	♋ ♑	♊ ♐ ♌ ♒	♉ ♏ ♍ ♓
I	13^h	16;27°	24;57°	21;26°	12;10°
II	13 1/2	23;51	26;15	22;32	12;46
III	14	30;22	22;57	23;53	13;33
IV	14 1/2	36; 0	30; 0	25;39	14;29
V	15	40;56	32;22	27;38	15;32
VI	15 1/2	45;34	34;53	29;42	16;38
VII	16	48;38	37;38	31;56	17;47

Tables with circular arrangement are not uncommon in Byzantine treatises; for example Marc. gr. 325 fol. 105^v[10] in this fashion tabulates the hourly and

[5] Almagest II, 3 Heiberg, p. 95, 6 to 13.

[6] Cf. p. 142.

[7] Ptolemy does not indicate his method of computation but it is easy to reconstruct it. As soon as the maximum rising amplitude η_0 is known the ortive amplitude $\eta(\lambda)=\mathrm{EH}$ of any point of longitude λ on the ecliptic (cf. Fig. 30) is given by $\sin\eta=\sin\lambda\cdot\sin r_2$. This follows immediately from p. 30 (5) in the triangle EHV, since $\sin\eta/\sin\varepsilon=\sin\lambda/\sin(90+\varphi)$; hence $\sin\eta=\sin\lambda\sin\varepsilon/\cos\varphi$ and with (6) the formula

$$\sin\eta=\sin\eta_0\cdot\sin\lambda.$$

[8] VI, 11 Heiberg, p. 543, 24f. and plate at the end of Vol. I.

[9] The same numbers are found also tabulated in the ordinary fashion; cf., e.g., Manitius I, p. 454 or Vat. gr. 208 fol. 122^v and Vat. gr. 1594 fol. 144^v.

[10] Cf. our Pl. I.

daily velocity, etc., for the sun in each zodiacal sign. Obviously, one came close to the idea of representation by polar coordinates though the decisive step was never taken. Ptolemy's circular table of rising and setting amplitudes as function of polar coordinates λ and φ would lead to our Fig. 32. Ptolemy's table makes all concentric circles equidistant and stretches the curves into symmetrically arranged radii.

5. Paranatellonta

It is not surprising that early astronomy is much more concerned with horizon-phenomena than later periods which have learned to refer the phenomena to much less obvious but far more convenient coordinates; namely the ecliptic or the equator. But, as in many other instances, astrology has preserved antiquated concepts and caused their survival even in strictly technical treatises. It is probably for this reason that the Almagest after the catalogue of stars discusses (in VIII, 5) the problem of determining the point of the ecliptic which rises simultaneously with a given star. Simultaneously rising stars are called "*paranatellonta*" and astrological doctrine associates their qualities and influences with the corresponding point of the ecliptic.[11] Thus Ptolemy explains (Alm. VIII, 5) a method by means of which one can determine the point of the ecliptic which rises (respectively sets) simultaneously with a star whose position is given by its ecliptic coordinates λ and β.

The problem can easily be reduced to the determination of the rising point E of the equator (Fig. 33). The position of the star Σ is supposed to be known by its coordinates $VG=\lambda$ and $G\Sigma=\beta$. From this one can find the equator coordinates,[12] in particular the right ascension $VF=\alpha$. If we were able to determine the point E of the equator which is in the horizon simultaneously with Σ we would know the arc EF and therefore also the arc $EV=\alpha+EF$. But the equator arc EV is the rising time $\rho(H)$ of the ecliptic arc VH. Therefore if we enter the tables of oblique ascensions with the value of EV we find as the corresponding argument the longitude $\lambda(H)$ of the point H of the ecliptic which rises simultaneously with the star Σ.

The arc EF can easily be found from the Menelaos configuration Fig. 34 by means of Theorem II (J being the south pole of the celestial sphere):

$$\frac{(m_2)}{(m_1)}=\frac{(r_2)}{(r_1)}\cdot\frac{(n_2)}{(n)}$$

where

$$m_1=90-\varphi \qquad m_2=\varphi$$

$$r_1=\delta \qquad r_2=90-\delta \qquad n=90.$$

Therefore $n_2=EF$ is given by the equivalent of $\sin n_2=\tan\varphi\tan\delta$.

[11] For the modifications and far reaching influences of this doctrine cf. the article "Paranatellonta" by W. Gundel in RE 18, 3 (1949), col. 1241–1275.

[12] Cf. above p. 32.

6. Length of Daylight; Seasonal Hours

An application of special importance of the concept of rising times is due to the connection between the variability of the length of daylight and the rising times at a given geographical latitude. Suppose that on a certain day the longitude of the sun is λ, represented by a point H of the ecliptic. At sunrise H is in the eastern horizon, whereas at sunset the point $\overline{\mathrm{H}}=\mathrm{H}+180$ is rising when H is setting.[13] Obviously the length of daylight, i.e. the time from sunrise to sunset, equals the rising time of the semicircle $\mathrm{H}\overline{\mathrm{H}}$ of the ecliptic. But the rising time $\rho(\mathrm{BA})$ of any ecliptic arc $\mathrm{AB}(\lambda(\mathrm{A})>\lambda(\mathrm{B}))$ is given by

$$\rho(\mathrm{BA})=\rho(\mathrm{A})-\rho(\mathrm{B}).$$

Thus the length of daylight $d(\lambda)$, when the sun has the longitude λ, is given by

$$d(\lambda)=\rho(\lambda+180)-\rho(\lambda). \tag{1}$$

Examples.

Length of daylight for Alexandria in (approximately) 10-day intervals after the vernal equinox. From the table of rising times in Alm. II, 8 one finds for "Lower Egypt":

λ	$\lambda+180$	$\rho(\lambda+180)$	$\rho(\lambda)$	$d(\lambda)$	
♈ 0°	♎ 0°	180°	0°	180°	
10	10	191;32	6;48	184;44	
20	20	203; 7	13;43	189;24	
♉ 0	♏ 0	214;47	20;53	193;54	etc.

For sphaera recta one finds, of course, always $d(\lambda)=180$, e.g.

$$\rho(♎10)-\rho(♈10)=189;10-9;10=180.$$

In the preceding formulae the length of daylight is expressed by means of arcs of the equator, measured in degrees. If we define one complete rotation of the equator as 24 "*equinoctial hours*" then we have the relation

$$15°=1^{\mathrm{h}}. \tag{2}$$

Because of this relation equatorial degrees are also called "*time degrees.*"

Outside of scientific context, however, "*seasonal hours*" (abbrev.: s.h.) were used in antiquity, defined as twelfths of the length of daylight (or night, respectively). If at a given day the solar longitude is λ, then we can find the length of daylight $d(\lambda)$ by means of (1) and therefore

$$1^{\mathrm{s.h.}}=(d(\lambda)/12)^{\circ}=(d(\lambda)/3,0)^{\mathrm{h}}. \tag{3}$$

The same result can be obtained without computing $d(\lambda)$ explicitly. The stereographic projection Fig. 35 shows that the length of daylight for a solar position at Σ is given by the motion of the sun from H to Δ, i.e. by a time

[13] We ignore here the change of the solar longitude between sunrise and sunset.

$d(\lambda) = 180° + 2\,\mathrm{EA}$. But $\mathrm{EA} = \mathrm{EB} - \mathrm{BA} = \alpha(\lambda) - \rho(\lambda)$ where $\alpha(\lambda)$ is the right ascension, $\rho(\lambda)$ the oblique ascension of Σ. Consequently

$$1^{\text{s.h.}} = d(\lambda)/12 = 15° + 1/6\big(\alpha(\lambda) - \rho(\lambda)\big) \qquad (4)$$

gives the length of one seasonal hour in degrees. The term $\alpha - \rho$ is an "*ascensional difference*," encountered before.[14]

Example. Find the length of 5 seasonal hours of night at a geographical latitude $\varphi \approx 41°$ (i.e. the Hellespont according to Alm. II, 8) when the sun is at $\lambda = ♑ 23°$. For this latitude one finds in Alm. II, 8

$$\rho(\lambda) = 315;6 \qquad \rho(180 + \lambda) = 94;38$$

hence for the length of the night $315;6 - 94;38 = 220;28°$ and for 5 seasonal hours of night $5 \cdot 220;28/12 = 91;51,40°$. Division by 15 gives as the corresponding equinoctial hours $6;7^{h} \approx 6;10^{h}$.

This is the result obtained by Ptolemy[15] in his discussion of an observation made by Menelaos in Rome (in A.D. 98, Jan. 14). The substitution of the clima of the Hellespont for Rome is approximately correct; according to Almagest II, 8 (or I, 13) one has for the Hellespont: $\varphi = 40;56°$, longest daylight 15^{h}; according to Ptolemy's Geography[16] for Rome: $\varphi = 41;40°$, longest daylight $15;12^{h}$. The seasonal hours in the two localities are therefore practically the same.

Applications[17]

1. Find for a given locality (geographical latitude φ) and a given date (i.e. given solar longitude λ) and hour the longitude $\lambda(\mathrm{H})$ of the rising point H of the ecliptic (in astrological context called "*horoscopos*" or "*ascendant*").

Let the given moment be defined as $t^{\text{s.h.}}$ after sunrise. For given λ we can find the length of daylight $d(\lambda)$. Therefore we know that $t \cdot d(\lambda)/12$ time degrees have elapsed since sunrise. At the moment of sunrise $\rho(\lambda)$ time degrees have passed since the rising of the vernal point. Therefore, when H is at the given moment in the eastern horizon $\rho(\lambda) + t \cdot d(\lambda)/12$ time degrees have elapsed since the rising of the vernal point. But this amount is by definition the rising time of the point H. Thus we have shown that

$$\rho(\mathrm{H}) = \rho(\lambda) + t \cdot d(\lambda)/12. \qquad (5)$$

To $\rho(\mathrm{H})$ we then can find the corresponding $\lambda(\mathrm{H})$ in the table of oblique ascensions (Alm. II, 8) for the given latitude φ.

Example. $\varphi = 36°$, $\lambda_{\odot} = ♌ 8;35°$, $t = 0;40^{\text{s.h.}}$ after sunrise. Thus from Alm. II, 8 in the column for Rhodes ($\varphi = 36$):

$$d(\lambda) = \rho(♒ 8;35) - \rho(♌ 8;35) = 325;4 - 117;5 = 207;59 \approx 3,28°.$$

Therefore

$$1^{\text{s.h.}} = 3,28/12 = 17;20° \qquad 0;40^{\text{s.h.}} = 0;40 \cdot 17;20 = 11;33,20$$

and

$$\rho(\mathrm{H}) = \rho(♌ 8;35) + 11;33,20 \approx 117;5 + 11;33 = 128;38.$$

[14] Above p. 36 (1).

[15] Alm. VII, 3 Heib. II, p. 33, 3ff. For the longitude cf. below p. 60.

[16] III, 1, 61 and VIII, 8, 3 (Nobbe, p. 151, 26 and p. 205, 7f.).

[17] Almagest II, 9.

To this value of $\rho(\mathrm{H})$ belongs for $\varphi=36$

$$\lambda(\mathrm{H}) = ♌17;55.$$

Thus H = ♌17;55 was rising at the horizon of Rhodes at the given moment.

2. Find for given φ and t the culminating point M of the ecliptic (also called "*midheaven*").

First Case. Given the longitude $\lambda(\mathrm{H})$ of the ascendent. Using stereographic projection (Fig. 36) the rule (6) given by Ptolemy becomes evident. Because

$$\mathrm{VE}=\rho(\mathrm{H}) \qquad \mathrm{CE}=90°$$

the right ascension $\alpha(\mathrm{M})$ of the culminating point M of the ecliptic is given by

$$\alpha(\mathrm{M})=\mathrm{VC}=\mathrm{VE}-\mathrm{CE}=\rho(\mathrm{H})-90°. \tag{6}$$

From $\alpha(\mathrm{M})$ can be found $\lambda(\mathrm{M})$ in the table for sphaera recta in Alm. II, 8.

Example.[18] Find M for $\varphi=36°$ when H = ♉0° is rising.

In Alm. II, 8 one finds for Rhodes ρ(♈30) = 19;12. Hence $90-\rho(\mathrm{H})=70;48=\alpha(-\lambda)$ to which belongs $-\lambda=72;19$ thus M = ♈0° − 72;19 = ♑17;41.

Remark. The occurrence of negative values for $\alpha(\lambda)$ when $\rho(\mathrm{H})<90$ makes it necessary, for the methods of antiquity, to operate in (6) with $360+\rho(\mathrm{H})$. Therefore the "Handy Tables" do not tabulate $\alpha(\lambda)$ but

$$\alpha'(\lambda)=\alpha(\lambda)+90°. \tag{7}$$

This "*normed right ascension*," as we shall call $\alpha'(\lambda)$,[19] leads to the very convenient formulation of the rule (6) namely,

$$\alpha'(\mathrm{M})=\rho(\mathrm{H}). \tag{8}$$

If the ascendant H is known one has only to determine from the table for the given geographical latitude the value $\rho(\mathrm{H})$ of the oblique ascension of H and then find the same value in the table for the normed right ascension $\alpha'(\lambda)$. The corresponding argument λ is the longitude of the culminating point M.[20]

Vice versa: if $\lambda(\mathrm{M})$ is known one enters with the value of the normed right ascension $\alpha'(\lambda)$ the table of oblique ascensions for the given geographical latitude and obtains in the corresponding argument the longitude of the ascendant H.

It is this modification of Ptolemy's rule (6) that is commonly used in mediaeval tables, both Byzantine and Islamic.

Second Case. Given a moment t. We may assume that we know for this moment the longitude $\lambda_\odot$ of the sun Σ (cf. Fig. 37) and the time difference τ of the moment t from noon. Whether τ is measured in seasonal or in equinoctial hours, we can always find its equivalent in degrees, i.e. the arc DC. We can furthermore obtain from the tables in II, 8 the right ascension $\alpha(\lambda_\odot)=\mathrm{VD}$ of the sun. Thus we have for the right ascension of the culminating point M

$$\alpha(\mathrm{M})=\alpha(\lambda_\odot)-\mathrm{DC} \tag{9}$$

which allows us to find the longitude $\lambda(\mathrm{M})$.

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[18] From Alm. II, 11.

[19] One can express this also in the form that α' is reckoned from the winter solstitial point ♑0°, because α'(♑0) = 0°.

[20] For an example cf. below p. 979.

Example. Find the culminating point M if $\lambda_{\odot} = ♑ 23°$, at Rome, 5 seasonal hours after midnight.

We have found before[21] that under these conditions 5 seasonal hours correspond to about 6;10 equinoctial hours. The arc since noon is therefore $18;10^h = 272;30°$. From Alm. II, 8 one finds for the right ascension of the sun

$$\alpha(\lambda_{\odot}) = 294;52°.$$

Hence

$$DC = 360 - 272;30 = 87;30°$$

and

$$\alpha(M) = \alpha(\lambda_{\odot}) - DC = 294;52 - 87;30 = 207;22°$$

which is very near to the right ascension 207;50° of ♎ 30°. Hence

$$M \approx ♎ 30°.$$

7. Geographical Latitude; Shadow Table

In Alm. II, 6 Ptolemy enumerates data which we here present in the form of a table (cf. Table 2). As column of arguments is to be considered the column of longest daylight M (column (d) in our table) which progresses in steps of $1/4^h$ from 12^h to 18^h Nos. 1 to 25) followed by increasingly longer intervals as one approaches the north pole. The next column (e) gives the corresponding geographical latitudes φ in degrees and minutes, computable from M by the method described above,[1] using for ε the value 23;51,20° as is clear from the entry No. 33. Our column (a) does not figure in Ptolemy's list but the "seven climata" are repeatedly mentioned in the Almagest.[2] All parallels from Nos. 1 to 21 appear again in the table of rising times of Alm. II, 8.[3]

To the first six parallels Ptolemy adds remarks about the two days in each year when the sun reaches the zenith. For these latitudes there will be an interval during which the noon shadows fall toward south. The limits are determined by the solar longitudes λ for which the declinations $\delta(\lambda)$ obtain the value φ. Example: to $\varphi = \delta = 12;30°$ corresponds according to Alm. I, 15[4] the solar longitude $\lambda = 32;20°$. Thus at points $\Delta\lambda = 57;40°$ distant from the summer solstice the sun will reach the zenith (cf. Table 2, No. 4). The values of $\Delta\lambda$ for localities with $\varphi \leqq \varepsilon$ are also given in the Geography Book VIII.[5]

Similarly $\delta(\lambda) = 90 - \varphi$ is the condition for the sun not to reach the horizon; this explains the last six entries in column (g) of Table 2. It is of interest that this problem also appears in India, in the Pañcasiddhāntikā,[6] and with essentially the same values for φ.[7]

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[21] Above p. 41.

[1] Cf. above I A 4, 3.

[2] Cf., e.g., the tables in Alm. II, 13 (below p. 50ff.).

[3] Cf. above I A 4, 1. A similar list, relating M and φ, is given in the "Geography" (I 23; Mžik, Ptol. Erdkunde, p. 65f.), all values of φ being rounded to the nearest multiple of 0;5°. The boundaries are $\varphi = -16;25°$ and $\varphi = +63°$; cf. below p. 935.

[4] Cf. above I A 3, 1.

[5] E.g. VIII 16, 3-14 (Nobbe, p. 221-223); several numbers are garbled.

[6] Probably written sometime between A.D. 500 and 600.

[7] Pc.-Sk. XIII 21-25, Neugebauer - Pingree II, p. 84.

Table 2

(a)	(b)	(c)	(d)	(e)		(f)		(g)
Clima	No.	Locality	M	φ	s_1	s_0	s_2	Dist. of sun fr. summer solst. in zenith
	1	Equator	12^h	0°	26;30 S	0	26;30 N	±90°
	2	Taprobane	12 1/4	4;15	21;20 S	4;25 N	32	79;30
	3	Sinus Aual.	12 1/2	8;25	16;50 S	8;50	37;55	69
	4	Sinus Adul.	12 3/4	12;30	12 S	13;20	44;10	57;40
I	5	Meroe	13	16;27	7;45 S	17;45	51	45
	6	Napata	13 1/4	20;14	3;45 S	22;10	58;10	31
II	7	Soene	13 1/2	23;51	0	26;30	65;50	0
	8	Ptolemais Herm.	13 3/4	27;12	3;30 N	30;50	74;10	
III	9	Lower Egypt	14	30;22	6;50	35; 5	83; 5	
	10	Mid-Phoenicia	14 1/4	33;18	10	39;30	93; 5	
IV	11	Rhodes	14 1/2	36	12;55	43;50?	103;20	
	12	Smyrna	14 3/4	38;35	15;40	47;50	114;55	
V	13	Hellespont	15	40;56	18;30	52;10	127;50	
	14	Massalia	15 1/4	43; 4	20;50	55;55	144	
VI	15	Mid-Pontus	15 1/2	45; 1	23;15	60	155; 5	
	16	Sources of Ister	15 3/4	46;51	25;30	63;55	171;10	
VII	17	Mouth of Borysth.	16	48;32	27;30	67;50	188;35	
	18	Maiotic Sea	16 1/4	50; 4	29;55	71;40	208;20	
	19	South Britain	16 1/2	51;30	31;25	75;25	229;20	
	20	Mouth of Rhine	16 3/4	52;50	33;20	79; 5	253;10	
	21	Mouth of Tanais	17	54;1	34;55	82;35	278;45	
	22	Brigantium	17 1/4	55	36;15	85;40	304;30	
	23	Mid-Great-Brit.	17 1/2	56	37;40	88;50	335;15	
	24	Katuraktonium	17 3/4	57	39;20	92;25	372; 5	
	25	South of Little-Br.	18	58	40;40	96	419; 5	
	26	Mid-Little-Brit.	18 1/2	59;30				
	27	North of Little-Br.	19	61				
	28	Ebudian Islands	19 1/2	62				
	29	Thule	20	63				
	30	Unknown Skythians	21	64;30				
	31		22	65;30				no sunset
	32		23	66				
	33		24^h	66;8,40				0°
	34		$\approx 1^m$	67				±15
	35		≈ 2	69;30				30
	36		≈ 3	73;20				45
	37		≈ 4	78;20				60
	38		≈ 5	84				75
	39	North Pole	6^m	90				90

In the "Handy Tables" the relation between M and φ is further refined[8] to steps of only 0;4 between $M = 12^h$ and $M = 18^h$; similar material is embedded in Book VIII of the "Geography."[9]

[8] Cf. below p. 980.
[9] Cf. below p. 937.

The entries in column (f) represent the lengths of the shadow at noon of a vertical gnomon of length 60 for the summer solstice (s_1), the equinoxes (s_0), and the winter solstice (s_2). The numbers are, of course, the equivalent of a sexagesimal table for $\tan(\varphi-\varepsilon)$, $\tan\varphi$, and $\tan(\varphi+\varepsilon)$, respectively, rounded here to 0;5 units because the results are expressed in terms of unit fractions 1/2 1/3 1/4 1/6 and 1/12.[10] The results agree very well with modern computations, usually remaining within 2 or 3 minutes of the given value of φ. The value $s_0=43\ 1/2\ 1/3\ (=43;50)$ for Rhodes in No. 11 is probably an error and should be replaced by 43 1/2 1/6.[11]

§ 5. Ecliptic and Horizon Coordinates

1. Introductory Remarks

The following deals with two major problems: determination of the angle between ecliptic and horizon and between ecliptic and circles of altitude. The first problem is met, e.g., if one needs information about the depression of the sun below the horizon as is the case if one is interested in the conditions of first and last visibility of the planets or of the moon, near sunset or sunrise. The second problem is of fundamental importance in the computation of eclipses because parallax displaces sun and moon vertically to the horizon and this change in altitude must be resolved into its longitudinal and latitudinal components. It is therefore not surprising that a great deal of effort has been devoted to the tabulation of the angles between ecliptic and circles of altitude. These angles depend not only on the geographical latitude but also on the distance from the meridian, i.e. on the hour before or after noon. The theoretical basis for such tables is developed in the Chaps. 10 to 12 of Book II of the Almagest while the final tabulation is given in II, 13.[1]

For the construction of this table one has to define which one of the four angles at the intersection P of the ecliptic with another great circle is meant when we speak about "the angle" at P. Ptolemy's definition amounts to the following. The ecliptic divides the celestial sphere into two hemispheres; the one which contains the north pole (cf. Fig. 38) we call the "northern" hemisphere. We then take on the circle which intersects the ecliptic at P, the direction which points into the northern hemisphere. Secondly the daily rotation of the celestial sphere defines a sense of rotation from east to west as seen from N. The opposite direction on the ecliptic is the direction of increasing longitudes. The "angle at P" is enclosed between the two positive directions on the ecliptic and on the intersecting circle (cf. Fig. 38).

[10] Converted to sexagesimal fractions in Manitius' translation and in our Table 2. In No. 8 $s_0=30;50$ is the correct value, found in MS D, whereas Heiberg and Manitius accepted the obviously wrong version 36;50. In No. 11 $s_0=43\ 1/2\ 1/3=43;50$ is taken from Heiberg p. 109, 9 whereas Ptolemy in Alm. II, 5 (Heiberg, p. 100, 15) had found (correctly) 43;36.

[11] The correct computation for $s_0=43;36$ and of s_1 and s_2 for $\varphi=36°$ is given in Alm. II, 5 (cf. above p. 24, No. 1) and agrees for s_1 and s_2 with the values in Alm. II, 6, No. 11.

[1] Cf. below p. 50.

2. Angles between Ecliptic and Horizon

We may restrict ourselves to the case of sphaera obliqua because the angles between ecliptic and horizon at sphaera recta are the same as between ecliptic and meridian at sphaera obliqua[1] (independent of φ) and this latter problem will be discussed in the next section.

For the equinoctial points the angle between ecliptic and horizon is obviously $90-\varphi+\varepsilon$ since ecliptic and equator intersect each other in this case at a point of the horizon.

If a point H of the ecliptic of longitude $\lambda>0$ lies in the horizon (cf. Fig. 40) we consider the circle of altitude $\mathrm{KG}\overline{\mathrm{Z}}$ which has H as its pole. Our goal is the determination of the arc $m_1=\mathrm{KG}$. Let E be the rising point of the equator and $\overline{\mathrm{SMZ}}$ the circle of altitude which has E as its pole. The intersection $\overline{\mathrm{M}}$ of the ecliptic with $\overline{\mathrm{SMZ}}$ is the lower culmination of the ecliptic. Since we are able to determine for given H the upper culmination M of the ecliptic we also know the longitude of $\overline{\mathrm{M}}=\mathrm{M}+180$. Consequently, the following parts in the Menelaos configuration with vertex K are known:

$$m=r=s=90$$
$$s_2=\mathrm{H}\overline{\mathrm{M}}=\lambda(\mathrm{M}+180)-\lambda(\mathrm{H})$$
$$r_1=(90-\varphi)-\delta(\overline{\mathrm{M}})$$

where $\delta(\overline{\mathrm{M}})$ is the declination of $\overline{\mathrm{M}}$ which is known through $\lambda(\overline{\mathrm{M}})$. Thus we can use Menelaos Theorem I

$$\frac{(m)}{(m_1)}=\frac{(r)}{(r_1)}\cdot\frac{(s_2)}{(s)}$$

to find (m_1) from

$$(m_1)=(m)\,\frac{(r_1)}{(s_2)}$$

which is the sine theorem for the right triangle $\bar{\mathrm{S}}\mathrm{H}\overline{\mathrm{M}}$.

Example. Find the angle which the ecliptic makes with the horizon at Rhodes ($\varphi=36°$) when ♉0° rises.

We know from p. 42 that M = ♑17;41 culminates when H = ♉0° rises. Thus $\overline{\mathrm{M}}$ = ♋17;41 and $s_2=\mathrm{H}\overline{\mathrm{M}}=77;41°$. To $\overline{\mathrm{M}}$ = ♋17;41 = ♎ − 72;19 we can find in Alm. I, 15 the declination

$$\delta(\overline{\mathrm{M}})=\delta(72;19)=22;39,47\approx 22;40.$$

Therefore $r_1=90-36-22;40=31;20$. Finally from the table of chords (Alm. I, 11):

$$(m)=\mathrm{Crd}(180)=2,0$$
$$r_1=\mathrm{Crd}(62;40)=62;24,7$$
$$s_2=\mathrm{Crd}(155;22)=117;14,16.$$

[1] At sphaera recta not only the meridian but also the horizon contains the north pole N; therefore both circles are perpendicular to the equator. Cf. Fig. 39 which depicts this situation in stereographic projection.

Hence

$$(m_1)=\mathrm{Crd}(2m_1)=\frac{2,4;48,14}{1,57;14,16}\approx 1,3;52,20$$

to which corresponds (Alm. I, 11) an arc of $2m_1=64;19,4°$. Hence we have found for the angle between ecliptic and horizon at Rhodes when ♉0° rises

$$m_1=32;9,32\approx 32;10°.$$

The angles between ecliptic and horizon are not explicitly tabulated in the Almagest. They can be obtained, nevertheless, from the tables in II, 13. The last entry in each of the columns which are headed "eastern angles" gives the angle between the ecliptic and the altitude circle at the rising point H of the ecliptic. Since the circle of altitude is perpendicular to the horizon at H we obtain the angle between ecliptic and horizon by subtracting 90° from the "eastern angle." This leads to the following Table 3 and to the corresponding graphical representation Fig. 41.[2]

Table 3

Clima			φ	♈	♉ ♓	♊ ♒	♋ ♑	♌ ♐	♍ ♏	♎
I	Meroe	13^h	16;27	49;42	52; 9	59;51	71;57	84;51	94; 9	97;24
II	Syene	13 1/2	23;51	42;18	44;41	51;53	63;46	76;53	86;41	90; 0
III	Lower Egypt	14	30;22	35;47	37;55	44;49	56;28	69;49	79;55	83;29
IV	Rhodes	14 1/2	36	30; 9	32; 7	38;36	50; 1	63;36	74; 7	77;51
V	Hellespont	15	40;56	25;13	26;59	33; 6	44;16	58; 6	68;59	72;55
VI	Mid Pontus	15 1/2	45; 1	21; 8	22;43	28;25	39;21	53;25	64;43	68;50
VII	Borysthenes	16	48;32	17;37	19;22	24;20	34;58	49;20	61;22	65;19

3. Ecliptic and Meridian

There exist obvious symmetry relations for the angles between ecliptic and meridian. First: the same angles occur at points of the ecliptic symmetrically located to an equinox (points A and B in Fig. 42); secondly: the sum of these angles is 180° for points symmetrically located to a solstice (points B and $\bar{\mathrm{A}}$ in Fig. 42).

Obviously the angle between ecliptic and meridian is 90° when the point of intersection is a solstice and $90\pm\varepsilon$ in the case of an equinox.

For the case of a general position we use the given culminating point M of the ecliptic as the pole for a great circle $\mathrm{K}\bar{\mathrm{K}}$ which must go through the east Point E of the equator (cf. Fig. 43) since M is a point of the meridian. E is therefore the pole of the meridian $\mathrm{KMC}\bar{\mathrm{K}}$. Using Menelaos Theorem II with $\bar{\mathrm{K}}$ as vertex

$$\frac{(m_2)}{(m_1)}=\frac{(r_2)}{(r_1)}\cdot\frac{(n_2)}{(n)}$$

in which the arcs

$$m_1=90-m_2 \qquad m_2=\delta(\mathrm{M})$$
$$r_1=90-r_2 \qquad r_2=180-\lambda(\mathrm{M})$$
$$n=90$$

[2] Instead of the above found angle $m_1=32;10°$ for Rhodes and ♉0° the table gives only $122;7-90=32;7°$.

are known, we can find $n_2 = \Theta E$ and from it the desired angle between ecliptic and meridian at M:

$$\Theta K = n_2 + 90°.$$

In order to obtain the numerical values of this angle for every endpoint of a zodiacal sign it suffices to compute it only for M = ♍ 0° and ♌ 0° because all other cases are covered by the above-mentioned symmetry relations. By this method Ptolemy finds for the angles between ecliptic and meridian for the endpoints of all signs the following values [1]:

♈ 0°	66; 9°	
♉	69; 0	♓ 0°
♊	77;30	♒
♋	90; 0	♑
♌	102;30	♐
♍	111; 0	♏
	113;51	♎

These values appear once more in the tables of Alm. II, 13 for the angles between ecliptic and circles of altitude for the position at noon.[2]

4. Ecliptic and Circles of Altitude

We now consider the angle made by the ecliptic with general circles of altitude, i.e. with circles which go through the zenith but not through the north pole.

Again symmetries simplify the practical computations. First: if a solstice T is in the meridian (cf. Fig. 44) and if AT = TB then

$$\alpha + \beta = 180°.$$

Secondly: let γ be the angle between the ecliptic and a circle of declination (cf. Fig. 45). Consider two positions P_1 and P_2 of the same point P of the ecliptic, one before, one after noon (or midnight), such that the distance from the meridian is the same, i.e., AM = MB. Then we have for the angles between ecliptic and circles of altitude P_1Z and P_2Z (Z being the zenith)

$$\alpha + \beta = 2\gamma$$

where γ is the angle between ecliptic and meridian if P is in the meridian;[3] thus $\alpha + \beta$ is independent of φ.

In order to determine the angle between the ecliptic and a circle of altitude in general position Ptolemy uses a combination of two Menelaos configurations. In order to have the required number of parts at one's disposal one must determine the culminating as well as the rising point of the ecliptic.

[1] The values for $2n_2$ are rounded to the nearest degree while 23;51° is taken for ε.

[2] Cf. below p. 51.

[3] For clima I ($\varphi < \varepsilon$) this rule has to be modified to $\alpha + \beta = 2\gamma \pm 180$ since Ptolemy counts angles in such a fashion as to avoid negative values. Cf. also below p. 992.

Culminating Point. In order to find the culminating point M (Fig. 46) from the given point P of longitude λ and from the given time difference t between P and the meridian ZCMS one has only to add (with the proper sign) to the right ascension α(P) the time difference $15 \cdot t^h$, expressed in degrees, in order to obtain α(M) and thus λ(M).

Rising Point. We have found (p. 42)

$$\alpha(\mathrm{M})+90=\rho(\mathrm{H})$$

where ρ(H) is the oblique ascension of the rising point H of the ecliptic for a given geographical latitude φ. From ρ(H) one can find λ(H) by means of the tables of rising times in Alm. II, 8.

Zenith Distance. We can now consider as known the longitudes of the points H, P, and M of the ecliptic. The Menelaos configuration Fig. 47 (which is a part of Fig. 46) with vertex S allows us to use Theorem I

$$\frac{(m_1)}{(m)}=\frac{(s)}{(s_2)}\cdot\frac{(r_1)}{(r)}$$

with the following parts known

$$m_1=90-\varphi+\delta(\mathrm{M}) \qquad m=r=90$$

$$s=\lambda(\mathrm{H})-\lambda(\mathrm{M}) \qquad s_2=\lambda(\mathrm{H})-\lambda(\mathrm{P}).$$

Thus the altitude $\mathrm{PA}=r_1$ can be found and hence also the zenith distance

$$\mathrm{ZP}=r_2=90-r_1.$$

Angle at P. We draw with P as pole the great circle KLΛ (cf. Fig. 46). For the Menelaos configuration Fig. 48 (vertex K) holds Theorem II

$$\frac{(m_2)}{(m_1)}=\frac{(r_2)}{(r_1)}\cdot\frac{(n_2)}{(n)}$$

for which we know

$$m_1=90-m_2 \qquad m_2=\mathrm{PA}\ \text{found in the preceding step}$$

$$r_2=\lambda(H)-\lambda(P) \qquad r_1=90-r_2 \qquad n=90.$$

Thus n_2 can be found and therefore also

$$\text{angle at P}=180-n_1=n_2+90.$$

Example. Find for the horizon of Rhodes ($\varphi=36$) the angle at P = ♊30 between the ecliptic and the circle of altitude when P is 1^h east of the meridian.

From α(♊30) = 90 and $t=1^h=15°$ we find $\alpha(\mathrm{M})=90-15=75$. Hence (from Alm. II, 8, sphaera recta) M = ♊16;12 is culminating.

From $\alpha(\mathrm{M})=75$ we obtain $\rho(\mathrm{H})=75+90=165$. Hence (from Alm. II, 8, $\varphi=36$) H = ♍17;37 is rising.

In Alm. I, 15 we find for the declination of M: δ(♊16;12) = 23;7,31 and therefore in Fig. 47 $m_1=90-36+23;7,31=77;7,31$. Furthermore s = ♍17;37 − ♊16;12 =

91;25 and $s_2 =$ ♍17;37 − ♊30 = 77;37. From these data one finds for P the altitude PA = 72;13 and therefore the zenith distance PZ = 17;47.

Finally in Fig. 48: $m_2 = PA = 72;13$ $m_1 = 17;47$ $r_2 = 77;37$ $r_1 = 12;23$. Theorem II then lets us find $n_2 = 43;14$ and the angle at P: $90 + n_2 = 133;14$. This is indeed the value listed in the tables of Alm. II, 13.

5. The Tables (Alm. II, 13)

The results of the computations discussed in the preceding sections are assembled in a set of seven large tables at the end of Book II of the Almagest. Each of these tables concerns one of the traditional *seven climata*, beginning with Meroe where the longest daylight is 13 hours ($\varphi = 16;27$) and proceeding in steps of half-hour increase of the longest daylight up to 16 hours (Borysthenes, $\varphi = 48;32$). Each of these seven tables contains twelve subtables for the twelve zodiacal signs from Cancer to Gemini, in four columns each. As an example may serve the subtable for clima IV, Rhodes ($M = 14\ 1/2^h$, $\varphi = 36°$) and Cancer (i.e. ♋0°):

hour	zenith distance[1]	eastern angle	western angle
noon	12; 9°	90;0°	
1	17;47	133;14	46;46
2	28;22	147;45	32;15
3	40;27	151;46	28;14
4	52;36	151;52	28; 8
5	64;36	149;54	30; 6
6	76;16	146;25	33;35
7	87;23	141;30	38;30
7;15	90; 0	140; 1	39;59

The first column gives the equinoctial hours before or after noon and as last entry the time of sunrise or sunset reckoned from noon. This is half the length of daylight when the sun is at the beginning of the sign in question. In our case we have $\lambda_\odot =$ ♋0° and therefore the last entry has the maximum value $1/2\,M = 7;15^h$. For other values of $\lambda_\odot$ one can easily check these final entries in the first column by means of the table of oblique ascensions in Alm. II, 8. For example for $\varphi = 36$ and $\lambda_\odot = 30°$ one finds for the length of daylight[2]

$$d(\text{♉}0°) = \rho(\text{♎}30) - \rho(\text{♈}30) = 216;28 - 19;12 = 197;16 = 13;9,4^h$$

and therefore $1/2\,d = 6;34,32^h$ which is rounded in the table for ♉0° to $6;35^h$.

The second column gives the zenith distances. Assuming that at noon ♋0° is in the meridian we obtain as zenith distance $90 - (90 - \varphi + \varepsilon) = 36 - 23;51 = 12;9$. If ♋0° is 1^h distant from the meridian we found p. 49 the zenith distance 17;47°,

[1] The Greek text has no technical term for "zenith distance" but says simply "arc."
[2] Above p. 40.

in agreement with the table. At the moment of rising and setting the entry is, of course, always 90°.

The third column begins with 90° because the ecliptic at ♋0° intersects the meridian perpendicularly. Also for all other zodiacal signs the first entry agrees with the value given in our little table on p. 48, independent of φ. The next value, 133;14°, is the angle of intersection when ♋0° is 1^h to the east of the meridian, in agreement with our computation.[3] The last entry in this column gives the angle at the ascendant between the ecliptic and the altitude circle. We have used this entry before for the tabulation of the angles between ecliptic and horizon.[4]

In the fourth column the angle is given for ♋0° being 1^h to the west of the meridian. Its value is obtained by subtracting the "eastern angle" in the preceding column from 180°. Indeed $180-133;14=46;46$.

The usefulness of the symmetry relations which we mentioned at the beginning (p. 48) becomes now evident, not only for the computation of the tables but also for the checking of their correctness. Let α be an "eastern angle" (i.e. an angle made at a point of the ecliptic located to the east of the meridian), β the corresponding "western angle."

We know that $\alpha+\beta=180$ for two positions symmetric to a solstice, e.g. ♌0° and ♊0°, at the same hour to the east in the first case, to the west in the second, or vice versa. Indeed we find in the tables

Lower Egypt:		♌0° E ♊0° W	
	noon	102;30 + 77;30 = 180	
	1^h	153;13 + 26;47 = 180	
	2^h	166;22 + 13;38 = 180	etc.

and similarly

		♌0° W ♊0° E	
	noon	102;30 + 77;30 = 180	
	1^h	51;47 + 128;13 = 180	
	2^h	38;38 + 141;22 = 180	etc.

for all geographical latitudes.

Also for all latitudes we must find $\alpha+\beta=2\gamma$ where γ is the angle tabulated for noon. Consequently we have, e.g. for ♌0°:

	Rhodes		Lower Egypt		
	α	β	α	β	
noon	102;30		102;30		
1^h	139;32	65;28	153;13	51;47	
2^h	155;19	49;41	166;22	38;38	etc.

Here we have always $\alpha+\beta=205=2\cdot 102;30$.

In spite of the fact that symmetries reduce considerably the number of values which must be computed independently it is obvious that it is no easy task to compile these tables for seven different geographical latitudes, for twelve zodiacal

[3] Above p. 50.
[4] Above p. 47.

signs, and all integer hours. Yet, these tables are rather inconvenient in practical use, particularly for the computation of eclipses. It is unlikely that a syzygy occurs exactly at an integer hour before or after noon (or midnight), or at a solar longitude which is a multiple of 30°. Finally φ need not to be near one of the seven selected values; for example Byzantium lies about half way between clima V and VI. In all such cases one is compelled to interpolate for 3 independent variables. This reduces considerably the accuracy of the results since in certain intervals quite large variations occur between the tabulated numbers. It is therefore not surprising that the Islamic astronomers computed special tables for the components of solar and lunar parallax, thereby saving the user the necessity of computing in each case the angles between ecliptic and circles of altitudes.

B. Lunar Theory

We feel no gratitude toward those whose assiduous toil has given us illumination on the subject of the moon, while owing to a curious disease of the human mind we are pleased to enshrine in history records of bloodshed and slaughter, so that persons ignorant of the facts of the cosmos may be acquainted with the crimes of mankind.

Gaius Plinius Secundus
Admiral of the Roman Imperial Fleet
(Nat. Hist. II, 43. Trsl. Loeb Class. Libr.)

§ 1. Solar Theory

The theory of the moon is historically as well as astronomically intimately related to the theory of the sun. For calendaric purposes, whence the interest in the motion of the moon originated, one needs to know the conjunctions and oppositions of the two luminaries. The occurrence of solar and lunar eclipses depends on the motion of the lunar nodes. And for both sun and moon geocentric coordinates are appropriate to describe the observed motions. On the other hand it is obvious that the lunar motion is by far less regular than the motion of the sun which describes an apparently fixed orbit among the stars. Consequently the solar orbit, the ecliptic, became the fundamental reference system for all celestial motions and the solar theory had to precede the theory of the motion of the moon and of the planets.

It is this type of reasoning that underlies the arrangement in the Almagest. The first two books which are devoted to trigonometry and spherical astronomy are followed by Book III on the solar motion. The Books IV to VI concern the lunar theory proper and the theory of eclipses. Only after this comes the catalogue of stars, the detailed discussion of precession and some theorems concerning stars and spherical astronomy (Books VII and VIII). The remaining five books deal with the planets.

To us it would seem more natural to begin with the catalogue of fixed star positions[1] since the stars determine the background against which the motions in our planetary system are observed. In antiquity, however, a direct determination of relative fixed star positions would have been much too inaccurate. Meas-

[1] Copernicus did make this change of arrangement but he felt it necessary to give a detailed motivation for this departure from the Almagest (De Revol. II, 14).

urements with simple sighting instruments needed to be related to the positions of sun and moon whose ecliptic coordinates could be computed for any given moment with much higher accuracy than individual observations might guarantee. In other words the fundamental coordinate axis, the ecliptic, is best located through solar and lunar positions of accurately computed longitude and latitude. It is with reference to lunar and solar positions that the positions of other celestial objects, stars and planets, can be most reliably obtained. Thus solar and lunar theory is of much greater importance for the whole of ancient astronomy than the theory of a single planet.

1. The Length of the Year

The concept of "solar year" obviously originated from the experience of the periodic recurrence of the climatic seasons. Mathematical astronomy had to replace such an intuitive concept by an accurate definition. The return of the sun to the same position with respect to the fixed stars must have appeared to be a much more reliable criterium than the slow seasonal variation of the length of daylight. A sidereal definition of the solar year is also suggested by instrumental considerations. Sighting instruments could be constructed with fair accuracy whereas sun dials and in particular water clocks were utterly inadequate for the determination of small changes in time intervals. Of course, the position of the sun cannot be directly compared with fixed stars but the development of an accurate theory of solar and lunar motion made it possible to relate solar positions to the stars at any given moment, thus lending precision to the concept which we would call "*sidereal year*."

The identification of the "solar year" with the sidereal year is characteristic for the Babylonian astronomy[1] of the hellenistic period. This happy state of affairs was disturbed when Hipparchus came to the conviction that no periodic time interval should be accepted as exactly constant without empirical confirmation through observations distant as far as possible from one another. We know from the Almagest that Hipparchus not only established the fact that the sidereal year had to be distinguished from the tropical year but also that he considered it possible that the latter was not of constant length.[2]

Ptolemy, about three centuries after Hipparchus, came to the conclusion that there was not sufficient evidence of fluctuations in the length of the tropical year and that its length was 1/300 of one day shorter than 365 1/4 day (the Alexandrian calendar year). Thus Ptolemy assumed a *tropical year* of $365;14,48^d$ length. For the *constant of precession* he accepted 1° per century,[3] i.e. the lower limit for possible values suggested by Hipparchus. Since precession can be interpreted as a motion of all "fixed" stars with respect to the vernal point, Ptolemy found it reasonable to define the concept "year" not by the return of the sun to the same

[1] Cf. below p. 529.
[2] Cf. also below p. 294.
[3] For an apparent confirmation of this constant of precession from the motion of the apsidal line of Mercury cf. below p. 160.

star but by the length of the tropical period. This tropical definition of the "year" and the corresponding reckoning of celestial longitudes from the vernal point remained valid ever since.

2. Mean Motion

Since all longitudes are reckoned from the vernal point, Ptolemy's determination of the length of the tropical year as $365;14,48^d$ implies that the solar longitude increases during this period exactly by 360°. Consequently he finds for the mean motion of the sun in longitude per day[1] the value

$$360°/365;14,48 = 0;59,8,17,13,12,31^{\circ/d}. \qquad (1)$$

This parameter is then used to build a convenient table of mean motions (Almagest III, 2) for other time intervals: hours, single days from 1 to 30, months (of 30 days each) from 1 to 12, Egyptian years (of 365 days each) from 1 to 18, and years in groups of 18 from 18 to 810. All entries are given in degrees and six sexagesimal digits for the fractions of degrees (cf. (1)).

The computation of the solar mean motion to so many places is not an "abus de calcul" as it has been called.[2] Since the tables should cover the order of magnitude of at least 1000 years, the epoch year Nabonassar 1 precedes Ptolemy's time by almost 900 years, one must require that about 2,0,0,0 days of mean motion will not introduce an error within seconds of longitude, which is the accuracy usually required in the computation of longitudes. Thus six sexagesimal places for the basic parameter (1) should be guaranteed. The explicit tabulation of so many places for each entry in the tables is another matter. It would have been sufficient to tabulate the accurate multiples of (1) to 3 places only and thus to obtain more compact tables. In fact this improvement was already adopted for the Handy Tables and is common practice in mediaeval works.

The use of 18-year steps has caused fruitless and misleading modern speculations although Ptolemy himself gave the explanation for this arrangement[3] which is common to all tables of mean motions in the Almagest (moon: IV, 4; planets: IX, 4). A column of 45 lines is a convenient arrangement for a papyrus sheet. Thus a table for 12 months followed by a table for 30 days gives the right number of 42 lines (plus headings). If one then combines a table for 24 hours with one for single years then there remains a space of $42-24=18$ lines. This is the sole basis for the choice of 18-year steps.

3. Anomaly

The discovery of the solar anomaly is not only one of the most remarkable achievements of early astronomy but also had a decisive influence on its whole

[1] The sidereal mean motion would be smaller since one tropical year would correspond only to a progress of 360–1/100°. One finds in this way 0;59,8,11,27, ...

[2] Tannery, AA, p. 163.

[3] I, 10 (p. 47, 3 Heib.) and III, 1 (p. 209, 13ff. Heib.).

methodology for the explanation and mathematical description of analogous phenomena in the motion of the moon and of the planets.

We do not know how one first became aware of the fact that the daily progress of the sun is not of a constant amount but is slightly greater than the mean value in one part of the year, slightly less in the other. The differences are much too small to make a direct discovery of a variability in the daily solar motion possible. It would be tempting to assume that it was the observation of the unequal length of the astronomical seasons that suggested as an explanation the corresponding inequality of the solar motion. In support of this genesis could be mentioned the well attested interest of early Greek astronomy (fifth century B.C.[1]) in the determination of equinoxes and solstices, probably for calendaric purposes.

On the other hand the anomaly of the solar motion is also a dominant element in Babylonian astronomy but in this case no plausible relation to the length of the seasons seems to exist. Extrapolating back from later Greek astronomy one could easily reconstruct the following argument: a circular motion with constant angular velocity requires equal times for each quadrant. Since experience shows that the four quadrants of the year are of unequal length the solar velocity cannot be constant. We have, however, no good evidence for ascribing the same type of reasoning to the Babylonian astronomers. The little we know about their dealing with solstices and equinoxes reveals only a schematic division of the year in four sections of equal length.[2] On the other hand solar longitudes are computed under full recognition of a variable velocity but using methods which allow no simple correlation with the seasonal intervals.[3]

In short, one faces two alternatives: either to conclude two independent but practically contemporary discoveries of the phenomenon of the solar anomaly, or to assume that the Greek discovery was recognized in its principal importance by Babylonian astronomers but re-interpreted in terms of arithmetical schemes which obliterated all traces of the original reasoning.[4] I find both alternatives about equally implausible.

A. Eccenter and Epicycles

At the time of Ptolemy the solution of the problem of describing an anomalistic motion by means of geometric models was, of course, well known. At least since Hipparchus the unequal length of the seasons had been used to determine the parameters of an appropriate circular orbit.

Before presenting Hipparchus' method Ptolemy gives a detailed discussion of certain characteristics of eccentric and epicyclic motions. We shall not follow him here but mention only those features which are of direct importance for the solar and lunar theory while other aspects will be introduced in later sections when actually needed.

[1] Cf. IV B 2, 1.

[2] Cf. p. 361.

[3] Cf. II Intr. 5.

[4] The inverse influence is practically excluded since it would mean the transformation of the arithmetical methods into a simple geometrical argument; but this is not feasible in a simple fashion.

Everywhere in the following it is assumed that circular motions are uniform with respect to the center of the orbit. We use the following notation (cf. Fig. 49 and 50):

Eccenter:	M	midpoint of a circle of radius R (usually $R=60$)
	O	observer
	MO$=e$	eccentricity, in units of R
	A	apogee, Π perigee
	P	celestial body, e.g. sun S
	$\bar{\kappa}$	mean, κ true eccentric anomaly (both reckoned from A)
	c	difference of anomaly, or "*equation.*" Greek term: *prosthaphairesis*, i.e. positive or negative (correction)
Epicycle:	C	center of epicycle
	r	radius of epicycle ($<R$)
	circle of radius $R=$OC	deferent
	$\bar{\alpha}$	(mean) epicyclic anomaly

Fig. 51 illustrates the main theorem: an eccenter motion can always be replaced by an equivalent epicyclic motion, and vice versa, under the assumption that $r=e$ and that the mean anomalies $\bar{\kappa}$ and $\bar{\alpha}$ are of the same absolute value but increase in opposite directions. The proof is obvious since P is always one vertex of the parallelogram OCPM.

All ancient and mediaeval astronomy under Greek influence made extensive use of this equivalence. The frequent shift from one cinematic model to the other is the best indication of the fact that none of these models implied that there existed in nature a corresponding mechanical structure. The constituent parts of any model are as unreal as the single terms in a modern series expansion, at least in principle when it is opportune to defend the purely mathematical aspect of the theory.

In practice the equivalence theorem is extremely useful to visualize in a convenient form certain qualities of a given model. For example, the statement that the equation c reaches its maximum amount when P is in quadrature to the apsidal line AΠ (cf. Fig. 52) is obvious for the epicyclic version and is therefore also true for an eccenter.

Similarly: in the same situation, which is called "*quadrature*" because P is 90° distant from the apsidal line (cf. Fig. 52), the velocity of P as seen from O is the "mean velocity," i.e. equal to the angular velocity of C. Ptolemy frequently makes use of this fact (e.g. in Alm. III, 3) but does not give a proof. Indeed it is obvious that in the motion of P only the longitudinal component is visible from O when the line of sight OP is tangential to the epicycle. But the longitudinal component in the motion of P is equal to the motion of C, i.e. to the mean motion.

B. Determination of Eccentricity and Apogee

The method for the determination of the eccentricity of the solar orbit and the position of its apsidal line, described in the following, has been utilized innumerable times from Ptolemy to Copernicus. Ptolemy does not say that Hipparchus invented the method; what we know from Almagest III, 4 is only that Hipparchus

used the following empirical data: it takes the sun

$$\begin{array}{l} 94\,1/2 \text{ days from vernal equinox to summer solstice} \\ 92\,1/2 \text{ days from summer solstice to autumnal equinox.} \end{array} \qquad (1)$$

This suffices to show that an eccentric orbit explains these data if its eccentricity is $e = 1/24\,R$ and its apogee A = ♊ 5;30. Ptolemy found the same values (1) for the seasons and therefore had to assume that the solar apogee remains at the same tropical longitude. In other words he concluded from the apparently unchanged validity of (1) that the anomalistic and the tropical year have the same length. Only Thābit b. Qurra (9th cent.) corrected this error and established the fact that the apsidal line of the solar orbit has also a motion in the direction of increasing longitudes, supposedly equal to precession.[1]

The problem to determine from (1) the parameters of an eccentric circular orbit is a special case of a problem which occurs in the theory of the moon[2] and of the planets[3]: three points on a circle appear under given angles from two points (observer O and center M, respectively; cf. Fig. 53); determine the position of O with respect to M. The two given angles at O are both 90° because the sun has travelled in the ecliptic, which has O as its center, 90° from the vernal equinox F to the summer solstice G, and again 90° from G to the autumnal equinox H. But also the angles $\bar{\alpha}_1$ and $\bar{\alpha}_2$ at M are known because with respect to M the sun moves with its mean velocity $0;59,8,\ldots^{\circ/d}$. We therefore find from (1)[4]:

$$\bar{\alpha}_1 = 93;9 \qquad \bar{\alpha}_2 = 91;11^\circ.$$

Since $\bar{\alpha}_1 > \bar{\alpha}_2 > 90$ the center M must be located in the first quadrant with respect to O. From $\bar{\alpha}_1 + \bar{\alpha}_2 = 180 + 4;20$ it follows that $\delta_1 = 2;10$ and therefore $\delta_2 = \bar{\alpha}_1 - 90 - \delta_1 = 0;59$. The coordinates of M with respect to O are therefore, with $R = 60$

$$\mathrm{OM}' = R \sin \delta_2 = 1;2 \qquad \mathrm{OM}'' = R \sin \delta_1 = 2;16$$

and thus

$$e = \mathrm{OM} = 2;30.$$

Finally the longitude of the apogee A can be found from $\sin \lambda_{\mathrm{A}} = \mathrm{MM}'/e = 2;16/2;30$ which leads to $\lambda_{\mathrm{A}} = 65;30$ or

$$\mathrm{A} = ♊\,5;30.$$

These are the elements for the solar orbit according to Hipparchus and Ptolemy.

C. The Table for the Solar Anomaly and its Use

With the dimension of the solar orbit determined one can now compute a table for the equation c as function of the mean eccentric anomaly $\bar{\kappa}$, reckoned from the apogee A. Ptolemy gives such a table in Almagest III, 6; his method of computation we have illustrated on p. 25 (Example 3). Obviously $c = 0$ for $\bar{\kappa} = 0$

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[1] Cf. Neugebauer [1962, 2], p. 267. Actually the motion is slightly faster than precession.

[2] Below p. 73 ff.

[3] Below p. 173 ff.

[4] The values given here are the ones used by Ptolemy in this computations. The tables of mean motions (Alm. III, 2) would give 93;8,33 and 91;10,16, respectively.

and 180; and the values of $|c|$ are symmetric to $\bar{\kappa}=0$ or 180. Ptolemy therefore arranged the arguments in his table in two columns, beginning with 6° and 354°, respectively and both ending with 180°. Until $\bar{\kappa}=90$ and 270 the steps are 6°, thereafter 3°.

It follows from Fig. 52, p. 1220 that the maximum equation c_{max} must satisfy the relation

$$\sin c_{max}=\frac{e}{R}. \tag{1}$$

Since it was found that $e=2;30$ for $R=60$ one has $\sin c_{max}=0;2,30$ and hence

$$c_{max}=2;23°. \tag{2}$$

In other words, the maximum equation occurs when the mean anomaly $\bar{\kappa}=92;23$. In the graphical representation of the values tabulated in Alm. III, 6 this is visible by a slight asymmetry of the curve for $|c|$ as function of $\bar{\kappa}$ (Fig. 54).

In order to find the true longitude $\lambda_{\odot}$ of the sun (cf. Fig. 55) one has to find c to the given mean anomaly $\bar{\kappa}$ and form the *true anomaly*

$$\kappa=\bar{\kappa}+c \qquad \begin{array}{lll} c<0 & \text{for} & 0<\bar{\kappa}<180 \\ c>0 & \text{for} & 180<\bar{\kappa}<360. \end{array} \tag{3}$$

The result has to be added to the longitude $\lambda_A=$ ♊ 5;30 of the apogee:

$$\lambda_{\odot}=\lambda_A+\kappa. \tag{4}$$

The mean anomaly $\bar{\kappa}$ increases linearly with time and its increment since a given time t_0 can be found in the tables of mean motions (III, 2). As epoch date Ptolemy uses in the Almagest the moment

$$t_0=\text{Nabonassar 1, Thoth 1, Alexandria noon}$$

which corresponds to

$$t_0=-746 \text{ Febr. } 26=\text{julian day } 1448,638.$$

For this date we need to know the value $\bar{\kappa}_0$ of the mean anomaly. It can be found as soon as $\bar{\kappa}$ is known for any other date t of given distance from t_0. Consequently Ptolemy uses an observation of an autumnal equinox, made by himself in the year Hadrian 17 Athyr 7 (A.D. 132 Sept. 25). On that day he found that the sun had the true longitude ♎ 0° at 2^h after Alexandria noon. From given λ_A and eccentricity $e=OM$ he found in the triangle OMS the equation $c=-2;10°$ (cf. p. 25, Example 2). Hence

$$\bar{\kappa}=\lambda_{\odot}-\lambda_A-c=180-65;30+2;10=116;40°.$$

We have now to find the difference $\Delta t=t-t_0$ in Egyptian years. Ptolemy introduces here two other fixpoints which are also used as astronomical eras (for the corresponding julian dates cf. p. 1066):

Nabonassar 1 to Death of Alexander (=era Philip):	424 years
Death of Alexander to Augustus 1:	294 years
Augustus 1 to Hadrian 17:	161 years
from Thoth 1 to Athyr 7, 2^h:	$66^d\ 2^h$
Total:	$\Delta t=879$ years $66^d\ 2^h$

Now one can obtain from the tables in III, 2 the mean motion $\Delta\bar{\kappa}$ of the sun during Δt. Ptolemy finds[1] $\Delta\bar{\kappa}=211;25°$; hence at epoch

$$\bar{\kappa}_0=\bar{\kappa}-\Delta\bar{\kappa}=116;40-211;25=265;15°$$

and for the mean longitude $\bar{\lambda}_0$ of the sun at epoch

$$\bar{\lambda}_0=\lambda_A+\bar{\kappa}_0=65;30+265;15=\text{♓}0;45. \tag{5}$$

The tables III, 6 for the equation c show the same value $2;23=c_{max}$ from $\bar{\kappa}=264$ to 270. Since the epoch value $\bar{\kappa}_0$ belongs to this interval we find for the true longitude λ_0 of the sun at epoch

$$\lambda_0=\bar{\lambda}_0+c_{max}=\text{♓}3;8°. \tag{6}$$

Fig. 56 shows the situation at epoch drawn to scale. The "*mean sun*" $\bar{S}$ is, according to this terminology, an ideal body which moves in the ecliptic with mean velocity about O and coincides with the true sun S in A and Π.[2]

Example. Find the true longitude of the sun for t = Nabon. 845 Mechir 18/19, 19 1/2^h after noon at Alexandria (= A.D. 98 Jan. 14, 4;30 a.m.) (Alm. VII, 3).

For $\Delta t=844$ years $5^m\ 17^d\ 19;30^h$ one finds from the tables in III, 2:

$$\Delta\bar{\kappa}=320;12,20°.$$

Adding this amount to the epoch value $\bar{\lambda}_0=\text{♓}0;45$ one finds $\bar{\lambda}=\text{♑}20;57$ as longitude of the mean sun. Its distance from the apogee is ♑20;57 − ♊5;30 = 225;27° = $\bar{\kappa}$.[3] Entering with this value as argument the tables for the equation (Alm. III, 6) one finds $c=+1;45$. Therefore the longitude of the true sun was ♑20;57 + 1;45 = ♑22;42 ≈ ♑23.

Note. This example is taken from Alm. VII, 3[4] where Ptolemy discusses an observation made by Menelaos in Rome (in the year A.D. 98, Jan. 14). The original record gave the moment in seasonal hours (5 seasonal hours after midnight). In order to change seasonal to equinoctial hours the solar longitude λ should be known and this requires the time Δt since epoch with respect to the meridian of Alexandria. In the above computation the value $\Delta t=\ldots 19;30^h$, reckoned from noon at Alexandria, actually includes not only the geographical transformation from Rome to Alexandria but also the change from seasonal to equinoctial hours which, in principle, presupposes the knowledge of λ. The rounding from ♑22;42 to ♑23 shows, however, that Ptolemy was satisfied with an approximate value of λ which he could obtain without knowing the exact equinoctial hour of the observation in Rome. Thus it is not necessary to know for the computation of λ that at the given moment 5 seasonal hours of night in Rome were equal to 6;10 equinoctial hours.[5]

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[1] Accurate computation with these tables results, however, in $\Delta\bar{\kappa}=211;25,43°$. Cf. p. 63 where it is shown that Ptolemy's result is exact if one includes the equation of time.

[2] It is important to realize that the "mean sun" in ancient terminology is not the same as the "mean sun" in modern astronomy. The latter moves in the equator and coincides with the true sun at $\lambda=0$.

[3] The same result can be obtained from $\Delta\bar{\kappa}+\bar{\kappa}_0=320;12+265;15=225;27$.

[4] Heiberg II, p. 33, 3ff. = Manitius II, p. 28, 14ff.

[5] For the solution of this problem, assuming $\lambda=$ ♑23° given, cf. above p. 41.

For the longitudinal difference between Alexandria and Rome Ptolemy reckons here $1\ 1/3^h = 20°$, which is nearly correct.[6] In the Handy Tables the difference is increased to $1;36^h = 24°$.[7] In the Geography the longitudinal difference is even $1\ 1/2\ 1/8^h = 24;22,30°$. Heron (around A.D. 60) in his "Dioptra" gave the estimate $2^h = 30°$.[8]

§ 2. Equation of Time

1. The Formulation in the Almagest (III, 9)

The civil calendars of antiquity reckon the days either from sunset to sunset or they begin and end with sunrise. The first mode prevails where actual lunar months are in use which begin with the first visibility of the new crescent at sunset. In Egypt, with its schematic 30-day months, the civil day began simply with sunrise.[1]

For astronomical computations both definitions of "days" are equally inconvenient since the moments of sunrise and sunset depend not only on the geographical location but also on the seasons of the year. One becomes independent of the geographical latitude if one reckons days from noon to noon. Also the gross effect of the seasonal variation in the length of daylight is eliminated by this norm. Nevertheless there remain effects which make the time between consecutive meridian transits of the sun, the so-called "*true solar day*", of unequal length. Since tables for the mean motions of the celestial bodies must be based on intervals of constant length, i.e. on "*mean solar days*", it is necessary to investigate the relation between the observed transits of the sun and the ideal mean solar day. The correction which transforms a true into a mean time interval is technically known as the "*equation of time*."[2] It is characteristic for the high level of hellenistic astronomy that a correct determination of this correction was achieved. We do not know to whom is due this important step in the theory of time reckoning; in the sources available to us the equation of time, or its equivalent, is first attested in the Almagest.

There are two effects which by their superposition produce what is now called the "equation of time." The first is the solar anomaly: the variability of the sun's velocity in its orbit influences the length of the time interval between consecutive transits. The second effect is due to the fact that the sun progresses not in the equator but in the ecliptic which intersects the meridian at greatly variable angles. Thus, even if the progress of the sun in the ecliptic were of a constant

[6] Correctly about $1;10^h = 17;30°$. Thus Ptolemy's error is only 2;30° (not 4° as Manitius II, p. 27 note a) says).

[7] Halma III, p. 34 and Halma I, p. 38 where $24° = 1\ 1/2\ 1/10^h$ appear as approximation of $24;10° = 60;30 - 36;20$ (correcting Halma's errors by means of Vat. gr. 208, fol. 52ᵛ and Vat. gr. 1291) (Honigmann, SK, p. 197, 94 and 198, 168).

[8] Cf. Neugebauer [1938], p. 22. Cf. below p. 848.

[1] Cf. p. 563, n. 3.

[2] I do not know where this term originated; it is found neither in Ptolemy nor in Theon. The Islamic term is "equation of day" (e.g. Battānī, Nallino II, p. 61) and similar in Byzantine tables (*ὄρθωσις τῆς ἡμέρας*) and in Latin works (Toledan Tables, verbatim from the Arabic: ***equationes dierum cum noctibus*** *suis*). Cf. also Wolf, Hdb. d. Astr. II, No. 494.

amount, nevertheless the time of meridian crossings of equal ecliptic arcs would not be constant.

Ptolemy's treatment in the Almagest of these facts is always connected with the comparison of two moments for which, by direct observation or by computation, the true longitudes of the sun are known. In the modern formulation the equation of time is simply considered as function of the solar longitude but this difference of definition is not essential.[3]

Let Δt be the difference between the two moments, determined "simply" (ἁπλῶς) as Ptolemy expresses it, i.e. without considering the above-mentioned effects. In other words Δt between consecutive transits of the meridian by the true sun is reckoned simply as integer days; we would say as "true solar days." To this Δt has to be found an "accurate" (ἀκριβής) interval $\Delta\bar{t}$ (our "mean solar days") which is needed for the accurate computation of mean motions that took place during the given interval Δt.

Ptolemy states, without proof, that the difference between Δt and $\Delta\bar{t}$ reaches a maximum of $8;20° = 0;33,20^h$ when the sun was at the first instant at the beginning of Scorpio, and at the second in the middle of Aquarius. Such a deviation of $\Delta\bar{t}$ from Δt by about half an hour is not negligible for the lunar motion which amounts to about one lunar diameter per hour. Indeed, a displacement by about one lunar radius will greatly influence the appearance of eclipses or the relative position of the moon and fixed stars.

Ptolemy gives rules, again without proof, for finding $\Delta\bar{t}$ from Δt. Let t_1 and t_2 be the two moments ($t_2 > t_1$), expressed in equinoctial hours with respect to true noon as given by observations; denote the corresponding true longitudes of the sun by λ_1 and λ_2, respectively. Compute from the tables of solar mean motions (Alm. III, 6) the mean longitudes $\bar{\lambda}_1$ and $\bar{\lambda}_2$ of the sun and (from Alm. II, 8) the right ascensions α_1 and α_2 which correspond to the true solar longitudes λ_1 and λ_2, respectively. Then the "accurate" interval $\Delta\bar{t}$ which corresponds to the "simply" taken interval $\Delta t = t_2 - t_1$ is given by

$$\Delta\bar{t} = \Delta t + \Delta\alpha - \Delta\bar{\lambda} \tag{1}$$

where $\Delta\alpha = \alpha_2 - \alpha_1$, $\Delta\bar{\lambda} = \bar{\lambda}_2 - \bar{\lambda}_1$.

2. Examples

A. Assume two observations, such that at the first observation the true sun had been found to be at $\lambda_1 = ♏ 0°$, at the second at $\lambda_2 = ♒ 15°$. We now have to find from the table of the solar equation c (Alm. III, 6) the corresponding mean longitudes. Since the solar apogee is located at ♊ 5;30 we have at the two observations the true eccentric anomalies $\kappa_1 = 144;30$ and $\kappa_2 = 249;30$. The next step consists in the determination of the mean eccentric anomalies $\bar{\kappa}_1$ and $\bar{\kappa}_2$ to which the given values κ_1 and κ_2 belong. From the tables we can excerpt the following relations:

$$\begin{array}{ll} \bar{\kappa}_1 + c = \kappa_1 & \bar{\kappa}_2 + c = \kappa_2 \\ 144 - 1;27 = 142;33 & 246 + 2;13 = 248;13 \\ 147 - 1;21 = 145;39 & 249 + 2;16 = 251;16. \end{array}$$

[3] Cf. below p. 66.

Linear interpolation for the given values of κ shows that to $\kappa_1 = 144;30$ belongs a $\bar{\kappa}_1 = 145;53,14$ and to $\kappa_2 = 249;30$ a mean position $\bar{\kappa}_2 = 247;15,44$. Consequently we have $\Delta\bar{\lambda} = \Delta\bar{\kappa} = 101;22,30$. For the right ascensions we find from the tables Alm. III, 8 a difference $\Delta\alpha = 317;31 - 207;50 = 109;41$. Hence

$$\Delta\bar{t} - \Delta t = \Delta\alpha - \Delta\bar{\lambda} = 8;18^\circ. \qquad (2)$$

This result agrees with Ptolemy's estimate of 8 1/3° as difference between mean and true interval.[1] It will be shown presently that this difference represents the maximum value.[2]

B. Of particular importance are time intervals which begin at the moment t_0 which is chosen as starting point, "epoch," of tables – in the case of the Almagest Nabonassar 1, Thoth 1, Alexandria noon. For this date Ptolemy computed the mean longitude $\bar{\lambda}_0$ by using the time difference $\Delta t = 879$ Eg. years $66^d\,2^h$ between t_0 and an observation of his own of an autumnal equinox at which moment the mean eccentric anomaly of the sun was $\bar{\kappa} = 116;40$.[3] In order to find $\bar{\kappa}_0$ at epoch one should, however, multiply the mean motion $\bar{v}$ of the sun not simply by Δt but by the accurate value $\Delta\bar{t}$.

For this we have to find the difference of right ascension $\Delta\alpha = \alpha - \alpha_0$. For Ptolemy's observation ($\lambda = ♎0°$) we have $\alpha = 180°$. In order to find α_0 we need the true longitude λ_0 of the sun at epoch. Using the value $\Delta\bar{\kappa}' = \Delta t \cdot \bar{v}$ as approximation for $\Delta\bar{\lambda} = \Delta\bar{\kappa}$ one finds $\Delta\bar{\kappa}' = 211;25,43$. Hence approximately

$$\bar{\kappa}'_0 = 116;40 - 211;25,43 = 265;14,17$$

to which belongs, according to Alm. III, 6, a true longitude

$$\lambda'_0 = ♊5;30 + 265;14,17 + 2;23 = ♓3;7.$$

The corresponding right ascension is $\alpha_0 = 335;6$ and therefore

$$\Delta\alpha = 180 - 335;6 = 204;54.$$

Since $\Delta\bar{\lambda} \approx \Delta\bar{\kappa}' = 211;26$ we obtain for the difference between mean and true interval the estimate

$$\Delta\bar{t} - \Delta t = \Delta\alpha - \Delta\bar{\lambda} = -6;32^\circ \approx -0;26^h.$$

Consequently the mean solar motion between the date of epoch and Ptolemy's observation was $0;26^h \cdot \bar{v}^{\circ/h}$ smaller than according to the computation simply with Δt. It is also clear that the small inaccuracies of the preceding deductions have no appreciable influence on the final solar longitude. Thus we have found for the corrected mean motion

$$\Delta\bar{\lambda} = \Delta\bar{\kappa}' - 0;26 \cdot \bar{v} \approx 211;26 - 0;1 = 211;25$$

that is exactly the value given by Ptolemy, whereas computation with Δt results in a mean motion of 211;25,43 thus nearer to 211;26.

In other words Ptolemy's epoch value $\bar{\lambda}_0 = ♓0;45$ seems to be in fact corrected for the equation of time although Ptolemy makes no mention of it in III, 7. One

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[1] Cf. above p. 62.

[2] Below p. 67.

[3] Cf. above p. 59.

must admit, however, the possibility that the accurate value is only the accidental result of less accurate roundings in the computation of the mean motion $\Delta\bar{\lambda}$ during Δt.

We can now give a general rule for finding the accurate length $\Delta\bar{t}$ from the simply computed interval Δt for all time intervals reckoned with the parameters of the Almagest from the epoch Nabonassar 1. We know that for the sun $\bar{\lambda}_0 =$ ♓ 0;45 to which belongs a true longitude $\lambda_0 =$ ♓ 3;8 and thus a right ascension $\alpha_0 = 335;8$. Consequently,

$$\Delta\bar{t} = \Delta t + (\alpha - 335;8) - (\bar{\lambda} - \text{♓}\,0;45) \tag{3}$$

where α and $\bar{\lambda}$ are right ascension and mean longitude, respectively, of the sun for the given moment Δt after epoch.

Example. For the moment $\Delta t = 844$ years $5^m\ 17^d\ 19;30^h$ after epoch we have found (on p. 60) for the mean and true longitude of the sun

$$\bar{\lambda} = \text{♑}\,20;57 \approx \text{♑}\,21 \qquad \lambda = \text{♑}\,22;42 \approx \text{♑}\,23$$

respectively. We wish to find $\Delta\bar{t}$ since epoch. From Alm. II, 8 one obtains for the right ascension of the sun $\alpha(\text{♑}\,23) = 294;52°$, and therefore from (3)

$$\begin{aligned}\Delta\bar{t} &= \Delta t + (294;52 - 335;8) - (\text{♑}\,21 - \text{♓}\,0;45)\\ &= \Delta t - 40;16 + 39;45 = \Delta t - 0;31° \approx \Delta t - 0;2^h.\end{aligned}$$

A correction of 2 minutes in time is negligible since it corresponds to only 0;1° of lunar motion. This result will find a direct confirmation by the construction of the curve of $\Delta E = \Delta t - \Delta\bar{t}$ (Fig. 57, p. 1222). Cf. also below p. 985.

C. For the determination of the parameters of the orbit of the moon Ptolemy makes use of a triple of carefully selected lunar eclipses, recorded in the years 27 and 28 of the era Nabonassar.[4] In this context an accurate determination of the mean motion of the moon during the intervals between the eclipses is of primary importance. The influence of the equation of time has therefore to be taken into consideration.

The moments to be compared, reckoned from epoch ($t_0 =$ Nabonassar 1), are:

$$t_1 = 26^y\ 28^d\ 8;40^h \qquad t_2 = 27^y\ 17^d\ 11;10^h \qquad t_3 = 27^y\ 194^d\ 7;40^h$$

and therefore the intervals, obtained "simply" by subtraction, are

$$\Delta_{12}t = 354^d\ 2;30^h \qquad \Delta_{23}t = 176^d\ 20;30^h.$$

First we have to determine for the given three moments mean and true longitudes of the sun. For this problem we may disregard the equation of time which would be associated with the time intervals beginning at t_0. This is evident for t_1 and t_2 because these intervals are almost exactly integer years, a fact which makes the influence of the equation of time equal to zero. For t_3 an additional half year occurs with endpoints similarly located as in the preceding example. Thus we know[5] that the equation of time changes the solar longitude by only about 0;1°, a difference which would have no significance for the subsequent computations.

[4] Almagest IV, 6; cf. below p. 77. The dates are −720 March 19, −719 March 8 and September 1 respectively.

[5] Cf. p. 63.

Knowing that the mean longitude of the sun at epoch was $\bar{\lambda}_0 = \text{♓}0;45°$[6] we find from the tables (Alm. III, 2) of the solar mean motion

$$\bar{\lambda}_1 = \bar{\lambda}_0 + t_1 \cdot \bar{v} = \text{♓}22;23 = \text{♊}5;30 + 286;53$$

($\bar{v}$ being the solar mean motion, ♊5;30 the solar apogee). Similarly

$$\bar{\lambda}_2 = \text{♓}11;24 = \text{♊}5;30 + 275;54$$
$$\bar{\lambda}_3 = \text{♍}5;43 = \text{♊}5;30 + 90;13.$$

The corresponding equations are (from Alm. III, 6)

$$c_1 = +2;15 \qquad c_2 = +2;21 \qquad c_3 = -2;23.$$

The resulting true longitudes would be

$$\lambda_1 = \text{♓}24;38 \qquad \lambda_2 = \text{♓}13;45 \qquad \lambda_3 = \text{♍}3;20$$

respectively. Ptolemy gives the values

approx. 24 1/2 13 1/2 1/4 approx. 3 1/4.

This is one of the many cases which one can find in the Almagest that final results expressed with unit fractions deviate badly (and unnecessarily much) from the accurately computed results.

Continuing our computation with the accurate values we find for the right ascensions of the sun (from Alm. II, 8)

$$\alpha_1 = 355;4,50 \qquad \alpha_2 = 345;3,8 \qquad \alpha_3 = 155;18,20.$$

Hence, with the above given mean longitudes, we have

$$\Delta_{1\,2}\bar{\lambda} = -10;59 \qquad \Delta_{1\,2}\alpha = -10;2$$
$$\Delta_{2\,3}\bar{\lambda} = 174;19 \qquad \Delta_{2\,3}\alpha = 170;15$$

and for the "accurately" computed intervals

$$\Delta_{1\,2}\bar{t} = \Delta_{1\,2}t + \Delta_{1\,2}\alpha - \Delta_{1\,2}\bar{\lambda} = \Delta_{1\,2}t + 0;57° \approx \Delta_{1\,2}t + 0;4^{\mathrm{h}}$$
$$\Delta_{2\,3}\bar{t} = \Delta_{2\,3}t + \Delta_{2\,3}\alpha - \Delta_{2\,3}\bar{\lambda} = \Delta_{2\,3}t - 4;4° \approx \Delta_{2\,3}t - 0;16^{\mathrm{h}}.$$

The corresponding mean motion of the moon is approximately 0;2° for the first interval and $-0;9°$ for the second.

If we had computed with Ptolemy's values for the true solar longitudes we would have obtained a correction of $+1;4° \approx 0;4^{\mathrm{h}}$ in the first case, $-4;8° \approx -0;17^{\mathrm{h}}$ in the second. Ptolemy himself gives as results $0;4^{\mathrm{h}}$ and $1/5^{\mathrm{h}} - 1/2^{\mathrm{h}}$ $(= -0;18^{\mathrm{h}})$ respectively. The effect on the mean longitude of the moon of these rounding errors is negligible.

3. Proof of Ptolemy's Rule

According to modern definition the "*equation of time*" E is the difference between the right ascension of the mean sun and of the true sun[1]

$$E = \bar{\alpha} - \alpha. \tag{1}$$

[6] Cf. above p. 63.

[1] Cf. p. 1081.

Since Ptolemy in the Almagest operates only with time intervals, $\Delta t = t_2 - t_1$ taken "simply" and $\Delta \bar{t}$ proportional to mean motions, we have to compare

$$\Delta E = E_2 - E_1 = \Delta \bar{\alpha} - \Delta \alpha \tag{2}$$

with his rule

$$\Delta t - \Delta \bar{t} = \Delta \bar{\lambda} - \Delta \alpha. \tag{3}$$

In both formulae $\Delta \alpha$ represents the same increment of right ascension of the true sun, taken at the moments t_1 and t_2. Hence

$$\Delta t - \Delta \bar{t} = \Delta E + \Delta \bar{\lambda} - \Delta \bar{\alpha}.$$

Here $\Delta \bar{\lambda}$ represents the increment of the mean longitudes of the sun $\bar{\lambda}_2 - \bar{\lambda}_1$ connected with the true longitudes λ_1 and λ_2 at t_1 and t_2, respectively by

$$\bar{\lambda}_2 - \bar{\lambda}_1 = \lambda_2 - c_2 - (\lambda_1 - c_1),$$

c_1 and c_2 being the corresponding equations of center. According to the definition of mean motion $\bar{\lambda}$ and therefore also $\Delta \bar{\lambda}$ increase proportionally with time, and the same holds for $\bar{\alpha}$ and $\Delta \bar{\alpha}$. According to modern definition $\bar{\alpha} = 0$ for $\lambda = 0$. According to the ancient concept of "mean sun" we have $\bar{\lambda} = -c_0$ for $\lambda = 0$. But the rate of change for $\bar{\alpha}$ is the same as for $\bar{\lambda}$, namely the mean solar velocity. Thus, when the modern mean sun has the right ascension $\bar{\alpha}$ the ancient mean sun has the longitude $\bar{\lambda} = \bar{\alpha} - c_0$ and thus we have for every interval

$$\Delta \bar{\lambda} = \Delta \bar{\alpha} \tag{4}$$

or

$$\Delta t - \Delta \bar{t} = \Delta E. \tag{5}$$

In other words Ptolemy's rule for finding from a "simply" expressed interval Δt the "accurate" interval $\Delta \bar{t}$ corresponds exactly to the transformation of an interval of "apparent solar time" to an interval of "mean solar time" in modern terminology.

4. The Equation of Time as Function of the Solar Longitude

If t_0 represents the epoch for a given set of tables, $\bar{\lambda}_0$, λ_0, α_0 mean and true longitude and right ascension of the sun at t_0, then, as we have seen, Ptolemy defines "accurate" time intervals $\Delta \bar{t}$ by means of

$$\Delta \bar{t} = t - t_0 + (\alpha - \alpha_0) - (\bar{\lambda} - \bar{\lambda}_0) = \Delta t - \Delta E \tag{1}$$

if we define ΔE by

$$\Delta E = \bar{\lambda} - \bar{\lambda}_0 - (\alpha - \alpha_0). \tag{2}$$

The right-hand side of (2) is a function of the solar longitude λ whose values we can find for any given λ by means of the tables in Alm. III, 6 for $c(\bar{\kappa}) = \lambda - \bar{\lambda}$[1] and in Alm. II, 8 for $\alpha(\lambda)$. Thus we can compute ΔE as function of λ from

$$\Delta E = \alpha_0 - \bar{\lambda}_0 + (\lambda - \alpha) - c. \tag{3}$$

[1] As always $\bar{\kappa} = \bar{\lambda} - ♊5;30 = \bar{\lambda} - 65;30°$.

This function ΔE of λ is not tabulated in the Almagest but a table for it was included by Ptolemy in his "Handy Tables" as we know from the preserved introduction.[2] A papyrus fragment containing a table for ΔE and written in the lifetime of Ptolemy or shortly after has been identified in the British Museum[3] but the first complete table is preserved only in Theon's version of the "Handy Tables" (written in the latter half of the fourth century) which then served as the prototype for the corresponding tables of Muslim and western astronomers until the 17th century.[4]

In order to get an impression how ΔE would look as function of λ under the assumption of the epoch t_0 = Nabonassar 1 of the Almagest and with the other elements on which this work is based we compute ΔE from (3) in steps of 10° for λ, at least as far as the term $\lambda - \alpha$ is concerned. In this way we obtain our Table 4 directly from Alm. II, 8. The equation of center c has to be found from Alm. III, 6 where, however, the independent variable is the mean eccentric anomaly $\bar{\kappa}$ and not λ. It is therefore more convenient to tabulate $c = \lambda - \bar{\lambda}$ as function of $\bar{\kappa}$, beginning at $\bar{\lambda}$ = ♊ 5;30, and to compute for values of $\bar{\kappa}$ at 10° intervals the corresponding values of λ (cf. Table 5). This then allows us to construct with sufficient accuracy a continuous graph of $-c$ as function of λ and thereafter to superimpose on it the graph of $\lambda - \alpha$ (cf. Fig. 57). Finally we have to add the constant $\alpha_0 - \bar{\lambda}_0$ for the date Nabonassar 1, Thoth 1. As we have found before[5] $\bar{\lambda}_0$ = ♓0;45 = 330;45 and from it $\alpha_0 = 335;8$. Hence

$$\alpha_0 - \bar{\lambda}_0 = +4;23°.$$

Since the sum of $\lambda - \alpha$ and $-c$ reaches a minimum of about $-4;48°$ in the middle of Aquarius we see that almost all values of ΔE are positive. ΔE has to be subtracted from Δt in order to find $\Delta \bar{t}$. Thus, ignoring a very small exception near ♒ 15°, we can say that for tables of mean motions computed with the epoch of the era Nabonassar the correction due to the equation of time is always negative.

Table 4

λ	$\lambda - \alpha$	λ
♈ 0♎	0	♎ 0♈
10	+0;50−	20
20	1;35	10
♉ 0♏	2;10	♍ 0♓
10	2;30	20
20	2;32	10
♊ 0♐	2;16	♌ 0♒
10	1;42	20
20	+0;55−	10
♋ 0♑	0	♋ 0♑

[2] Ptolemy, Opera II, p. 162, 23–163, 6 ed. Heiberg. Cf. also below p. 984f.
[3] Cf. Neugebauer [1958], p. 97ff.
[4] Flamsteed's treatise "De inaequalitate dierum solarium" (London 1672) is supposedly the first modern treatment of the subject; cf. Wolf, Handbuch 2, p. 261.
[5] Cf. p. 63.

Table 5

λ	$\bar{\lambda}$	$\bar{\kappa}$	$c=\lambda-\bar{\lambda}$	$\bar{\kappa}$	$\bar{\lambda}$	λ
♊ 5;30	♊ 5;30	0	0	360	♊ 5;30	♊ 5;30
15; 6,40	15;30	10	−0;23,20+	350	25;30	25;53,20
24;43,20	25;30	20	0;46,40	340	15;30	16;16,40
♋ 4;21	♋ 5;30	30	1; 9	330	♉ 5;30	♉ 6;39
14; 1,40	15;30	40	1;28,20	320	25;30	26;58,20
23;43,40	25;30	50	1;46,20	310	15;30	17;16,20
♌ 3;29	♌ 5;30	60	2; 1	300	♈ 5;30	♈ 7;31
13;18	15;30	70	2;12	290	25;30	27;42
23;11	25;30	80	2;19	280	15;30	17;49
♍ 3; 7	♍ 5;30	90	2;23	270	♓ 5;30	♓ 7;53
13; 8,20	15;30	100	2;21,40	260	25;30	27;51,40
23;13,20	25;30	110	2;16,40	250	15;30	17;46,40
♎ 3;24	♎ 5;30	120	2; 6	240	♒ 5;30	♒ 7;36
13;37,40	15;30	130	1;52,20	230	25;30	27;22,20
23;55	25;30	140	1;35	220	15;30	17; 5
♏ 4;16	♏ 5;30	150	1;14	210	♑ 5;30	♑ 6;44
14;39,20	15;30	160	0;50,40	200	25;30	26;20,40
25; 3,20	25;30	170	−0;26,40+	190	15;30	15;56,40
♐ 5;30	♐ 5;30	180	0	180	♐ 5;30	♐ 5;30

The graph of Fig. 57 confirms Ptolemy's statement[6] that the maximum effect of the equation of time will be found for a pair of observations made at solar longitudes of approximately ♏0° and ♒15°, respectively.

We will come back to the discussion of the equation of time in connection with Theon's Handy Tables (below V C 4, 3 B).

§ 3. Theory of the Moon. First Inequality; Latitude

1. Introduction

Book IV of the Almagest is devoted to the "simple" lunar theory, that is to say to a cinematic model which assumes the moon moving (in retrograde direction) on an epicycle which moves (in the direct sense) on a deferent in whose center the observer is located. This model would be the exact equivalent of the cinematic model for the solar motion[1] were it not for two features which are characteristic for the theory of the moon: a) the apsidal line of the lunar orbit is rotating (in the direct sense), and b) the lunar orbit, i.e. the common plane of the deferent and of the epicycle, is inclined to the ecliptic. Since, however, the angle of this inclination is small (5°), it will be ignored for the computation of longitudes. The latitudes can then be computed independently by the same method as the solar declinations.

[6] Cf. above p. 62.

[1] Because of the equivalence theorem (p. 57) we need not distinguish between an eccenter- and an epicycle-model.

This "simple" theory of the lunar motion is undoubtedly the largest body of theoretical astronomy inherited by Ptolemy from his predecessors. Nevertheless, a careful analysis of the foundations on which this classical theory was built led him to significant numerical and methodological improvements.

The theory developed so far rests on data obtained from lunar eclipses. In Book V of the Almagest it is shown that lunar longitudes outside of syzygies are not represented with sufficient accuracy and that there exists a "second anomaly" of the lunar motion depending on the moon's elongation from the sun. This second anomaly is zero at syzygies, so that the simple theory from Book IV remains valid for the theory of eclipses. The discovery of the second anomaly, in modern terminology known as "evection," had a far reaching influence on the techniques of ancient and medieval astronomy through Ptolemy's invention of movable eccenters by means of which he succeeded in describing correctly the observed deviations from the simple theory.

As a consequence of these refinements in the theory of the motion of the moon lunar positions provided the most accurately known data for the determination of nearby positions of planets and fixed stars. On the other hand the parameters of the lunar theory are derived from eclipses because only through the solar theory are ecliptic coordinates known with sufficient accuracy. This illustrates the decisive importance of the theoretical models and their interrelations in the order sun-moon-planets and fixed stars (which, in turn, influence the determination of the sidereal as well as of the tropical coordinates), an importance far greater than the role played by individual observations. This imbalance between theoretical structures and direct observations became even more accentuated in the time after Ptolemy and remained characteristic for all pre-telescopic astronomy.

In Chap. 1 of Book IV Ptolemy explains that only lunar eclipses are suitable for the determination of lunar longitudes because they are independent of parallax. Chap. 2 gives a short historical summary concerning the determination of the fundamental period relations of the lunar motion, followed by a critical discussion of the method to determine the length of the anomalistic period. Chap. 3 derives the numerical values of the different mean motions which then are tabulated in Chap. 4.

We shall postpone the discussion of Ptolemy's historical remarks to the section on Hipparchus.[2] We then can quickly dispose of the tables of mean motions and their computations but we will have to go into more detail in the discussion of Ptolemy's arguments concerning the correct determination of a period of the lunar anomaly.

2. Mean Motions

Ptolemy inherited from Hipparchus the following parameters:

$$\text{mean synodic month: } \bar{m} = 29;31,50,8,20^{d} \tag{1}$$

$$251 \text{ synodic months} = 269 \text{ anomalistic months} \tag{2}$$

$$5458 \text{ synodic months} = 5923 \text{ periods of latitude,} \tag{3}$$

[2] Below I E 5, 1 A.

or, written sexagesimally

$$4{,}11 \text{ synodic months} = 4{,}29 \text{ anomalistic months} \tag{2}$$

$$1{,}30{,}58 \text{ synodic months} = 1{,}38{,}43 \text{ periods of latitude.} \tag{3}$$

In order to find the moon's mean motion per day Ptolemy makes use of the mean motion of the sun [1]

$$\bar{v}_{\odot} = 0;59{,}8{,}17{,}13{,}12{,}31^{\circ/d} \tag{4}$$

and finds, using (1), for the travel of the moon during one mean synodic month

$$6{,}0 + \bar{m} \cdot \bar{v}_{\odot} = 6{,}29;6{,}23{,}1{,}24{,}2{,}30{,}57^{\circ}.$$

Hence the daily motion:

$$\bar{v}_{☾} = (6{,}0 + \bar{m} \cdot \bar{v}_{\odot})/\bar{m} = 13;10{,}34{,}58{,}33{,}30{,}30^{\circ/d}. \tag{5}$$

From (2) and (1) one obtains for the mean motion in anomaly

$$4{,}29 \cdot 6{,}0/4{,}11 \cdot \bar{m} = 13;3{,}53{,}56{,}29{,}38{,}38^{\circ/d}. \tag{6}$$

As we shall see later on [2] Ptolemy modified this result on the basis of his own observations to

$$13;3{,}53{,}56{,}17{,}51{,}59^{\circ/d}. \tag{6a}$$

Similarly (3) and (1) give for the daily motion of the moon with respect to the node

$$1{,}38{,}43 \cdot 6{,}0/1{,}30{,}58 \cdot \bar{m} = 13;13{,}45{,}39{,}40{,}17{,}19^{\circ/d}, \tag{7}$$

a value which is again corrected by Ptolemy to [3]

$$13;13{,}45{,}39{,}48{,}56{,}37^{\circ/d}. \tag{7a}$$

In modern theory one prefers not to use the motion of the moon with respect to the node but to consider the motion of the nodal line by itself. This parameter is obtainable by subtracting (7a) from the lunar motion in longitude (5), resulting in a retrograde motion of

$$-0;3{,}10{,}41{,}15{,}26{,}7^{\circ/d} \tag{8}$$

for the nodes.

For many applications it is also convenient to know the relative velocity of sun and moon, i.e. the mean "*elongation*"

$$\bar{v}_{☾} - \bar{v}_{\odot} = 12;11{,}26{,}41{,}20{,}17{,}59^{\circ/d} \tag{9}$$

as derived from (4) and (5).

On the basis of the values (5), (6a), (7a), and (9) tables of mean motions are given in Alm. IV, 4 for mean longitude, anomaly, latitude, and elongation, respectively, in exactly the same arrangement as the tables of solar mean motions [4] (and later on, in Alm. IX, 4, for the planets), i.e. for multiples of 18 years, for single years, months, and days. To find the mean parameters for a given date

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[1] Above p. 55.
[2] Below p. 79.
[3] Below p. 81.
[4] Above p. 55.

requires the addition of the accumulated mean motions to the values at epoch, i.e. at Nabonassar 1, Thoth 1. For the respective values and their derivations cf. below p. 79 and p. 82.

3. Period of the Lunar Anomaly

We have seen how the solar anomaly was determined in Greek astronomy from the inequality of the seasons.[1] No such simple method exists in the case of the moon with its short anomalistic period of about 27 1/2 days. It is hard to imagine any other way of discovering the variability of the lunar velocity than by direct observation of the day by day progress of the moon with respect to the fixed stars. The existence of such observations is abundantly attested in the Babylonian "Diaries" of the Persian and Seleucid period[2] and it is therefore not surprising to find very accurate values for the length of the anomalistic months embedded in the computations of Babylonian ephemerides.[3]

That the Babylonian parameters had been known to Hipparchus is evident from Ptolemy's historical introduction to Book IV of the Almagest.[4] For him the problem arose of checking these Babylonian-Hipparchian parameters and of obtaining values as accurate as possible for the mean motion in anomaly. In this context he gave (in IV, 2) a careful analysis of the conditions under which one could detect the exact completion of a period of the lunar anomaly.

Let us assume that τ is such a period; let $\lambda_0 = \lambda(t_0)$ be the true longitude of the moon at some moment t_0 and $\lambda_1 = \lambda(t_0 + \tau)$ at the later moment $t_0 + \tau$ and $\Delta\lambda = \lambda_1 - \lambda_0$ the corresponding increment in longitude. Since τ is a period of the anomaly the equation at t_0 and at $t_0 + \tau$ is the same, independent of t_0. Therefore also $\Delta\lambda$ will be independent of the moment at which the interval τ begins. This is the same as saying: for an interval τ to be a period of the anomaly it is a necessary condition that the true motion during τ is the same as the mean motion $\Delta\bar{\lambda}$ during τ, independent of the moment t_0 at which τ begins (cf. Fig. 58).

Ptolemy considers only lunar eclipses as sufficiently accurate for the determination of lunar longitudes. In order to find a period τ of the anomaly we should find two intervals $\Delta\lambda$ and $\Delta\lambda'$ of true lunar longitudes which are equal; this implies that we should find pairs of eclipses, separated from each other by the same time interval $\Delta t = \tau$, so that the mean motion of the moon is the same for each pair. In order also to establish the equality of the true motions $\Delta\lambda$ for two pairs of eclipses special conditions must be satisfied in order to eliminate the influence of the solar anomaly while the effect of the lunar anomaly should be as outspoken as possible.

In order to show that one cannot conclude the equality of $\Delta\lambda$ from the equality of Δt Ptolemy assumes that a first eclipse occurs at E when the sun is exactly at quadrature to the apsidal line AΠ (Fig. 59) where the equation has almost its maximum value $c = 2;23°$. If τ has exactly the length of $n + 1/2$ years a second eclipse would be located at E′, diametrically opposite to E. Thus $\Delta\lambda = 180 + 2c =$

[1] Above p. 57f.
[2] Cf. below p. 546.
[3] Cf. p. 481f.
[4] Below p. 310.

184 3/4°. If we now assume that a second pair of eclipses exists such that the first eclipse occurs at E′, the second $n+1/2$ years later at E we would have only a longitudinal difference of $\Delta\lambda' = 180 - 2c = 175\ 1/4°$ in spite of the equality of the time intervals.

In order to eliminate the influence of the solar anomaly the two pairs of eclipses should satisfy one of the following four conditions (cf. Fig. 60, cases 1 to 4, in which $E_1 E_1'$ represents the first pair of eclipses, $E_2 E_2'$ the second pair, assuming that $\Delta_1 t = \Delta_2 t = \tau$ and therefore also $\Delta_1 \bar{\lambda} = \Delta_2 \bar{\lambda} = \Delta \bar{\lambda}$): The sun travelled

1. complete rotations (hence $\tau = n$)
2. $n+1/2$ rotations, in the first pair beginning at the apogee, in the second at the perigee (or vice versa)
3. beginning at the same longitude in both pairs
4. in the first pair beginning $\alpha°$ before the apogee and ending in the second pair $\alpha°$ after it.

In all these cases we will have $\Delta_1\lambda = \Delta_2\lambda$ as required.

For the moon certain situations must be avoided since it is possible that two pairs of eclipses $E_1 E_1'$ and $E_2 E_2'$ of equal time interval τ and equal true longitudinal difference $\Delta\lambda$ nevertheless do not belong to a period of lunar anomaly α because $\alpha(E_1) = \alpha(E_1')$ and $\alpha(E_2) = \alpha(E_2')$ is not satisfied. Indeed, from $\Delta_1 t = \Delta_2 t = \tau$ it follows that $\Delta_1 \bar{\lambda} = \Delta_2 \bar{\lambda}$; therefore one also obtains equality of the true longitudinal increments $\Delta_1\lambda = \Delta_2\lambda$ only if the differences of the equations are the same: $\Delta_1 c = \Delta_2 c$. Equality of anomaly, however, requires that both $\Delta_1 c = 0$ and $\Delta_2 c = 0$ at the endpoints of the intervals $\Delta_1 t$ and $\Delta_2 t$, respectively. Thus the following cases of $\Delta_1 c = \Delta_2 c \neq 0$ must be avoided (cf. Fig. 61):

1. $\alpha(E_1) = \alpha(E_2)$ because for all position of E_1' and E_2' one obtains $\Delta_1 c = \Delta_2 c$ without $E_1' = E_1$ and $E_2' = E_2$.
2. $\alpha(E_1) = -\alpha(E_2')$ because E_2 and E_1' will be in symmetric positions to the apsidal line, such that again $\Delta_1 c = \Delta_2 c$ without $E_1' = E_1$ and $E_2' = E_2$.
3. $\Delta_1 c = \Delta_2 c = 0$ is possible without $E_1' = E_1$ and $E_2' = E_2$. This occurs, e.g., if $E_1' = A$, $E_1 = \Pi$ and $E_2' = \Pi$, $E_2 = A$ (and similar combinations with the apogee A the perigee Π of the epicycle).

Ptolemy says that it is advisable, not merely to avoid such situations, but to choose pairs of eclipses which will make deviations from a return to the same anomaly as visible as possible[5].

We again assume that $\Delta_1 t = \Delta_2 t$ (and hence $\Delta_1 \bar{\lambda} = \Delta_2 \bar{\lambda}$), and we select eclipses such that, if there is not an integer number of revolutions of anomaly in both intervals, $\Delta_1\lambda$ will differ as much as possible from $\Delta_2\lambda$. Thus we will be able to tell that $\Delta\alpha \neq 360°$, and can conclude that τ is not a period of return in anomaly even though $\Delta_1 t = \Delta_2 t$.

The most favorable cases are the following (cf. Fig. 62):

1. $E_1 = A$, $E_1' \neq \Pi$, and $E_2 = \Pi$, $E_2' \neq A$. In the most extreme situation (shown in Fig. 62) $\alpha(E_1') \approx 90°$, hence $\alpha(E_2') \approx 270°$, or $\alpha(E_1') \approx 270°$, hence $\alpha(E_2') \approx 90°$. Thus $\max|\Delta_1\lambda - \Delta_2\lambda| = 2c$.
2. $\alpha(E_1) \approx 270°$, $\alpha(E_2) \approx 90°$. If, as illustrated, $E_1' \approx A$, and hence $E_2' \approx \Pi$, then $|\Delta_1\lambda - \Delta_2\lambda| = 2c$. The most extreme situation occurs when $\alpha(E_1') \approx 90°$, and hence $\alpha(E_2') \approx 270°$. Then $\max|\Delta_1\lambda - \Delta_2\lambda| = 4c$.

[5] I owe the explanation of this difficult passage to my colleague G. J. Toomer.

Ptolemy says that Hipparchus was aware of all these conditions and that he selected eclipses accordingly, in particular such that the last mentioned case 1 took place. He did not succeed, however, in eliminating simultaneously the anomaly of the sun, insofar as in each pair of eclipses about 7 1/2° were missing from a return to the same solar longitude[6] (cf. above p. 72 case 1). Unfortunately Ptolemy does not give the dates of these eclipses which would be of great interest for the investigation of Hipparchus' own observations in relation to the parameters which he adopted from Babylonian astronomy.[7] Nor does Ptolemy's own procedure for the determination of the mean lunar anomaly[8] rest on the selection of two pairs of eclipses. It seems as if the whole preceding analysis of the difficulties to obtain an exact period of the anomaly is meant as an explanation of the fact that Hipparchus' parameters were not final but required certain corrections, in spite of the care and competence shown in their determination.

4. Radius and Apogee of the Epicycle

A. Summary of the Method

We assume that the moon P moves on an epicycle of center C, observed from the center O of the deferent (Fig. 63). Let $\Delta\lambda$ be the difference in true longitude between two observed positions P_1 and P_2, $\Delta\bar{\lambda}$ the corresponding change in mean longitude during the time Δt between the two observations. Obviously the difference $\Delta\lambda - \Delta\bar{\lambda}$ depends on the size of the epicycle and on the position of P with respect to the apsidal line OA. This suggests using conversely $\Delta\lambda - \Delta\bar{\lambda}$ for the determination of the radius of the epicycle and of the distance of P from A.

It is essential for the approach to this problem as described in the Almagest to realize that one need not consider different positions C_1, C_2 of the epicycle (cf. Fig. 63). If we put P_1' in the same position with respect to A_2 as P_1 occupies with respect to A_1 then the angular distance from P_1 to P_1' is the same as from C_1 to C_2, that is to say $\Delta\bar{\lambda}$. But the angular distance from P_1 to P_2 is $\Delta\lambda$. Therefore the angle $P_1'OP_2$ equals $\Delta\lambda - \Delta\bar{\lambda}$. The angle at C_2 between P_1' and P_2 is the increment $\Delta\alpha$ of the anomaly during the time Δt. Thus we see that $\Delta\lambda - \Delta\bar{\lambda}$ is the angle under which the arc $\Delta\alpha$ appears from O.

We now add a third point (P_3) to the two preceding ones on the epicycle. We shall prove that this suffices for the determination of the ratio $r:R$ ($R = \mathrm{OC}$) and of the positions of P_1, P_2, P_3 with respect to the apsidal line OCA.

In order to obtain three accurately defined lunar positions Ptolemy uses three lunar eclipses. Thus we may consider to be known: the differences $\Delta\lambda$ of true longitudes, determined by the computed solar positions at the moments of the middle of the eclipses, together with the time intervals Δt. Hence we also know, from the tables of mean motions, the corresponding increments $\Delta\bar{\lambda}$ in mean longitude and $\Delta\alpha$ in epicyclic anomaly. Using the procedure of Fig. 63 we can say that we have three points on a circle of radius r, subtended by given angles

[6] It seems possible that the relation (5), quoted below p. 310, is the result of these observations.

[7] Cf. p. 310.

[8] Cf. below p. 78.

$\Delta\alpha$ at its center C. We should find the position of a point O outside this circle such that the three points appear from O under given angles $\Delta\lambda - \Delta\bar{\lambda}$.

We shall first make sure that this problem has a unique solution. The locus of all points O from which a given chord PQ appears under a given angle δ is a circle in which PQ subtends the angle 2δ at its center K (Fig. 64). Now we are given three points on the circle with angles $\bar{\delta}_1$, $\bar{\delta}_2$ at C such that the corresponding chords are seen from O under the angles δ_1 and δ_2. A third angle $\bar{\delta}_3$ at C is given by $\bar{\delta}_1 + \bar{\delta}_2 + \bar{\delta}_3 = 360°$ and correspondingly a third angle δ_3 at O has to satisfy $\delta_1 + \delta_2 + \delta_3 = 0$ (cf. Fig. 65). To each angle δ belong two symmetric loci, (δ) and $(-\delta)$, from which a given chord appears under the same angle $|\delta|$ (cf. Figs. 64 and 65). Hence we should also investigate intersections which occur when one reflects (δ_1) on P_1P_2, (δ_2) on P_2P_3, and (δ_3) on P_1P_3. If we reflect only one or two of these circles we do not obtain a common intersection of all three loci as is required for O. But if we reflect all three loci then there exists a common intersection of $(-\delta_1)$, $(-\delta_2)$, $(-\delta_3)$ in a point O′ as shown in Fig. 65.

The astronomical conditions allow us to distinguish between O and O′. Returning to Fig. 64 we remark that angles at O are counted positive for increasing longitudes (i.e. counter-clockwise) while angles at C are reckoned positive for increasing anomalies (i.e. clockwise). In the arrangement of Fig. 64 PQ, in this order, is seen both from O and from C under positive angles. We express this by saying that O is located in the positive halfplane. If K and O were located in the symmetric position with respect to the line PQ the positive angle $\bar{\delta}$ at C would appear as $-\delta$ in O; then we say that O is in the negative halfplane. Since we have three points on the epicycle we have six halfplanes of opposite signs defining six regions outside the epicycle with different combinations of three signs. Thus each intersection is associated with three signs according to the halfplanes to which it belongs. The astronomical data also provide signs associated with the increments which belong to the progress from the first to the second eclipse and from the second to the third. A third angle is determined, with its signs, from the condition $\delta_1 + \delta_2 + \delta_3 = 0$. Although the angles at O and at O′ are identical all signs in O are opposite to the signs in O′ and hence the astronomical data will exclude one of the two combinations of signs.

As an example we use the data from three Babylonian lunar eclipses, from the years Nabonassar 27 and 28, postponing the discussion of the remaining elements to later.[1] We denote these eclipses in their chronological order by I, II, III, respectively. From the observations and from the given time intervals one finds for the true longitudes λ, for the mean longitudes $\bar{\lambda}$, and for the anomaly α the following increments[2]

	$\Delta\lambda$	$\Delta\bar{\lambda}$	$\Delta\alpha$
from I to II:	349;15	345;51	306;25
from II to III:	169;30	170;7	150;26.

In order to place these three points in their proper relative position on the epicycle (Fig. 66) we remark that the arc from II to I is given by

$$\bar{\delta}_1 = 360 - 306;25 = 53;35$$

[1] Below p. 76ff.

[2] Cf. also below p. 77.

and the arc from I to III by

$$\bar{\delta}_2 = 150;26 - \bar{\delta}_1 = 96;51$$

(and hence $\bar{\delta}_3 = 360 - \bar{\delta}_1 - \bar{\delta}_2 = 209;34$). From O the motion from I to II appears under the angle

$$\delta_1' = 349;15 - 345;51 = +3;24,$$

the motion from II to III under the angle

$$\delta_2' = 169;30 - 170;7 = -0;37.$$

The change of anomaly from II to I on the epicycle is the result of a motion from II to III minus the motion from I to III which is the sum of the motions from I to II and from II to III. Therefore the chord II $\rightarrow$ I is seen from O under the angle

$$\delta_1 = \delta_2' - (\delta_1' + \delta_2') = -\delta_1' = -3;24°.$$

Similarly the chord from I to III is seen from O under the angle which corresponds to the motion from I to II and from II to III:

$$\delta_2 = \delta_1' + \delta_2' = +2;47°.$$

Finally

$$\delta_3 = -(\delta_1 + \delta_2) = +0;37°.$$

We now consider the six regions into which the sides of the triangle I, II, III divide the exterior of the circle which represents the epicycle (Fig. 67). If O were located in the region (1) δ_1 would be positive because O belongs to the halfplane from which the motion II $\rightarrow$ I is counted positive at O. The angles δ_2 and δ_3, however, would be negative since I $\rightarrow$ III and III $\rightarrow$ II would be seen moving in the sense of decreasing longitudes. Thus the region (1) obtains the signature $+ - -$. Since the given data require a signature $- + +$ we see that O cannot belong to (1). Fig. 67 gives the signature for all six regions,[3] always in the order δ_1, δ_2, δ_3. This shows that the signature $- + +$ belongs to region (4). Thus we know that O must be located in the sector with vertex III. The point O′ of our preceding discussion (p. 74) belongs to region (1) which has a signature exactly opposite to region (4). Hence O represents the solution of our problem and not O′.

Except for the abstract formulation we have given here it is exactly by this type of arguing that Ptolemy determined the location of O with respect to the epicycle and established the numerical values of the angles which he had to use for the trigonometric solution.

The trigonometric part of the problem presents no serious difficulty and it suffices here to sketch the general plan of the solution.

We are given three points P_1, P_2, P_3 on a circle of radius r at given angular distances $\bar{\delta}_1$, $\bar{\delta}_2$, $\bar{\delta}_3$ (cf. Fig. 68[4]); we should find a point O outside the circle such that the chords P_1P_2 and P_2P_3 appear under given angles δ_1 and δ_2, respectively. Let us assume that OP_1 intersects the circle at Q. Then the angles at Q are known: $P_1QP_2 = 1/2\,\bar{\delta}_1$ and $P_2QP_3 = 1/2\,\bar{\delta}_2$. All three sides of the triangle P_2P_3Q can be

[3] The interior of the triangle would have the signature $- - -$ (or $+ + +$ since we are dealing in fact with the projective plane).

[4] Fig. 68 does not give the positions of P_1, P_2, P_3 as required in the case of the eclipses I, II, III (for which cf. Figs. 66 and 67).

found: $P_2P_3 = r \operatorname{crd} \bar{\delta}_2$; P_2Q from the triangle P_2QO, in which all angles are known, in units of the side $QO = x$; similarly P_3Q from the triangle P_3QO, also in terms of x. Since $P_3Q = r \operatorname{crd} \varepsilon$ we can find the angle ε and therefore also the angle

$$\eta = 360 - (\bar{\delta}_1 + \bar{\delta}_2 + \varepsilon).$$

Consequently $y = P_1Q = r \operatorname{crd} \eta$ is known.

By means of these steps both $OQ = x$ and $QP_1 = y$ can be expressed in terms of r, and therefore also the product $x \cdot (x+y)$. But this product has a constant value for all positions of P_1 on the circumference and for a fixed position of O. Therefore, using the point A, we have

$$x \cdot (x+y) = (R+r)(R-r) \tag{1}$$

where $R = OC$ is the radius of the deferent. From (1) one can find the ratio r/R and hence the value of r in units of $R = 60$.

After having thus found the radius of the epicycle one still needs the anomaly α of one of the three points on the epicycle, e.g. of P_1 (cf. Fig. 69). If we make CK perpendicular to OP_1 we know $OK = x + 1/2\,y$. Since $R = 60$ we can find the angle c (which is the equation of P_1). Therefore the anomaly of P_1

$$\alpha = 90 + c - 1/2\,\eta \tag{2}$$

is known, since η has been found before.

The true longitude λ of P_1 is known from observation; having found the equation c we know also the mean longitude $\bar{\lambda} = \lambda - c$ of P_1.

In order to find the anomalies of P_2 and P_3 one has only to add $\bar{\delta}_1$ and $\bar{\delta}_2$ to α. The mean longitudes then result from the use of the increments $\Delta\bar{\lambda}$ which are given through the known time intervals Δt between the observations.

Thus all parameters of the epicyclic model of the lunar motion can be found on the basis of the observation of three lunar eclipses. The whole procedure is a beautiful example of mathematical analysis applied to a complex astronomical problem.

B. Numerical Data and Results

Ptolemy uses two triples of lunar eclipses for the determination of the radius of the moon's epicycle, one from the years Nabonassar 27/28,[1] one from his own time, Nabonassar 880/883. In the first case he finds $r = 5;13$, in the second $r = 5;14$.[2] Thus he accepted the convenient rounding $r = 5\ 1/4$ as the final result which is used in all subsequent computations. The values obtained for the anomaly of the moon at the second eclipse in each triple are used for the determination of the mean motion of anomaly[3] and also for the epoch value.[4]

We now can supplement our description of Ptolemy's procedures by a discussion of the observational data upon which the preceding computations were based.

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[1] Cf. above p. 74.

[2] As usual these computations contain many small inaccuracies such that $r = 5;13$ would be the nearest common solution.

[3] Below p. 79.

[4] Below p. 79.

The earlier triple of eclipses undoubtedly goes back to Babylonian records from the years 1 and 2 of the king "Mardokempados," i.e. Marduk-apal-iddina,[5] though the dates are given by Ptolemy in the Egyptian calendar. The hours, however, refer to local Babylon time, which Ptolemy transfers to Alexandrian time by subtracting 50 minutes, as is nearly correct.[6] For the given moments, modified in this way, Ptolemy finds for the longitudes of the moon, diametrically opposite to the true solar positions, found by computation

Eclipse	time since epoch	$\lambda_☾$	$\Delta\lambda$	$\Delta\bar{t}$
I	$26^y\ 28^d\ 8;40^h$	♍ 24;30		
			349;15	$354^d\ 2;34^h$
II	$27^y\ 17^d\ 11;10^h$	♍ 13;45		
			169;30	$176^d\ 20;12^h$
III	$27^y\ 194^d\ 7;40^h$	♓ 3;15		

where $\Delta\bar{t}$ includes the equation of time. We have discussed its computation on p. 64 and noted inaccuracies committed in the roundings. On the basis of these time intervals Ptolemy finds for the mean motions in longitude and anomaly

	$\Delta\bar{\lambda}$	$\Delta\alpha$
I → II	345;51	306;25
II → III	170; 7	150;26

but checking reveals again small deviations: 170;8 for $\Delta\bar{\lambda}$ and 306;24 for $\Delta\alpha$. The values for the angles δ computed without all these small errors would be $\delta_1 = -3;16°$ and $\delta_2 = +2;43°$ instead of Ptolemy's $-3;24°$ and $+2;47°$, respectively which he used for the determination of the radius of the epicycle.[7] The effect on the final result would probably not produce another rounded value than $r = 5\ 1/4$ but practically all numerical steps in Ptolemy's computations should be slightly changed.

The second triple of lunar eclipses, observed by Ptolemy himself, leads to the following set of data:

	t	$\lambda_☾$	$\Delta\lambda$	$\Delta\bar{t}$
I′	Hadrian 17, Payni 21	♏ 13;15		
	(= 133 May 6)		161;55	$1^y\ 166^d\ 23\ 5/8^h$
II′	Hadrian 19, Choiak 2	♈ 25;10		
	(= 134 Oct. 20)		138;55	$1^y\ 137^d\ 5\ 1/2^h$
III′	Hadrian 20, Pharm. 19	♍ 14; 5		
	(= 136 March 6)			

Both the longitudes and the corrections for equation of time show only negligible inaccuracies and also the increments, derived from $\Delta\bar{t}$, are practically correct

	$\Delta\alpha$	$\Delta\bar{\lambda}$	$\Delta\lambda - \Delta\bar{\lambda}$
I′ → II′	110;21	169;37	−7;42
II′ → III′	81;36	137;34	+1;21

and hence also the angles at O.

[5] That is "Marduk gave an heir"; biblical distortion: Merodach-baladan.

[6] Accurate value: 57 minutes.

[7] Above p. 75 f.

Supplementary Remarks. For the observational data as recorded by Ptolemy we give in Table 6 a comparison with the results of modern computations, for the Babylonian eclipses taken from P.V. Neugebauer's "Kanon der Mondfinsternisse," for Ptolemy's observations from Oppolzer's "Canon." Furthermore we give for the Babylonian eclipses in Fig. 70 a graphical representation of the relative positions of the moon and of the shadow at first and last contact.[8] The arrows represent the total motion of the center of the shadow relative to the moon during the eclipse. The diameter NS of the moon coincides with the hour-circle which passes through the center of the lunar disk.[9]

Table 6

No.	Date	Phases						$\lambda_{☾}$		magnitude	
		Text			modern						
		beg.	middle	end	beg.	middle	end	Text	modern	Text	mod.
I	−720 March 19	$20;30^h$	$22;30^h$	–	19.6^h	$21;30^h$	23.4^h	♍24;30	♍21;50	total	18.2
II	−719 March 8/9	–	24	–	22.9	23;40	0.3	♍13;45	♍11; 0	3″ from S	1.5
III	−719 Sept. 1	19	20;30	22	18.8	20; 0	21.2	♓ 3;15	♓ 1;10	>6″ from N	6.1

No.	Date	middle		$\lambda_{☾}$		magnitude	
		Text	mod.	Text	modern	Text	mod.
I′	133 May 6	$23;15^h$	$23;\ 7^h$	♏13;15	♏14;24	total	12.9
II′	134 Oct. 20	23; 0	23; 1	♈25;10	♈26;28	10″ from N	10.1
III′	136 March 6	4; 0	3;43	♍14; 5	♍14;56	6″ from S	5.5

C. Check of the Mean Anomaly; Epoch Values

The greatest time interval between two consecutive eclipses in either one of the two triples which Ptolemy used for the determination of the radius and the apogee of the lunar epicycle amounts to less than 1 1/2 years. For so short an interval the Babylonian-Hipparchian values for the mean motion of the lunar anomaly fully suffice to give an accurate value for the change $\Delta\alpha$ of the mean anomaly between two eclipses in each triple. On the basis of this knowledge it was possible to find for each one of the six eclipses its distance in anomaly from the apogee of the epicycle at the given moment. This information can now be used to determine the change of anomaly over the whole interval of 8 1/2 centuries which separate Ptolemy's observations from the Babylonian eclipses. Ptolemy found in this way an accumulated error of 0;17° for the Hipparchian mean value. The corresponding correction is reflected in the change from (6) to (6a) listed above p. 70.

The details of this computation are as follows. Ptolemy compares the Babylonian eclipse II with his own II′. The dates are:

[8] From P.V. Neugebauer, Kanon d. Mondf.

[9] Cf. P.V. Neugebauer, Astr. Chron. II, p. 128.

II: Mardokem. 2 (=Nab. 28) Thoth 18 [$=-719$ March 8=jul. d. 1458510] 11;10^h Alex. noon

II′: Hadrian 19 (=Nab. 882) Choiak 2 [=A.D. 134 Oct. 20=jul. d. 1770294] 11;0^h Alex. noon.

The time difference is

$$\Delta t = 854 \text{ Eg. y. } 73^d\, 23;50^h$$

in simple reckoning. The equation of time reduces the hours to 23;20. Thus one has finally

$$\Delta \bar{t} = 311783^d\, 23;20^h.$$

For the two eclipses the anomalies had been found to be[1]

$$\alpha_{II} = 12;24^\circ \qquad \alpha_{II'} = 64;38^\circ \tag{1}$$

hence $\Delta\alpha = 52;14^\circ$. Using the Hipparchian mean value for the daily increase of anomaly the result of multiplication by $\Delta\bar{t}$ is, according to Ptolemy, $\Delta\alpha = 52;31^\circ$, thus 0;17° too high. Division of 0;17° by $\Delta\bar{t}$ results in Ptolemy's correction of Hipparchus' value by $-0;0,0,0,11,46,39^{\circ/d}$, i.e. to $13;3,53,56,17,51,59^{\circ/d}$.

Checking these statements shows, however, that the multiplication of Hipparchus' mean value (6), p. 70 by $\Delta\bar{t}$ leads to $\Delta\alpha = 52;32,11$. Consequently Ptolemy's own tables (Alm. IV, 4) do not lead to $\Delta\alpha = 52;14$ as required according to (1) by the eclipses but to 52;15,11°, i.e. 0;17° less than the correctly computed Hipparchian value.

In a similar fashion Ptolemy recomputed also the mean motion in longitude. Since the mean longitudes of the moon are

$$\bar{\lambda}_{II} = ♍\, 14;44 \qquad \bar{\lambda}_{II'} = ♈\, 29;30$$

respectively, one has an increment of $\Delta\bar{\lambda} = 224;46^\circ$. Ptolemy says that this is in agreement with Hipparchus' mean value and he therefore built his own tables on the same parameter. Actually one finds from the tables $\Delta\bar{\lambda} \approx 224;47,12$ thus a deviation which would affect the two last places (5th and 6th) of the accepted mean motion. This is a very small error but it should have been avoided since it lies within the pretended accuracy of the tables.

In order to obtain the epoch values for the tables of the mean motions of the moon Ptolemy computes back to Nabonassar 1 Thoth 1 from the nearby eclipse II. For it one has[2] $\Delta t = 27$ Eg. y., $17^d\, 11;10^h$ and $\bar{\lambda} = ♍\, 14;44$ $\alpha = 12;24^\circ$. From the tables in IV, 4 Ptolemy obtains[3]

$$\Delta\bar{\lambda} = 123;22^\circ \qquad \Delta\alpha = 103;35^\circ$$

and therefore the epoch values

$$\bar{\lambda}_0 = ♉\, 11;22 \qquad \alpha_0 = 268;49 \tag{2}$$

and hence an initial elongation from the mean sun (♓0;45)[4] of

$$\bar{\eta}_0 = 70;37^\circ.$$

These are the parameters listed in the tables Alm. IV, 4.

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[1] For the method cf. above p. 76 (2).

[2] Since Δt is close to an integer number of years the influence of the equation of time can be ignored.

[3] Actually the tables give $\Delta\bar{\lambda} = 123;22,33$ and $\Delta\alpha = 103;35,23$.

[4] Above p. 63.

5. The Tables for the First Inequality

The "first anomaly" of the moon is the equation c which corresponds to the anomaly α of the position of the moon P on the epicycle (cf. Fig. 71). Obviously c is negative for $0 \leqq \alpha \leqq 180$ and positive for $180 \leqq \alpha \leqq 360$. With $r = 5;15$ and $R = 60$ given it is easy to compute a table of $c(\alpha)$.

The results are tabulated twice in the Almagest. Once in IV, 10 by themselves, a second time in V, 8 as column 4 which gives the anomaly uninfluenced by the "second anomaly" (which is caused by a variation of the distance OC[1]). In both tables α proceeds in steps of 6° between 270° and 90° and in steps of 3° in the other semicircle which contains the perigee of the epicycle. The maximum equation

$$|c_{\max}| = 5;1^\circ \tag{1}$$

is found for $\alpha = 96$ and 264.

For an example for the computation of the lunar longitude cf. below p. 84.

6. Latitude

Ptolemy's theory of the lunar latitude is concerned with three problems: first, to establish as accurately as possible the mean value of the change of the argument of latitude; secondly, to determine the epoch value of the argument of latitude for Nabonassar 1 Thoth 1; thirdly, to give a table of the latitude as function of the argument of latitude. Certainly the solution of the third problem does not contain any new contribution by Ptolemy. His approach to the two other problems, however, represents an important methodological progress over the procedure used by Hipparchus and is therefore of particular historical interest.

A. Mean Motion of the Argument of Latitude

According to modern usage the argument of latitude measures the angular distance of the moon in its orbit from the ascending node. The ancient norm, however, does not use the node as zero point but, for reasons unknown, the northernmost point N of the lunar orbit (cf. Fig. 72). Since the plane of the epicycle coincides with the inclined orbital plane (which is the plane of the deferent) we can distinguish between a *mean* argument of latitude $\bar{\omega}'$ of the center C of the epicycle, and a *true* argument of latitude ω' of the moon P. The difference $c = \omega' - \bar{\omega}'$ is nothing but the equation $c = \lambda - \bar{\lambda}$ which connects mean and true longitude and which is tabulated as function of the anomaly α in Alm. IV, 10.

According to definition $\bar{\omega}'$ increases proportionately with time. In order to determine the mean value of this increase Ptolemy tries to find two eclipses which are as similar to each other as possible and therefore as near as possible to an exact period of $\bar{\omega}'$.

A pair of such eclipses would have to satisfy the following four conditions:

(a) both eclipses are of the same magnitude

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[1] Cf. below I B 4, 1.

(b) both eclipses belong to the same type of node (ascending, respectively, descending)

(c) the moon is eclipsed from the same side (south or north)

(d) both eclipses occur at the same geocentric distance OP, because otherwise one cannot conclude from (a) that both eclipses have the same distance from the node.

If these conditions are satisfied then we are sure that the true moon is at both eclipses in exactly the same situation with respect to a node of the same type. From a common value of ω' it is then easy to conclude the value of $\bar{\omega}'$ at each eclipse. Since the time difference is known we can find the mean value of the change of $\bar{\omega}'$.

The eclipses which Ptolemy found nearly satisfying all these conditions are the two following ones:

A. Darius 31 (= Nab. 257) Tybi 3/4, $1/2^h$ before midnight Babylon[1]

B. Hadrian 9 (= Nab. 872) Pachon 17/18, $3\,3/5^h$ before midn. Alexandria[2].

The conditions (a) and (c) are satisfied because both eclipses have the magnitude 2′ from the south.[3] Any reasonably accurate estimate of the nodal motion will show that the nodes must be of the same kind[4]; thus also (b) is satisfied. That (d) is satisfied, or nearly satisfied, follows from the computation of the anomalies of the moon for the two given dates[5]; one finds for the first eclipse $\alpha_1 = 100;19$, for the second $\alpha_2 = 251;53 = -108;7$, i.e. positions at nearly the same distance from O.

The corresponding equations are, according to Alm. IV, 10, $c_1 = -5;0°$ and $c_2 = 4;53°$, respectively. Since all four conditions are satisfied we know that the moon has been observed in both eclipses in the same position with respect to the node. Therefore we see (cf. Fig. 73[6]) that the center C of the epicycle moved during the time interval $\Delta\bar{t}$ between the two eclipses by the amount $\Delta\bar{\omega}' = -5;0 - 4;53 = -9;53°$. For $\Delta\bar{t}$ one finds (including the equation of time) $615^y\ 133^d\ 21;50^h$. If one multiplies this value of $\Delta\bar{t}$ by Hipparchus' value for the mean motion of the argument of latitude $13;13,45,39,40,17,19^{\circ/d}$ one finds (mod. 360) $\Delta\bar{\omega}' = -10;2$ instead of the $-9;53$ required by the observations just analyzed. Consequently we find an accumulated error of $-0;9°$ which, distributed over the interval $\Delta\bar{t}$, leads to Ptolemy's value (7a) given in his mean motion tables (cf. p. 70).

B. Epoch Value for the Argument of Latitude

For the determination of the mean motion of the argument of latitude it was only necessary to know that the two eclipses were in the same position with

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[1] $= -490$ Apr. 25.

[2] = A.D. 125 Apr. 5.

[3] Modern values: 1.7″ and 2.0″, respectively.

[4] If we assume, e.g., a nodal motion of $-0;3,10,40^{\circ/d}$ we find from multiplication by the time interval $\Delta t = 1,2,23,29^d$ a motion of about $-15°$ (mod. 360). This suffices to exclude opposite nodes for two lunar positions of nearly the same longitude (cf. the dates given in notes 1 and 2). Cf. also below p. 82 n. 4.

[5] Ptolemy's corrections for the equation of time are slightly inaccurate because of the use of unit fractions of hours but the effect on the anomalies is negligible.

[6] Drawn to scale.

respect to the node but it was not necessary to know the actual distance from the node. In the case of the epoch value, however, a specific position must be known from which one can compute back to the beginning of the era.

Thus Ptolemy selects two eclipses which satisfy the conditions (a), (c), and (d) (p. 81) but belong to opposite nodes. If these conditions are satisfied we know that the positions P_1 and P_2 of the true moon at the two eclipses had equal distance from the nodes toward the same side of the ecliptic (cf. Fig. 74).

Two such eclipses, selected by Ptolemy, are:

C. Mardokem. 2 (= Nab. 28) Thoth 18/19, $5/6^h$ before midn. Alexandria [1]

D. Darius 20 (= Nab. 246) Epiph. 28/29, $1;15^h$ before midn. Alexandria. [2]

Both eclipses have a magnitude of 3 digits from the south [3] but belong to opposite nodes. [4] That also condition (d) is nearly satisfied follows from the anomalies $\alpha_1 = 12;24$ and $\alpha_2 = 2;44$, respectively, which show that the moon was in both cases near the apogee of the epicycle.

The corresponding equations are $c_1 = -0;59°$ and $c_2 = -0;13°$. With respect to the (rotating) nodal line we have therefore a situation as schematically represented in Fig. 76. The mean distance $\Delta\bar{\omega}'$ can be computed from the mean time interval $\Delta\bar{t}$ between the eclipses; Ptolemy finds $\Delta\bar{\omega}' = 160;4$. [5] Thus Fig. 76 shows that

$$180° = x - 0;13 + 160;4 + 0;59 + x.$$

Hence one finds for x, the true nodal distance at both eclipses, the value 9;35°. The nodal distance of C_1, the center of the epicycle at the first eclipse, is therefore given by $x + c_1 = 9;35 + 0;59 = 10;34$. The argument of latitude, reckoned from N, is $270 + 10;34 = 280;34°$. We also know that this eclipse occurred $27^y\ 17^d\ 11;10^h$ after the epoch date Nab. 1 Thoth 1 [6]; during this time the argument of latitude increased by 286;19°. Thus at epoch the argument of latitude had the value

$$\bar{\omega}'_0 = 280;34 - 286;19 = -5;45 \equiv 354;15 \quad (\text{mod. } 360). \tag{1}$$

This is the epoch value given in the tables of mean motion (Alm. IV, 4).

The essential progress in Ptolemy's method for the determination of the mean motion in latitude and of its epoch consists in the fact that it does not involve a specific value for the apparent diameter of the moon or of the shadow, values which are difficult to determine accurately and, of course, only by means of eclipse observations. Ptolemy succeeded in going directly from the observable elements of the eclipses to the mean motion in latitude by the above described elegant procedure. [7] But an enormous amount of computational work must have preceded it in order to have all the data available which are necessary for a proper selection among the recorded eclipses.

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[1] This is the same eclipse used before as No. II for the determination of r (cf. p. 74, p. 77, and Fig. 70, p. 1228). The julian date is -719 March 8.

[2] $= -501$ Nov. 19/20. Compare Fig. 75 with Fig. 70 II, p. 1228.

[3] Actually only 1.5″ and 2.1″ respectively.

[4] To show this one has to remark that the nodal motion during $\Delta t = 218^y\ 310^d$ amounts to nearly 90°. At the eclipse (C) the moon, and thus one node, was near ♍ 15 (cf. above p. 77). Consequently, this node was at the eclipse (D) near ♐ 15. But the moon was at (D) near ♉ 23 thus near the opposite node.

[5] Actually one finds 160;3,9.

[6] Cf. above p. 77.

[7] For the more primitive method used by Hipparchus cf. below p. 313 f.

C. The Lunar Latitude; Example

The moon moves in a plane of fixed inclination $i=5°$[1] with respect to the ecliptic. In other words the plane of the eccenter as well as of the epicycle coincide with the orbital plane[2] and the "latitude" of the moon is given by the arc between the moon and the ecliptic of the great circle through the pole of the ecliptic and the moon. The fact that all motions of the lunar cinematic model take place in the same plane make the theory of the lunar latitude much simpler than the theory of the planetary latitudes where the planes of the deferents and of the epicycles are inclined to each other.[3]

As we have seen all basic parameters of the lunar theory were derived from eclipse observations. What the lunar tables accordingly provide are positions of the moon in its orbit with respect to the nodes. The longitude of the moon should then be obtained by projecting the position M in the orbit onto the ecliptic (M′ in Fig. 77). If Ω denotes the longitude of the ascending node (which recedes uniformly at a rate of about $0;3,11^{\circ/d}$) and if λ_1 is the projection of the arc ω onto the ecliptic, we should find $\lambda_☾$ from

$$\lambda_☾=\Omega+\lambda_1=\Omega+\operatorname{arc}\tan(\tan\omega\cdot\cos i). \tag{1}$$

Ptolemy, however, simplifies his computations of longitudes by assuming[4]

$$\lambda_1\approx\omega \tag{2a}$$

i.e. by computing $\lambda_☾$ from

$$\lambda_☾=\Omega+\omega. \tag{2b}$$

Consequently he applies the same equation c to $\lambda_☾$ and to ω:

$$\lambda_☾=\bar{\lambda}+c \qquad \omega=\bar{\omega}+c \tag{3}$$

where c is found from the cinematical model in the orbital plane.[5]

The computation of latitudes $\beta_☾$, however, is based on the correct solution of the spherical triangle with hypotenuse ω, inclination $i=5°$, and $\beta_☾=\mathrm{MM}'$ from

$$\sin\beta_☾=\sin\omega\cdot\sin i, \tag{4}$$

of course using the Menelaos theorem as in the case of the solar declinations.[6] The deviations in the table Alm. V, 8 column 7 from modern values remain below 0;1°.

In all these cases Ptolemy reckons the argument ω' from the northernmost point N of the lunar orbit,[7] not from the node. Hence

$$\omega'=\omega-90 \tag{5}$$

which gives $\omega'=270$ for the ascending node, $\omega'=90$ for the descending one.

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[1] Observational data which supposedly confirm this round value for the extremal latitude of the moon are mentioned only later by Ptolemy (Alm. V, 12; cf. below p. 101).

[2] Cf., e.g., Almagest IV, 6 (Man. I, p. 218f.) or V, 2 (Man. I, p. 260ff.).

[3] Below I C 7.

[4] Alm. IV, 6 (Man. I, p. 219).

[5] The error thus committed reaches only about 0;6° for ω between about 30° and 60°.

[6] Above p. 30.

[7] Cf. above p. 80 and Fig. 72.

Example. Find longitude and latitude of the moon for Nab. 845 Mechir 18/19, 19 1/2^h after noon at Alexandria (= A.D. 98 Jan. 14, 7;30 a.m.).

The tables of mean motions (Alm. IV, 4) give for $\Delta t = 844^y\,5^m\,17^d\,19\,1/2^h$[8] the following results:

$$\begin{array}{lrrr} & \Delta\bar{\lambda} = 167;28 & \Delta\alpha = 31;\,5 & \Delta\bar{\omega}' = 293;55 \\ \text{Epoch:} & \bar{\lambda}_0 = ♉\ 11;22 & \alpha_0 = 268;49 & \bar{\omega}'_0 = 354;15 \\ \hline \text{mean moon:} & \bar{\lambda} = ♎\ 28;50 & \alpha = 299;54 & \bar{\omega}' = 288;10 \\ \multicolumn{2}{l}{\text{from Alm. IV, 10 the equation:}} & c = +4;\,8. & \end{array}$$

Thus for the true moon

$$\lambda = \bar{\lambda} + c = ♏\,2;58 \qquad \omega' = \bar{\omega}' + c = 292;18$$

and with the latter value from Alm. V, 8 by interpolation

$$\beta = +1;52°.$$

As a check for the value of $\bar{\lambda}$ we can also compute the elongation $\bar{\eta}$ and find

$$\bar{\eta} = \Delta\bar{\eta} + \bar{\eta}_0 = 207;16 + 70;37 = 277;53.$$

As longitude of the mean sun for the same date we have found before (p. 60) $\bar{\lambda}_\odot = ♑\,20;57$. Therefore

$$\bar{\lambda}_☾ = \bar{\lambda}_\odot + \bar{\eta} = ♑\,20;57 + 277;53 = ♎\,28;50$$

as by direct computation.

§ 4. Theory of the Moon. Second Inequality

1. Empirical Data and Ptolemy's Model

As we have seen in the preceding section the determination of the characteristic elements of the lunar theory had been derived from eclipse observations. In particular the Babylonian-Hipparchian values for the mean motions in longitude, anomaly, and latitude had nearly six centuries of eclipse records at their disposal. It is therefore not surprising that both Hipparchus and Ptolemy found good agreement between predicted and observed data for lunar eclipses.[1] In other words the simple epicyclic model which produces the "first" lunar inequality proved satisfactory as far as eclipses were concerned. But, as we know from the introduction to Chap. 2 of Book V of the Almagest, Hipparchus found serious deviations from the predicted longitudes for lunar positions which were not associated with syzygies, in particular near the quadratures. Apparently, however, these discrepancies were too irregular with respect to amount and location to lead Hipparchus to a consistent theoretical picture of the lunar motion.

[8] As we have shown on p. 64, the equation of time has only a negligible effect.

[1] Solar eclipses remain outside of these discussions because they depend also on geographical elements.

Ptolemy, being aware of these deviations, undertook a systematic study of the observational evidence and succeeded in correctly uncovering the pattern of these perturbations. Such a study must have involved a great deal of numerical work since he could not have reached his conclusions without computing in each single case the mean longitudes of sun and moon, the lunar anomaly and the effect of the longitudinal component of the lunar parallax.

The pattern which Ptolemy derived from the numerical data is expressed in the following experiences:

(1): no deviations (or at most deviations within the limits explicable by parallax) in the syzygies

(2a): no, or only small, deviations occur in the quadratures under the condition that the moon is simultaneously near the apogee or the perigee of the epicycle

(2b): maximum deviations are found in the quadratures when the moon is simultaneously near maximum equation caused by the epicyclic anomaly; the deviations are always increasing the effect of the epicyclic anomaly.

Ptolemy realized that exactly this type of discrepancies from the simple theory would result from an increase of the size of the epicycle when in quadrature as compared with its size at the syzygies, as previously determined from eclipses. A change in actual size would contradict the whole spirit of cinematical models of Greek astronomy but the same effect could be obtained by bringing the epicycle nearer to the observer when in quadrature, thus increasing its apparent size. Consequently Ptolemy gave the deferent of the lunar orbit an eccentric motion of its own, depending on the elongation from the sun. At conjunction and opposition the center C of the epicycle had to remain at the original distance $R=60$ from the observer O, whereas at elongations 90° and 270° the distance OC had to be reduced to a value required by the greatest observed increment of the epicyclic equation.

The cinematic device invented by Ptolemy to achieve such a periodic variation of distance consists in a crank mechanism (cf. Fig. 78) in which the leg OM of length e rotates backwards with the same angular velocity with respect to the direction observer → mean sun as the elongation $\bar{\eta}$ between the mean sun and the mean moon (i.e. the center C of the epicycle) increases. At the syzygies $\bar{\eta}$ is 0° or 180°, therefore $2\bar{\eta}=0$ and the distance $OC=R$. In the quadratures, however, $\bar{\eta}=90°$ or 270°, hence $2\bar{\eta}=180$ and the distance OC at its minimum $R-2e$, thus making the epicyclic anomaly as effective as possible. Obviously this type of motion accounts for all the above-mentioned empirical data (1), (2a), and (2b). There remains only the determination of the eccentricity e on the basis of properly selected observations.

The effect produced by this new model on the longitude of the moon, beyond the equation resulting from the simple epicyclic model, is known as the "*second lunar inequality*" and commonly identified with the periodic inequality known in modern celestial dynamics as "*evection*," to which Tycho Brahe added another term, the "*variation*," which reaches its maximum between the syzygies and the quadratures, being zero at both these points.[2] We shall see, however, that a similar term occurs also in Ptolemy's model of the lunar motion.[3]

[2] Cf. below p. 1109f.

[3] Below p. 88ff. For a comparison with modern theory cf. p. 1108.

A word is still to be said about Ptolemy's terminology. He calls the circle with center M and radius $R-e$ the "*eccenter*" and E its "*apogee*" (cf. Fig. 78). It should be noted that the center C of the epicycle does not rotate with constant angular velocity about the center M of the eccenter but progresses with mean velocity only as seen from O, since the angle COE increases as the "*double elongation*" $2\bar{\eta}$, i.e. twice as fast as the amount tabulated in the last column of the table of mean motions in Alm. IV, 4. It is interesting to find here in the lunar theory the roots of the concept of the "*equant*" which plays such an important role in Ptolemy's planetary theory,[4] that is, the idea that uniformity of motion may refer to a point different from the geometric center of a circular orbit.

The motion of the moon itself on the epicycle requires a special investigation[5] which has its parallels in planetary theory. It hardly can be doubted that it was the theory of the lunar motion where the modifications of the simple epicyclic motion originated. Exactly the same transfer of fundamental concepts from the lunar theory to the planetary models can be observed in the late Islamic and Copernican development. And the same is true, long before Ptolemy, in the mathematical astronomy of Mesopotamia where the lunar theory contains all the guiding principles for the computation of planetary phenomena.

2. Determination of the Parameters

Ptolemy derives the values for the characteristic parameters of his new model of the lunar motion from observational data which he presents without much further discussion. For the moment we shall follow him in this procedure in order not to obscure the method which leads from the observations to the determination of the basic features of the model. Only at the end[1] shall we return to the investigation of the details of the empirical elements.

A. Maximum Equation; Eccentricity

As we have seen[2] the maximum equation which results from the simple lunar theory is 5;1°. Obviously the first goal must be to find how much greater this maximum appears to be at quadratures of the moon. The corresponding observations should therefore satisfy as nearly as possible the following conditions:

(1) The observed lunar motion should equal the mean motion[3] because this indicates that the line of sight to the moon is tangential to the epicycle and hence that the equation is of maximum amount;

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[4] Below p. 155.

[5] Cf. below p. 88.

[1] Below, p. 91 ff.

[2] Cf. p. 80.

[3] In his discussion of two observations (below p. 87) Ptolemy does not make use of this criterium but simply computes the epicyclic anomaly α for the given dates and finds them near $\pm 90°$. I do not see the practical advantage of the formulation (1) over the direct computation of α.

(2) the elongation from the sun should be near $\pm 90°$ (quadrature);
(3) the longitudinal component of the lunar parallax should be zero.

The last condition is very essential since one does not know yet how much nearer to the observer the moon will be at elongations $\bar{\eta} = \pm 90°$ in comparison with the distance at $\bar{\eta} = 0$. Hence one does not know how much of the observed longitude is due to parallax and it is therefore necessary to select observations near the highest point of the ecliptic where a change in geometric distance influences only the observed latitude but not the longitude of the moon.

Ptolemy selects, obviously from a larger number of observations, two typical examples which very nearly satisfy these conditions.

(I). Observation by Ptolemy in the year Antoninus 2, Phamenoth 25,[4] 5 1/4^h before noon, Alexandria. Observed: $\lambda_\odot = ♒18;50$ $\lambda_☾ = ♏9;40$. For this moment one finds ♐4 culminating, thus the moon $24;20° \approx 1\ 1/2^h$ west of the meridian. The tables for the angles between ecliptic and circles of altitude (Alm. II, 13[5]) show for this time and longitude an angle very near to 90°; thus the longitudinal parallax is zero. From the tables of mean motions (Alm. V, 4) Ptolemy finds the anomaly $\alpha = 87;19$ thus a position of the moon on the epicycle near maximum equation. Similarly he finds for the mean longitudes $\bar{\lambda}_\odot = ♒16;27$ $\bar{\lambda}_☾ = ♏17;20$ thus a mean elongation $\bar{\eta} \approx 90°$.

From these elements it follows that the first inequality (Alm. IV, 10) amounts to about $-5°$. Thus the expected true longitude of the moon would be about ♏12;20 instead of the observed ♏9;40. This shows that the maximum of the first inequality, found from the simple lunar theory, has to be modified by an amount of about $-2;40°$, called the "*second inequality*."

(II). A similar result for the other half of the epicycle is derived from a Hipparchian observation. Its date is[6] year 51 of the third Callippic period, Epiphi 16,[7] 2/3 of the first (seasonal) hour of the day at Rhodes. Observed: the sun in ♌8 1/2 1/12, the moon in ♉12 1/3. Since ♉9 is culminating the longitudinal parallax is negligible. As mean longitudes are found $\bar{\lambda}_\odot = ♌10;27$ $\bar{\lambda}_☾ = ♉4;25$, hence an elongation $\bar{\eta} \approx 96°$. Finally the anomaly is $\alpha = 257;47$, causing a first inequality of about $+5°$ (Alm. IV, 10). Hence the true moon should have been, according to the simple model, in ♉9;25 instead of in ♉12;20. This would mean a second inequality of about 3°. But this value has to be modified slightly because Hipparchus' placing the sun in ♌8;35 does not quite agree with the solar theory which gives for the true sun at the given moment only the longitude ♌8;20. Thus the computed true elongation is ♌8;20 − ♉9;25 = 88;55° whereas Hipparchus observed ♌8;35 − ♉12;20 = 86;15°. In other words the moon appeared 2;40° nearer to the sun than predicted on the basis of the first inequality. Hence the maximum amount of the second inequality has again been found to be 2;40°.

On the basis of these observations it is now easy to determine the eccentricity $e = \mathrm{OM}$ of the deferent (cf. Fig. 78, p. 1229, and Fig. 79). In the quadratures the double elongation is 180° and OMC form a straight line; therefore $\mathrm{OC} = R - 2e$. In the two observations the equation c has its maximum value, found to be

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[4] A.D. 139 Febr. 9.

[5] Cf. above p. 50.

[6] For some textual difficulties cf. below p. 92.

[7] −127 Aug. 5.

$5+2;40=7;40°$. Hence

$$R-2e=\frac{r}{\sin c} \tag{1}$$

with $r=5;15$ for the previously determined radius of the epicycle[8] ($R=60$). Substitution of these values in (1) leads to $R-2e=39;22$ hence to

$$e=10;19 \tag{2}$$

for the eccentricity OM. The radius of the eccenter is therefore

$$R-e=49;41 \qquad (R=60).^{9} \tag{3}$$

These results lead to very unpleasant consequences for the geocentric distances of the moon. Obviously the maximum distance is $R+r=65;15$, the minimum $R-2e-r=34;7$. These extrema have almost the ratio 2:1 which should be reflected in almost the same range for the apparent diameter of the moon and for the lunar parallax. This is in flagrant contradiction to directly observable facts and shows that Ptolemy's model must contain some incorrect elements. Nevertheless the longitudes are so well represented by the new theory that it was not replaced by another cinematic model (epicycle-epicycles) before the late Islamic period[10] and then again by Copernicus.[11] Ptolemy himself never mentions this difficulty although he cannot have overlooked it.

B. "*Inclination*"

The motion of the moon on the epicycle must progress, in the mean, with an amount tabulated as anomaly in the tables of mean motions in Almagest IV, 4. In the simple lunar theory this angle $\bar{\alpha}$ of the anomaly was reckoned from the apogee of the epicycle. But the apogee is no longer uniquely defined in the new model (Fig. 78) in which one could consider as "apogee" either the point in the continuation of the direction OC or in the direction MC. Ptolemy shows (in Alm. V, 5) that neither one of these two possibilities agrees with the empirical data but that mean anomaly $\bar{\alpha}$ must be reckoned from a point $\bar{\text{A}}$ on the epicycle (henceforth called the "*mean apogee*" of the epicycle) such that the direction $\bar{\text{A}}$C always points to a point B, located on the circle of radius e around O, diametrically opposite to the center M of the eccenter (cf. Fig. 80). In other words also the "mean motion" of the moon on its epicycle does not proceed with respect to a fixed direction but varies with the elongation. The variable direction $\bar{\text{A}}$CB of the line of reference for the mean anomaly $\bar{\alpha}$ is called the "*inclination*" (πρόσνευσις)[12] of the lunar epicycle.

It is obvious from Fig. 80 that the difference between the "mean apogee" $\bar{\text{A}}$ and the "*true apogee*" A is particularly noticeable when the double elongation $2\bar{\eta}$ is about 90°, i.e., in the octants of the lunar orbit with respect to the sun. For

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[8] Above p. 76.

[9] In the Canobic Inscription (below p. 903) the parameters r and e are renormed such that $R-e$ obtains the value 60.

[10] Thirteenth and fourteenth century; cf. Roberts [1957].

[11] Copernicus, De Revol. IV, 3, IV, 8, IV, 9. Cf. also Neugebauer [1968, 2].

[12] The same term also occurs in the theory of eclipses (below p. 141) but with totally different meaning.

$2\bar{\eta}=0°$ or $180°$, however, A and $\bar{A}$ coincide; hence, the effect of the inclination vanishes in the syzygies and in the quadratures.

I think one can recognize here a systematic trend in the development of the lunar theory. First the eclipses serve for the determination of the mean motions and of the radius of the epicycle. Then the quadratures lead to the discovery and numerical evaluation of the major component in the second inequality. Finally the octants show the need for a modification of the law of motion for the moon on the epicycle itself. For all three groups Ptolemy adduces observations by Hipparchus, and it is clear that observations of this type must have convinced Hipparchus that the simple lunar theory was not satisfactory outside of the syzygies. Yet first Ptolemy succeeded in the discovery of the dependence of all observed discrepancies on the elongation from the sun. That indeed perturbations by the sun are their cause could not become clear before Newton's theory of gravitation.

In order to find out which point on the circumference of the epicycle should serve as zero point for the mean anomaly Ptolemy discusses two observations made by Hipparchus in the year -126 (incidentally: this is the last year from which observations by Hipparchus are recorded). These observations are selected such that the amount $|\bar{\eta}|$ of the elongation is near 45° whereas the mean anomaly is close to 0° or 180°. The latter condition implies that the moon is near apogee or perigee of the epicycle, a situation which allows most easily to conclude from the observed position P of the moon the position of the point $\bar{A}$ from which the mean anomaly $\bar{\alpha}$ has to be counted (cf. Figs. 81 and 82 which give a schematic representation of the two Hipparchian observations).[13]

Both observations were made in Rhodes in the year 197 after the death of Alexander, the first (I) in Pharmouthi 11 at the beginning of the second seasonal hour of the day, the second (II) in Payni 17,[14] 9 1/3 seasonal hours of the day. Hipparchus[15] found:

	I	II
$\lambda_{\odot}$:	♈ 7 1/2 1/4	♋ 11 − 1/10
$\lambda_{☾}$:	♓ 21 2/3	♌ 29;0
corrected for parallax:	♓ 21 1/3 1/8	same.

Using his tables Ptolemy found for the given moment[16]

	I	II
$\bar{\lambda}_{\odot}$:	♈ 6;41	♋ 12; 5
$\lambda_{\odot}$:	♈ 7;45	♋ 10;40
$\bar{\lambda}_{☾}$:	♓ 22;13	♌ 27;20
$\bar{\alpha}$:	185;30	333;12.

These data suffice to show that $\bar{\alpha}=0°$ can be neither A nor in the continuation of MC. In case I the mean moon, and therefore C and A, have the longitude

[13] Angles are drawn nearly to scale but the eccentricity, and particularly the radius of the epicycle, are exaggerated.

[14] -126 May 2 and July 7, respectively.

[15] Cf. below p. 92.

[16] Including the equation of time.

♓22;13 whereas the true moon P is in ♓21;40. Consequently the arc AP on the epicycle is $<180°$ (cf. Fig. 81) whereas $\bar{\alpha}=185;30°$. Hence $\bar{\mathrm{A}}$ and P cannot be located on the same side of the diameter ΠA. In a similar fashion one can exclude A also in case II. Since $\lambda(\mathrm{A})=\lambda(\mathrm{C})=\bar{\lambda}_{☾}=$ ♌27;20° and $\lambda_{☾}=$ ♌29° the arc from A to P must be much nearer to 360° than $\bar{\alpha}=333;12°$ (cf. Fig. 82).

It is now easy to verify these facts numerically. The double elongations $2|\bar{\eta}|$ are

$$\text{for I:}\quad 88;56° \qquad \text{for II:}\quad 90;30°. \tag{1}$$

With $2|\bar{\eta}|$, e, and $R-e$ known one can find in the triangle OMC the distance $\rho=\mathrm{OC}$

$$\text{for I:}\quad \rho=48;48 \qquad \text{for II:}\quad \rho=48;31 \qquad (R=60). \tag{2}$$

In order to determine the position of $\bar{\mathrm{A}}$ one has to find the difference γ of the epicyclic anomaly between the moon P and the nearest perigee (Π) or apogee (A), respectively. Thus we must solve the triangle OCP in which the sides $\rho=\mathrm{OC}$ and $r=\mathrm{CP}$ are known and the equation c can be determined as follows. As secure observational elements Ptolemy considers the true elongations

$$\eta=\lambda_{☾}-\lambda_{\odot}=\begin{cases} ♓21;27-♉\ 7;45=-46;18° & \text{for I}\\ ♌29\quad -♋10;54=\ \ 48;\ 6 & \text{for II.}\end{cases}$$

From his own tables follows

$$\eta'=\bar{\lambda}_{☾}-\lambda_{\odot}=\begin{cases} ♓22;13-♉\ 7;45=-45;32 & \text{for I}\\ ♌27;20-♋10;40=\ \ 46;40 & \text{for II;}\end{cases}$$

hence for the equation

$$c=\lambda_{☾}-\bar{\lambda}_{☾}=\eta-\eta'=\begin{cases} -0;46° & \text{for I}\\ +1;26 & \text{for II.}\end{cases}$$

With c determined we can now find γ in the triangle OCP:

$$\text{for I:}\quad \gamma=6;21° \qquad \text{for II:}\quad \gamma=14;43°.$$

This fixes the position of P on the epicycle and therefore the angle $\mathrm{AC\bar{A}}$:

$$\begin{aligned}&\text{For I: } \mathrm{AC\bar{A}}=\gamma+\bar{\alpha}-180=6;21+185;30-180=11;51°\\ &\text{for II: } \mathrm{AC\bar{A}}=360-(\gamma+\bar{\alpha})=360-14;43-333;12=12;5°.\end{aligned} \tag{3}$$

We see that the two cases lead to practically the same angle between A and $\bar{\mathrm{A}}$. In (2) we have found that the distances $\rho=\mathrm{OC}$ are almost identical. Consequently O will have the same distance from the straight line which continues the radius $\bar{\mathrm{A}}$C in both cases. Since the double elongations $2|\bar{\eta}|=\mathrm{MOC}$ are also almost equal (cf. (1)) it is natural to ask at what distance OB the continuation of MO meets the continuation of $\bar{\mathrm{A}}$C. In the triangle OCB we know the base $\mathrm{OC}=\rho$ and the two angles $180-2\bar{\eta}$ at O and ACA at C. Thus one finds

$$\text{for I: OB}=10;18 \qquad \text{for II: OB}=10;20$$

i.e. values nearly identical with the eccentricity $e=10;19$.

The agreement is close enough to justify the definition of the "inclination" of the epicycle by the direction $B\bar{C}\bar{A}$ such that B is always the second endpoint of the diameter MOB in the circle of radius $e = 10;19$.

C. Critical Remarks

In his discussion of the second lunar inequality and of the "inclination" of the epicycle Ptolemy cites two examples each of which confirm the value $e = 10;19$ for eccentricity of the deferent and the symmetric location of the points B and M (cf. Fig. 80, p. 1230). One of these observations was made by Ptolemy himself (A.D. 139 Febr. 9[17]), three by Hipparchus (-127 Aug. 5 and -126 May 2 and July 7[18]). Checking Ptolemy's numerical analysis of these observations (which essentially concern solar longitudes and elongations) one finds the usual small discrepancies, caused by unsystematic roundings in arithmetical operations and in the use of tables.

One slightly larger error occurred in the discussion of Hipparchus' last observation[19] where Ptolemy found $\bar{\lambda}_{☾} = ♌\,27;20$ instead of 27;7 and $\bar{\alpha} = 333;12$ for the mean anomaly instead of 333;1. The cause of these deviations is a simple oversight: Ptolemy determined correctly the equation of time as about $-1/3^h$ but forgot to apply this correction in the computation of mean motions.

Of astronomical interest is the question whether or not it is permissible to ignore the longitudinal component of the parallax. The answer to this question can be found in two ways.

First method: from the given solar longitude and from the time of observation one can find the culminating point M of the ecliptic. The difference of right ascensions $|\alpha(M) - \alpha(☾)|$ represents the distance of the moon from the meridian, thus $0;4 \cdot |\alpha(M) - \alpha(☾)|$ the same distance in equinoctial hours. With this time difference and with the longitude of the moon as arguments one enters the tables in Alm. II, 13 which give the angle between the ecliptic at the moon's place and the circle of altitude.[20] If this angle is sufficiently near to 90° the horizontal component of the parallax will be negligible.

This procedure (obviously followed by Ptolemy) has, however, two disadvantages. First, the tables Alm. II, 13 are only constructed for integer hours, for zero-points of zodiacal signs, and for greatly different geographical latitudes. Thus they require usually double, if not triple, interpolation. Secondly, it is difficult to estimate the effect which small changes in the given data would cause in the critical angle. Hence it is for the present problem more convenient to follow another procedure.

Second method: find, as before, the culminating point M of the ecliptic and from it the rising point H.[21] Consequently the highest point of the ecliptic has the longitude $\lambda(H) - 90$. At that point the longitudinal component of the parallax is exactly zero because the tangent to the ecliptic is parallel to the horizon. Conse-

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17 Above p. 87.

18 Above p. 87 and p. 89.

19 Above p. 89, No. II.

20 Above p. 48 ff.

21 From $\rho(H) = \alpha(M) + 90$; cf. p. 42.

quently one has only to compare the longitude $\lambda_{☾}$ of the moon with $\lambda(\mathrm{H})-90$ to be able to estimate whether or not the longitudinal parallax need be computed or not and in what sense changes in $\lambda_{☾}$ will influence the result.

Applying this procedure to the four observations recorded by Ptolemy[22] we obtain the following results:

Observation	M	H	H−90	$\lambda_{☾}$	$\lvert\Delta\rvert$
Ptol. +139 Febr. 9	♐ 4;35	♒21;21	♏21;21	♏ 9;40	11;41°
Hipp. −127 Aug. 5	♉11; 8	♌17;55	♉17;55	♉12;20	5;35
Hipp. −126 May 2	♒ 8;55	♉29;21	♒29;21	♓21;40	22;19
Hipp. −126 July 7	♍10;30	♏28;49	♌28;49	♌29; 0	0;11

It is immediately clear from this list that the longitudinal parallax will be negligible in the second and fourth case. The third case clearly shows too large a distance of the moon and indeed Hipparchus corrected the observed longitude by the amount of 21;40−21;27,30=0;12,30°.[23] This is not far from the value (0;15,53°) one finds if one computes with tables in the Almagest[24], of course assuming the simple lunar theory for the geocentric distance of the perigee of the epicycle.[25]

The first case shows a value of Δ about one-half of the value in the third case; it does not sound convincing when Ptolemy declares the longitudinal parallax to be negligible. Indeed, computation with the tables in the Almagest shows, assuming the simple model, a total parallax of about 0;46°, a western angle of about 80;33°,[26] hence a longitudinal component of about 0;8°. Since, according to the modified model, the lunar epicycle is for the given observation at its smallest geocentric distance (the amount being in principle still unknown) one should allow for a correspondingly greater parallax (perhaps near 0;20° in longitude) which would by no means be negligible for the evaluation of the maximum difference between the simple theory and observations in the case of quadratures.[27] Also in the observation of the solar longitude Ptolemy accepts ♒18;50° although his theory would lead to only ♒18;43°.

In the case of the first Hipparchian observation one meets textual difficulties. All manuscripts give year 50 of the third Callippic period but the longitude of the moon requires the emendation of 50 to 51.[28] More difficult to explain is the quotation from Hipparchus[29] *δρόμος … ἦν σμα'*. The term *δρόμος* must mean "motion in anomaly" but the tables for mean motions in the Almagest give 257;47° and not 241°. I do not see any plausible emendation of this error. Also the culminating point of the ecliptic, given by Ptolemy as ♉9°, is not accurately

[22] Above pp. 87, 89.

[23] Above p. 89.

[24] Alm. III, 13 for the angle between ecliptic and altitude circle, Alm. V, 18 for parallax.

[25] For the Hipparchian theory of parallax cf. below I E 5, 3.

[26] Ptolemy's rounded values ($\lambda_{☾}$=♏10°, 1;30^h west) would give an angle of 83°.

[27] Also numerically Ptolemy's data are not very accurate. With M=♐4 (actually ♐4;35) and $\lambda_{☾}$=♏9;40 the moon is not 1 1/2^h west of the meridian but 1;39^h (or 1;42^h).

[28] Ideler, Astron. Beob., p. 217 and Chron., p. 345. Also Ginzel, Hdb. II, p. 410.

[29] Heiberg, p. 363, 18f.

determined; actually one finds M = ♉ 11;8°. Finally the equation of time should be $0;16^h$ instead of Ptolemy's $0;5^h$.

In view of these facts one can consider the maximum effect of the second lunar inequality only approximately evaluated, at least on the basis of the two examples of observations adduced by Ptolemy.

The two remaining observations which concern the octants of the lunar elongation are affected only by some insignificant deviations without influence on the parameters of the model.

3. Computation of the Second Inequality; Tables

As in the case of the simple lunar theory one has now to find for a given moment t the equation c which leads from the mean longitude $\bar{\lambda}_{☾}$ to the true longitude $\lambda_{☾} = \bar{\lambda}_{☾} + c$. With all characteristic parameters of the model known (cf. Fig. 83), i.e.

$$r = 5;15 \qquad R - e = 49;41 \qquad e = 10;19$$

and with $\bar{\lambda}_{☾}$, $\bar{\alpha}$, $\bar{\eta} = \bar{\lambda}_{☾} - \bar{\lambda}_{\odot}$ found from the tables (Alm. IV, 4) for the given t one can readily compute the angle c under which the radius CP of the epicycle appears from the observer at O. First one finds in the triangle OMC the geocentric distance $\rho = \text{OC}$ of the center C of the lunar epicycle. With ρ known one can find in the triangle OBC the angle θ at C, i.e. the difference in anomaly between the "true apogee" A and the "mean apogee" $\bar{\text{A}}$ from which the mean anomaly $\bar{\alpha}$ is to be counted. With the "inclination" θ of the epicycle we also know the anomaly $\alpha = \theta + \bar{\alpha}$ of the moon P with respect to the true apogee A. Making use once more of the distance ρ one can solve the triangle OCP and determine the angle c between mean and true moon.

This whole procedure presents no theoretical difficulty and provides us with the exact equation as it follows from the accepted model. Ptolemy carries out these computations in a numerical example for which he chooses the date of the last Hipparchian observation.[1] The result agrees, as to be expected, with the equation $c = +1;26°$ deduced previously from Hipparchus' observations.[2]

For practical computing of lunar longitudes and latitudes this procedure is by far too long-winded. Ptolemy therefore constructed tables (Alm. V, 8) which permit us to find a sufficiently accurate solution in a few simple steps. The main idea for this procedure consists in tabulating the equation as function of the true anomaly for the two extremal distances of the epicycle, i.e. for $\rho = R = 60$ (syzygies) as well as for $\rho = R - 2e = 39;22$ (quadratures), and to use a proper interpolation device for intermediate elongations.

The first two columns of the tables in V, 8 are given to the arguments from 6° to 180° and from 354° to 180° respectively, in 6° steps up to 90° (and 270°), in 3° steps for the rest.

Column 3 gives the absolute values c_3 of the angle θ between mean and true apogee ($\bar{\text{A}}$ and A, respectively, in Fig. 83) as function of the double elongation $2\bar{\eta}$.

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[1] Above p. 89 (−126 July 7).

[2] Cf. p. 90.

The true anomaly α is then given by

$$\alpha=\bar{\alpha}\begin{cases}+c_3 & \text{for} \quad 0\leqq 2\bar{\eta}\leqq 180\\ -c_3 & \text{for} \quad 180\leqq 2\bar{\eta}\leqq 360\end{cases} \tag{1}$$

(cf. Fig. 84 and Fig. 85).

The value of α obtained from (1) is the argument for the values $c_4(\alpha)$ and $c_5(\alpha)$ tabulated in the fourth and fifth columns. Column 4 gives the equation $c_4(\alpha)$ under the assumption that the epicycle is at maximum distance from O, i.e. $2\bar{\eta}=0°$ (syzygies). Consequently the values given in this column are identical with the values of the first inequality tabulated in Alm. IV, 10.[3] Column 5 assumes minimum distance OC, i.e. $2\bar{\eta}=180°$ (quadratures). But $c_5(\alpha)$ is not the equation itself but only increment of the equation $c(\alpha)$ caused by the approach of the center of the epicycle from maximum to minimum distance. As maximum equation at minimum distance ($\rho=R-2e=39;22$) Ptolemy found the value 7;40°.[4] Since the maximum equation at maximum distance was 5;1[5] Ptolemy takes as the maximum value of the increment the difference 7;40 − 5;1 = 2;39°, in good agreement with the empirical data.[6] In a similar fashion $c_5(\alpha)$ can be computed for any value of α, representing the increment of the equation at the given anomaly α at minimum distance over the corresponding equation $c_4(\alpha)$ at maximum distance (for the result see Fig. 85).

There remain the positions between syzygies and quadratures. For such elongations one enters again column 1 or 2 with the double elongation $2\bar{\eta}$ as argument and finds in column 6 a coefficient $c_6(2\bar{\eta})$ which is zero at $2\bar{\eta}=0°$ and 1 at $2\bar{\eta}=180°$, increasing in a sinusoidal fashion from 0 to 1 (cf. Fig. 85). With this coefficient of interpolation one finds

$$c(2\bar{\eta},\alpha)=c_4(\alpha)+c_6(2\bar{\eta})\cdot c_5(\alpha) \tag{2}$$

as the final equation at elongation $\bar{\eta}$ and true anomaly α. Hence

$$\lambda=\bar{\lambda}+c \qquad \omega'=\bar{\omega}'+c$$

for the true longitude and for the argument of latitude. With the latter value as argument one finds in column 7 the latitude of the moon.[7]

The computation of the coefficients c_6 requires some discussion. Let $c_{\max}(2\bar{\eta})$ be the maximum equation for an intermediate position of the epicycle (at mean elongation $\bar{\eta}$), while $c_{4,\max}=5;1°$ is the maximum equation at syzygy, $c_{4,\max}+c_{5,\max}=5;1+2;39=7;40°$ is the maximum equation at quadrature. The coefficient $c_6(2\bar{\eta})$ is then defined as the ratio

$$c_6(2\bar{\eta})=\frac{c_{\max}(2\bar{\eta})-c_{4,\max}}{c_{5,\max}}. \tag{3}$$

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[3] Cf. p. 80.

[4] Cf. above p. 88.

[5] Cf. p. 80.

[6] Above p. 87f. It is, however, not quite correct to subtract the first maximum equation from the second because they do not belong to the same value of α. At the syzygies the maximum occurs near $\alpha=96°$, at the quadratures near 102°.

[7] Cf. p. 83.

As an example Ptolemy computes c_6 for the mean elongation $\bar{\eta}=60°$. For this configuration (cf. Fig. 86) he finds in the triangle OMC the distance $\rho=\mathrm{OC}=$ 43;43 and from it and $r=5;15$ the maximum equation $c_{\max}(120)=6;54°$. Hence

$$c_6(120)=\frac{6;54-5;1}{2;39}=0;42,38.$$

This is indeed the number given in column 6 of the table Alm. V, 8 for the argument $2\bar{\eta}=120°$.[8]

The use of (2) for the computation of the equation $c(2\bar{\eta},\alpha)$ for any elongation $\bar{\eta}$ and anomaly α is only a convenient approximation. Actually one should have coefficients $c_6'(2\bar{\eta},\alpha)$ which depend both on anomaly and elongation such that

$$c(2\bar{\eta},\alpha)=c_4(\alpha)+c_6'(2\bar{\eta},\alpha)\cdot c_5(\alpha). \tag{4}$$

Hence one should compute c_6' from

$$c_6'(2\bar{\eta},\alpha)=\frac{c(2\bar{\eta},\alpha)-c_4(\alpha)}{(c_4(\alpha)+c_5(\alpha))-c_4(\alpha)}. \tag{5}$$

If we approximate the small angle c, c_4, c_5 by their respective sines we would have (cf. Fig. 87):

$$\begin{aligned} c(2\bar{\eta},\alpha)&\approx\frac{r}{\rho}\sin\big(\alpha-c(2\bar{\eta},\alpha)\big)\\ c_4(\alpha)&\approx\frac{r}{R}\sin\big(\alpha-c_4(\alpha)\big)\\ c_4(\alpha)+c_5(\alpha)&\approx\frac{r}{R-2e}\sin\big(\alpha-(c_4(\alpha)+c_5(\alpha))\big). \end{aligned} \tag{5a}$$

If we apply to Ptolemy's definition

$$c_6(2\bar{\eta},\alpha)=\frac{c_{\max}(2\bar{\eta})-c_{4,\max}}{(c_{4,\max}+c_{5,\max})-c_{4,\max}}$$

the same approximation of the equations by their respective sines we have

$$c_{\max}(2\bar{\eta})\approx\frac{r}{\rho}\qquad c_{4,\max}\approx\frac{r}{R}\qquad c_{5,\max}\approx\frac{r}{R-2e}. \tag{6a}$$

If we substitute (5a) in (5) we will obtain the same result as from the substitution of (6a) in (6) if we ignore in (5a) the small increments from $c_4(\alpha)$ to $c(2\bar{\eta},\alpha)$ and to $c_4(\alpha)+c_5(\alpha)$. Allowing for such a small error one obtains coefficients c_6 which are independent of α, hence one avoids the construction of tables with double interpolation. In short one can formulate the underlying approximation as the assumption that the equations for any anomaly α at variable elongations (or geocentric distances) show the same ratios as the corresponding maximum equations.[9]

[8] The differences for the tabulated values c_6 (Alm. V, 8) show several constant stretches. This indicates linear interpolation between values accurately computed for greater intervals than 3° or 6°.

[9] It is again only approximately correct to deal with the maximum equations as if they belonged to the same epicyclic anomaly, independent of the elongation.

Example. Find longitude and latitude of the moon for Nab. 845 Mechir 18/19, 19 1/2^h after noon, Alexandria.[10]

For this moment one obtains from Alm. IV, 4 the following mean positions[11]

$$\bar{\lambda} = ♎ 28;50 \quad \bar{\alpha} = 299;54 \quad \bar{\omega}' = 288;10 \quad \bar{\eta} = 277;53.$$

Using the tables Alm. V, 8 one finds with $2\bar{\eta} = 195;46$ for the inclination of the epicycle

$$c_3(2\bar{\eta}) = -5;22°$$

and hence for the true anomaly

$$\alpha = \bar{\alpha} + c_3 = 294;32°.$$

The corresponding equations are

$$c_4(\alpha) = 4;23° \quad c_5(\alpha) = 2;10°.$$

Again with the argument $2\bar{\eta} = 195;46$ one finds as coefficient of interpolation

$$c_6(2\bar{\eta}) = 0;58,25$$

and with it the final equation

$$c = c_4 + c_5\, c_6 = 4;23 + 2;6 = +6;29.$$

Hence the true longitude of the moon is

$$\lambda = \bar{\lambda} + c = ♎ 28;50 + 6;29 = ♏ 5;19$$

rounded by Ptolemy to ♏ 5 1/3.[12]

For the argument of latitude we obtain

$$\omega' = \bar{\omega}' + c = 288;10 + 6;29 = 294;39$$

and hence, in column 7, the latitude[13]

$$\beta = +2;5°.$$

Using the simple lunar theory we had found[14]

$$\lambda = ♏ 2;58 \quad \beta = +1;52°$$

respectively. The increase in longitude by 2;21° caused by the second inequality is not surprising since the mean moon is only about 8° distant from quadrature.

Fig. 88 will give an impression of the lunar orbit according to the new model. The excessive flattening of the orbit is very outspoken. In the simple theory the distance of the moon can only deviate by $2r$ from the maximum distance at point 0.

Fig. 89 is a graphical representation of lunar longitudes and latitudes computed for the 16 days from −126 Apr. 26 to May 11 at 6 a.m. Alexandrian time.[15] Table 7 shows the underlying computations and the values obtained for the same

[10] A.D. 98 Jan. 14.

[11] Cf. p. 84.

[12] Alm. VII, 3 (Heib. II, p. 33, 19).

[13] Ptolemy 2;10° as Heiberg and Manitius interpret the text (cf. below p. 117, note 7).

[14] Above p. 84.

[15] The seventh day, marked by H, corresponds to Hipparchus' observation of May 2 (above p. 89). Our computation includes the equation of time, using Ptolemy's approximation −0;20^h for the whole interval. Actually it would change from −0;21^h in No. 1 to −0;24^h in No. 16.

Table 7

No.	Date: Nab.	Date: julian	from Alm. IV, 4: $\bar{\lambda}$	$\bar{\alpha}$	$\bar{\omega}'$	$2\bar{\eta}$	from Alm. V, 8: $c_3(2\bar{\eta})$	α	$c_4(\alpha)$	$c_5(\alpha)$	$c_6(2\bar{\eta})$	$c_5 \cdot c_6$	c	λ: Alm.	mod.	Alm. − mod.	β: ω'	Alm.	mod.	Alm. − mod.	No.
	621	−126																			
1	VIII 5	Apr. 26	273; 7°	107; 5°	344; 9°	124;50°	+12;54°	119;59°	−4;31°	−2;32°	0;44,55	−1;54°	−6;25°	266;42°	266;53°	−0;11°	337;44°	+4;37°	+4;50°	−0;13°	1
2	6	27	286;18	120; 9	357;23	149;13	+ 9;32	129;41	−4; 5	−2;19	54,40	−2; 7	−6;12	280; 6	280; 8	−0; 2	351;11	+4;56	+5;10	−0;14	2
3	7	28	299;28	133;13	10;37	173;36	+ 2;19	135;32	−3;44	−2;10	59,36	−2; 9	−5;53	293;35	293;40	−0; 5	4;42	+4;58	+5;14	−0;16	3
4	8	29	312;39	146;17	23;50	197;59	− 6; 3	140;14	−3;26	−2; 0	57,55	−1;56	−5;22	307;17	307;28	−0;11	18;28	+4;44	+5; 0	−0;16	4
5	9	30	325;49	159;21	37; 4	222;22	−11;32	147;49	−2;53	−1;41	50,36	−1;25	−4;18	321;31	321;31	0	32;46	+4;12	+4;27	−0;15	5
6	10	May 1	339; 0	172;25	50;18	246;45	−13; 8	159;17	−1;55	−1;10	0;39, 3	−0;46	−2;41	336;19	335;49	+0;30	47;37	+3;21	+3;38	−0;17	6
H.7	11	2	352;10	185;28	63;32	271; 8	−11;54	173;34	−0;37	−0;22	26, 0	−0;10	−0;47	351;23	350;19	+1; 4	62;45	+2;17	+2;33	−0;16	7H.
8	12	3	5;21	198;32	76;45	295;31	− 9;10	189;22	+0;54	+0;32	14,25	+0; 8	+1; 2	6;23	4;58	+1;25	77;47	+1; 4	+1;20	−0;16	8
9	13	4	18;31	211;36	89;59	319;54	− 5;51	205;45	+2;22	+1;24	6,30	+0; 9	+2;31	21; 2	19;40	+1;22	92;30	−0;13	−0; 1	−0;14	9
10	14	5	31;42	224;40	103;13	344;16	− 2;19	222;21	+3;36	+2; 6	0,59	+0; 2	+3;38	35;20	34;19	+1; 1	106;51	−1;28	−1;20	−0; 8	10
11	15	6	44;52	237;44	116;27	8;39	+ 1;17	239; 1	+4;29	+2;31	0; 0,17	+0; 1	+4;30	49;22	48;48	+0;34	120;57	−2;34	−2;33	−0; 1	11
12	16	7	58; 3	250;48	129;40	33; 2	+ 4;49	255;37	+4;57	+2;39	3,58	+0;11	+5; 8	63;11	63; 2	+0; 9	134;48	−3;31	−3;36	+0; 5	12
13	17	8	71;14	263;51	142;54	57;25	+ 8;15	272; 6	+4;58	+2;34	11,33	+0;30	+5;28	76;42	76;56	−0;14	148;22	−4;15	−4;23	+0; 8	13
14	18	9	84;25	276;55	156; 8	81;48	+11;13	288; 8	+4;38	+2;18	22,22	+0;51	+5;29	89;54	90;27	−0;33	161;35	−4;44	−4;55	+0;11	14
15	19	10	97;35	289;59	169;22	106;11	+12;59	302;58	+4; 0	+1;59	35,16	+1;10	+5;10	102;45	103;34	−0;49	174;32	−4;58	−5;11	+0;13	15
16	20	11	110;46	303; 3	182;36	130;34	+12;26	315;29	+3;18	+1;40	0;47,33	+1;19	+4;37	115;23	116;19	−0;56	187;13	−4;57	−5;11	+0;14	16

moments from modern tables.[16] It is seen in this example that the Ptolemaic longitudes deviate from the modern ones by an amplitude of about $\pm 1°$ and a period of about half a draconitic month. The deviations in latitude are the consequence of the use of $\beta_{max} = 5°$ instead of about 5 1/3°.

The fact that Ptolemy's new lunar theory was highly successful in the prediction of lunar longitudes is the obvious reason for disregarding its glaring inadequacy to account for the smallness of the variations in apparent diameter and lunar parallax. The situation has a certain parallel in the development of Kepler's theory of Mars when he used one model to correctly predict the longitudes of the planet, another for its distances. Kepler did not rest until he was able to reconcile both aspects of the problem whereas Ptolemy's theory was accepted for centuries without any attempt to eliminate its deficiencies.

4. Syzygies

One might ask why Ptolemy discusses the second inequality before the theory of eclipses (in Book VI) since the refined lunar model coincides with the simple theory in the case of the syzygies. In fact, however, this agreement is only exact

[16] The entry "modern" needs some explanation. The Nautical Almanac Office of the Naval Observatory in Washington computed a lunar ephemeris, beginning with −300 Dec. 30 and ending at +189 July 14. These tables which were put at my disposal give the day by day longitudes (λ_N) and latitudes (β_N) of the moon for Greenwich, midnight, to hundredths of one degree. Comparison with the values (λ_T and β_T) found by Tuckerman for 7 p.m. Babylon (to one tenth of a degree) show that approximately

$$\lambda_T = \lambda_N - 2;50°. \tag{1}$$

Since the moment for which λ_T is computed precedes the time for λ_N by 8 hours one would expect values λ'_N satisfying roughly

$$\lambda_T = \lambda'_N - 4;23°. \tag{2}$$

Hence

$$\lambda'_N - \lambda_N = 1;33° \tag{3}$$

measures the deviation between the Nautical Almanac tables and the tables of Tuckerman which represent ancient data very satisfactorily. The reason for this apparent discrepancy lies in the fact that the aforementioned tables are computed for "ephemeris time," the latter for "universal time" (cf. below p. 1070).

In our example (Fig. 89 and Table 7) we wish to obtain longitudes λ of the moon for 6 a.m. Alexandria = 7 p.m. Babylon + 12^h. Assuming 6;35° for the lunar motion during 12^h we should find, because of (1)

$$\lambda = \lambda_T + 6;35° = \lambda_N + 3;45°. \tag{4}$$

A check with the longitudes λ explicitly computed in the Tuckerman tables (i.e. for Apr. 30, May 5, and May 10) shows that (4) is satisfied within an error of $\pm 0;15°$. Hence as "modern" longitudes are tabulated values λ obtained by means of $\lambda_N + 3;45°$. The results can be expected to represent within $\pm 0;15°$ the longitudes obtainable by Tuckerman's procedure, i.e. essentially by means of the parameters used in P. V. Neugebauer's tables.

For the "modern" values of β the following procedure has been used. Formula (4) shows that the "modern" longitudes λ for 6 a.m. Alexandria differ from the tabulated values λ_N by about 6^h of lunar motion. Consequently I added to the tabulated values β_N in each case $1/4\,\Delta\beta$. The results are tabulated as "modern" values of β and should represent a fair approximation of the Tuckerman values.

when the mean elongations are either 0° or 180° whereas eclipses require that the true elongations are 0° or 180°. Furthermore solar eclipses are greatly influenced by the lunar parallax, i.e. by the moon's geocentric distance which, in the revised lunar model, varies much more than according to the simple theory. Consequently the influence of the second inequality must be known before solar eclipses can be handled properly.

The chapter Alm. VI, 10 deals with the first problem, the influence of the second inequality on lunar eclipses. Since the radius of the epicycle as well as the mean motions and epoch values had all been determined from data furnished by lunar eclipses it is of primary importance for Ptolemy to show that the use of the simple lunar theory in the analysis of these eclipses did not introduce appreciable errors in comparison with the refined theory.

Ptolemy therefore investigates two extremal situations. In the first case the moon is assumed to be near quadrature of the epicyclic anomaly (cf. Fig. 90). In this position the addition or subtraction of a few degrees in anomaly does not change appreciably the equation; hence the effect of the inclination of the epicycle is negligible. We then assume that the mean elongation $\bar{\eta}$ is as great as possible. For an eclipse we know that $\lambda_{\odot} \equiv \lambda_{☾}$ (mod. 180). Hence $|\bar{\eta}| = |\bar{\lambda}_{☾} - \bar{\lambda}_{\odot}|$ will be a maximum when the equations are as great as possible[1] and of opposite signs, i.e.

$$\bar{\eta} = 5;1 + 2;23 = 7;24°.$$

Computing for this elongation the distance $\rho = \mathrm{OC}$ one finds that it only decreases to 59;36 as compared with $R = 60$ in the simple theory. The corresponding maximum equation is 5;3° or only 0;2° more than under the original assumptions.[2] The extremal error amounts therefore to less than $1/16^{h}$ of lunar motion.

In the second case the influence of the "inclination" is made as great as possible. This requires the moon to be located near the perigee Π of the epicycle (cf. Fig. 91). Hence the mean longitude $\bar{\lambda}_{☾}$ is practically the same as the true longitude $\lambda_{☾}$. Therefore the greatest possible elongation can only be due to the maximum equation of the sun, hence

$$\bar{\eta} = 2;23°.$$

The change in geocentric distance is practically negligible ($\rho = 59;58$) but the inclination of the epicycle is found to move the moon from Π to P by a distance 0;4 ($R = 60$). This displacement appears from O under an angle $c = 0;4°$. The corresponding lunar motion requires less than $1/8^{h}$.

Thus we have found that the maximum error in the equation of the moon at lunar eclipses is 0;2° at the quadratures of the epicyclic anomaly, and 0;4° at the perigee (of course less at the apogee). It is interesting to see that Ptolemy considers these errors as being within the limits of tolerance of observations or theoretical predictions, since these limits are generally set at about $\pm 0;10°$ in longitude. Consequently he has demonstrated to his satisfaction that lunar eclipses may be analyzed on the basis of the simple model for the lunar motion.

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[1] The maximum of $c_{☾} = 5;1$ (cf. p. 80), of $c_{\odot} = 2;23$ (cf. p. 59).

[2] Here, as well as in the next case, Ptolemy computes all corrections ab ovo from the given parameters of the model. Actually the tables in V, 8 give the same result.

§ 5. Parallax

1. Introduction

The concluding chapters (11 to 19) of Book V of the Almagest concern the determination of the parallax of the moon and of the sun, of the relative distances and sizes of these bodies and of the apparent diameters of sun, moon, and shadow as well as tables for the practical computation of parallaxes.

The (unfortunately very short) introductory chapter (11) contains almost all that is known about Hipparchus' approach to the problem[1] plus some general remarks, e.g. the statement that even from given parallaxes absolute distances are unobtainable. It seems to me clear from this remark that (rightly) neither Hipparchus nor Ptolemy had any confidence in the measurement of the circumference of the earth by Eratosthenes.[2]

In theoretical respect the problem of parallaxes is extremely simple: when the radius r_e of the earth is not negligible in comparison with the distance of a celestial object P (cf. Fig. 92) one must distinguish the center E of the earth from the position O of an observer. Up to now the coordinates of the celestial bodies were computed without any reference to the geographical coordinates of a specific observer, that is to say the celestial coordinates obtained in this way assume E as origin. If ζ is the zenith distance of P with reference to the point E an observer in O will find a zenith distance $\zeta' > \zeta$. The difference

$$p = \zeta' - \zeta \tag{1}$$

is the "*parallax*" of P. It also can be described as the angle under which the radius EO appears from P. Obviously this angle is non-negative and belongs to the plane of the circle of altitude of P. Assuming ζ and p known one can compute the ratio of $\mathrm{EO} = r_e$ and EP; in other words one can find the geocentric distance of P, measured in earth radii.

In contrast to the theoretical simplicity of the problem of parallax the practical difficulties of its measurement are very great indeed. The angular differences are so small that only lunar parallax could be determined with a reasonable degree of accuracy; we shall see in the following which factors misled the ancient and mediaeval astronomers in their evaluation of the solar parallax.

Chap. V, 12 of the Almagest gives us a glimpse of the instrumental difficulties of the seemingly simple problem of measuring a zenith distance. Ptolemy uses a vertical pole AB, about 4 cubits long, and a diopter BC of the same length pivoted at B (cf. Fig. 93). But instead of using a graded circular arc AC he use a straight ruler AC to measure the chord of ζ and finds ζ itself from the table of chords.[3]

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[1] Cf. below I E 5, 3 and I E 5, 4 B.

[2] The basic assumption made by Eratosthenes that Alexandria and Syene lie on the same meridian does not agree with Ptolemy's Geography IV, 15, 15 where Syene is placed 1/10 of one hour to the east of Alexandria, i.e. 1;30° in longitude (in agreement with Geogr. IV, 5, 9 and IV, 5, 73).

[3] Fig. 93 does not pretend to reconstruct the technical details of the instrument. This has been attempted by A. Rome [1927], certainly successfully in the main elements. The fact that the use of an arm AC of the same length as AB restricts the instrument to zenith distances $<60°$ might have to do, according to Rome, with an intentional avoidance of refraction. It remains difficult, however, to see how this instrument could have produced results of greater accuracy than direct readings on a quadrant.

Obviously he relies on linear divisions rather than on divisions of circles. Since AB is about 4 cubits long we may assume that 1/60th part corresponds to about 36 mm. The smallest unit conveniently shown would then be $0;5^p$, about 3 mm long.

The same distrust for the direct measurement of small angles appears also in Ptolemy's use of eclipses instead of the direct measurement of apparent diameters.[4] In fact, however, this method introduces new, and equally uncontrollable inaccuracies; consequently the whole treatment of parallaxes is one of the most unsatisfactory topics in the whole Almagest, still further aggravated by quite unnecessary trigonometric inaccuracies in the determination of the components of parallax.[5]

2. The Distance of the Moon

As a preliminary step Ptolemy motivates (in V, 12) the value $i=5°$ for the maximum latitude (β_{max}) of the moon. Supposedly this is the result of a measurement of the zenith distance ζ of the moon when it had a longitude near ♋0°, a northern latitude near β_{max} (as is always easy to conclude from the known motion of the nodal line), and was culminating at the meridian of Alexandria. For such a situation Ptolemy found $\zeta=2\ 1/8°$ which is a position so near to the zenith that one may ignore parallax. Consequently he finds (with $\varphi=30;58$ and $\varepsilon=23;51$) from

$$\varphi=\zeta+\beta_{max}+\varepsilon$$

the value

$$\beta_{max}=30;58-(2;7,30+23;51)\approx 5°.$$

This measurement, for which no further circumstances are given, does not look very real. Observation with Ptolemy's instrument should not have given ζ itself but only $AC=\mathrm{crd}\,\zeta$. Thus $\zeta=2\ 1/8$ would suppose the very unlikely reading $AC=2;13,20$. If one assumes as nearest direct reading $AC=2;15$ one would find $\zeta=2;9°$ which is nearer to $2\ 1/6°$ than to $2\ 1/8°$. Probably $\beta_{max}=5°$ is not more than a convenient round value[1] which was found to be in fair agreement with zenith distances observed near the zenith. In fact this result is about 0;18° too small, i.e. more than one lunar radius in error.

Accepting, however, $\beta_{max}=5°$ as a secure value Ptolemy determined the zenith distance of the culminating moon when near ♑0° and at maximum northern latitude.[2] These conditions were nearly satisfied for an observation of the moon in Alexandria, Hadrian 20, Athyr 13, $5;50^h$ after noon.[3] The accurate time interval of this moment from epoch Ptolemy found to be

$$\Delta\bar{t}=882^y\ 72^d\ 5;20^h$$

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[4] Below p. 104.

[5] Below p. 115.

[1] $\varphi=30;58°$ corresponds exactly to an equinoctial noon shadow 5:3, mentioned by Vitruvius (Arch. IX, 7). In the Geography Ptolemy gives $\varphi=31°$ (IV, 5, 9, p. 251, Nobbe); the same value is found in the Handy Tables (Halma I, p. 119). "Lower Egypt" has $\varphi=30;22°$ (Alm. II, 8). Actually Alexandria is at $\varphi=31;13°$.

[2] A maximal southern latitude would place the moon at a zenith distance of about 60°, thus at the limit of Ptolemy's instrument and perhaps too near to the horizon to ignore refraction (cf. above p. 100, note 3).

[3] A.D. 135 Oct. 1.

and from it the following mean values for the moon

$$\bar{\lambda} = ♐25;44 \qquad \bar{\alpha} = 262;20 \qquad \bar{\omega}' = 354;40 \qquad \bar{\eta} = 78;13$$

with $\bar{\lambda}_{\odot} = ♎7;31$ not far from quadrature. This fact is of interest because Ptolemy applies in the following the refined lunar theory which brings the moon at quadrature much too close to the earth. Using the tables in V, 8 he finds the equation $c = +7;26$, thus

$$\lambda_{☾} = ♑3;10 \qquad \omega' = 2;6 \quad \text{hence} \quad \beta = +4;59.$$

Ptolemy observed as chord of the zenith distance the value 51;35 to which corresponds the angle

$$\zeta' = 50;55°. \tag{1}$$

Without parallax the zenith distance would be given by

$$\zeta = \varphi - \delta - \beta$$

where δ is the declination of the ecliptic point $\lambda_{☾}$ (which must be close to ♑0° in order to allow us to reckon δ and β as arcs of the circle of altitude). With Ptolemy's values this would lead to

$$\zeta = 30;58 + 23;49 - 4;59 = 49;48° \tag{2}$$

(δ(♑3;10) being $-23;49$), hence to the parallax

$$p = \zeta' - \zeta = 50;55 - 49;48 = 1;7°. \tag{3}$$

Using these values in the configuration of Fig. 92 (p. 1235) Ptolemy finds[4] for the geocentric distance of the moon

$$\text{EP} = 39;45\, r_e. \tag{4}$$

With this distance known it is easy to also express all other dimensions in the model for the lunar motion in the same units. In the given configuration (cf. Fig. 94) we know that $\bar{\alpha} = 262;20$ and $2\bar{\eta} = 156;26$. The latter angle is, of course, computed with reference to the center E of the earth. Using $R = 60$, $e = 10;19$, $r = 5;15$ one can compute EP and finds the value 40;25. But we know from (4) that the same distance amounts to 39;45 r_e. Hence we have to multiply all dimensions of our model by the factor

$$\frac{39;45}{40;25} = 0;59 \tag{5}$$

if we wish to express them in earth radii. In this way one obtains

$$\begin{array}{lr} \text{radius of the epicycle} & r = 5;10\, r_e \\ \text{mean distance at syzygies} & R = 59\, r_e \\ \text{mean distance at quadratures} & R - 2e = 38;43\, r_e. \end{array} \tag{6}$$

Actually the mean distance of the moon is about 60.4 r_e.

It is interesting to compare the values (1) and (2) for the observed and for the computed zenith distance with modern data since all the dimensions (6) depend finally on the values of ζ' and ζ.

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[4] Accurate computation with the tables of chords (Alm. II, 11) leads to $\text{EP} = 39;49,31\, r_e \approx 39;50 r_e$ (modern tables: 39;49,48).

According to the tables of Tuckerman one has for the moon $\lambda \approx$ ♑ 5°, $\beta = +5;18°$ ($\lambda_{\odot} =$ ♎ 6;50). Computing the latitudinal parallax with the tables of P.V. Neugebauer[5] one finds the following data:

	modern	Ptolemy	Ptol.-mod.
Alexandria	$\varphi =$ 31;13°	30;58°	−0;15°
$\lambda_{☾} =$ ♑ 5	$\delta = -23;35$	23;49	+0;14
	$\varphi - \delta =$ 54;48	54;47	−0; 1
	$-\beta = -$ 5;18	− 4;59	+0;19
	$\zeta =$ 49;30	49;48	+0;18
	$-p_{\beta} =$ 0;46[6]	$-p =$ 1; 7	+0;21
	$\zeta' =$ 50;16	50;55	+0;39

This shows that the errors in φ, $\lambda_{☾}$, and δ compensate each other. The error in the inclination of the lunar orbit, however, amounts to almost 1/3° and the observed zenith distance is about 2/3° too great but these errors counteract each other such that the lunar parallax is only about 1/3° too great. Combining this error with the exaggerated approach of the moon according to the refined cinematic model produces finally an almost correct mean distance for the syzygies.

3. Apparent Diameter of the Moon and of the Sun

Since the successful prediction of eclipses depends on an accurate determination of the apparent diameters of the moon, the sun, and the earth's shadow it is not surprising to see that many attempts were made to obtain reliable data for these quantities. Ptolemy, in a short introduction to Alm. V, 14, mentions measurements with waterclocks (of the time of transits of the solar or lunar disk?) and observations of the rising time of sun and moon at the equinoctial points of the ecliptic.[1]

Ptolemy, rightly considers these methods as inaccurate. Instead of attempting to measure directly such small angles and their minute variations[2] he deduces these quantities from the analysis of eclipse observations. We shall see to what extent this procedure, though in principle sound, leads to incorrect results.[3]

By direct observations (which, however, do not require specific numerical results) Ptolemy convinced himself that

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[5] P. V. Neugebauer, Astron. Chron. I, p. 72f.

[6] For p_{λ} one finds about $-0;3,36°$ whereas Ptolemy assumes $p_{\lambda} = 0$.

[1] This condition simplifies the problem insofar as the angle is directly known under which the center of the disk crosses the horizon, namely $90 - \varphi$. The diameter is then given by $\Delta t \cos \varphi$ when Δt is the time required for the rising or setting of the whole disk (obviously an extremely ill-defined quantity).

A popular (and obviously meaningless) version of this procedure is the story that it takes the sun a 1/720th part of one day to cross the horizon, from which one concludes that the solar diameter is 1/2°. Cf., e.g., P. Oslo 73 (for the literature cf. Neugebauer [1962], No. 24); also Hultsch [1899], p. 193 (but misleading hypotheses).

[2] The instrumental problems are discussed by Hultsch [1897], [1899], [1900]; by Rome, Pappus Comm., p. 87ff.; by Lejeune, Euclide et Ptol., pp. 131, 151.

[3] Below p. 106.

(a) the apparent diameter of the sun is practically constant, i.e. the influence of the eccentricity of its orbit is negligible,

(b) the lunar diameter equals the solar diameter when the moon is at its maximum geocentric distance, whereas previous astronomers had assumed equality for the moon at mean distance.

The second statement implies that Ptolemy, in contrast to earlier astronomers (unfortunately not named[4]), denied the possibility of annular solar eclipses. He also mentions that the values obtained by himself for the diameters were considerably smaller than the traditionally accepted ones. Again no details are given.

A. Ptolemy's Procedure

Two pairs of lunar eclipses are used for the determination of the apparent diameter of the moon at greatest and at smallest distance at a syzygy (Alm. V, 14 and VI, 5, respectively).

The first pair consists of two eclipses, observed at Babylon, but reduced by Ptolemy to Alexandria:

I: Nabon. 127 Athyr 27/28 $4;45^h$ after midn. (-620 Apr. 22)

II: Nabon. 225 Pham. 17/18 $9;50^h$ after noon (-522 July 16/17).

For the mean anomalies Ptolemy finds

$$\bar{\alpha}_{\mathrm{I}} = 340;7 \qquad \bar{\alpha}_{\mathrm{II}} = 28;5$$

respectively, i.e. positions of the moon near the apogee of the epicycle. The true arguments of latitude are

$$\omega'_{\mathrm{I}} = \ 80;40^{5} \text{ i.e. } 9;20^\circ \text{ before the desc. node}$$

$$\omega'_{\mathrm{II}} = 262;12 \quad \text{i.e. } 7;48^\circ \text{ before the asc. node.}$$

From these nodal distances Ptolemy finds the distances b (perpendicular to the lunar orbit) from the center of the moon to the center of the shadow at the middle of the eclipse (cf. Fig. 95):

$$b_{\mathrm{I}} = 0;48,30^\circ \qquad b_{\mathrm{II}} = 0;40,40.$$

The eclipse magnitudes, according to the observations recorded, were $1/4\, d_{☾}$ and $1/2\, d_{☾}$, respectively. Hence an approach of the moon toward the center of the shadow by $b_{\mathrm{I}} - b_{\mathrm{II}} = 0;7,50^\circ$ corresponds to an increase of obscuration by $1/4\, d_{☾}$. Hence

$$d_{☾} = 4 \cdot 0;7,50 = 0;31,20^\circ. \tag{1}$$

The second eclipse shows that $b_{\mathrm{II}} = s$, the radius of the shadow. Thus

$$s = 0;40,40^\circ \approx 2\ 3/5\ r_{☾}{}^{6} \tag{2}$$

at the maximum distance of the moon.

[4] Proclus, Hypotyp. I, 19 Manitius, p. 10, 18) mentions the observation of annular eclipses by "earlier" astronomers. In IV, 98 (Manitius, p. 130, 18) Sosigenes (teacher of Alexander of Aphrodisias, thus before A.D. 200) is said to have observed one. The only eclipse possible is the one of A.D. 164 Sept. 4, annular for Greece, cf. Ginzel, Spez. Kanon pl. XI.

[5] Accurate computation, however, gives only 80;35, mainly because the equation of time amounts to $-0;20^h$ instead of Ptolemy's $-0;15^h$. Pappus in his commentary (Rome p. 102, 7) accepts Ptolemy's number without checking.

[6] From (1) it follows that $2\ 3/5\, r_{☾} = 0;40,44^\circ$.

A similar procedure is followed for the moon at the perigee of the epicycle (Alm. VI, 5). The eclipses are

III: Nabon. 574 Pham. 27/28 (−173 May 1)
IV: Nabon. 607 Tybi 2/3 (−140 Jan. 27).

The corresponding anomalies

$$\bar{\alpha}_{\mathrm{III}}=163;40 \qquad \bar{\alpha}_{\mathrm{IV}}=178;46$$

show that the moon is near the perigee of the epicycle, while

$$\omega'_{\mathrm{III}}=\ 98;20 \text{ i.e. } 8;20^\circ \text{ beyond the desc. node}$$
$$\omega'_{\mathrm{IV}}=280;36 \text{ i.e. } 10;36 \text{ beyond the asc. node}$$

lead to the following distances at the eclipse middle (cf. Fig. 96)

$$b_{\mathrm{III}}=0;43,3^\circ \qquad b_{\mathrm{IV}}=0;54,50^\circ.$$

The observed magnitudes were

$$m_{\mathrm{III}}=7 \text{ digits from N} \qquad m_{\mathrm{IV}}=3 \text{ digits from S}.$$

Hence the increment of obscuration is given by

$$(7/12-3/12)\,d_{\leftmoon}=1/3\,d_{\leftmoon}$$

resulting from an approach to the center of the shadow

$$b_{\mathrm{IV}}-b_{\mathrm{III}}=0;11,47^\circ.$$

Hence

$$d_{\leftmoon}=3\cdot 0;11,47\approx 0;35,20^\circ \tag{3}$$

at minimum distance for syzygies.

The diameter of the shadow can be deduced, e.g., from the eclipse IV (cf. Fig. 96):

$$b_{\mathrm{IV}}=s+(1/2-3/12)d_{\leftmoon}=s+1/4\,d_{\leftmoon}.$$

Hence one obtains for the radius of the shadow at minimum distance

$$s\approx 0;54,50-0;8,50=0;46^\circ. \tag{4}$$

Combining the results obtained for the maximum distance with the values found for the absolute dimensions of the lunar orbit Ptolemy computed[7] the radius r_m of the moon in units of the earth radius r_e.

For the maximum distance $R+r$ of the moon he had found[8]

$$\mathrm{EM}=59r_e+5;10r_e=64;10r_e.$$

At this distance the diameter d_m of the moon appears under the angle $d_{\leftmoon}=0;31,20^\circ$.[9] Hence (Fig. 97)

$$r_m=\frac{64;10}{120}r_e\cdot \mathrm{crd}\,0;31,20=0;32,5\cdot 0;32,49\,r_e=0;17,33\,r_e \tag{5}$$

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[7] Alm. V, 15.
[8] Above p. 102.
[9] Cf. (1), p. 104.

gives the absolute size of the moon in earth radii. This result is very near to the correct value ($\approx 0;16,20\, r_e$).

Ptolemy could have improved his result still further by applying the same process also to the minimum distance $R-r=53;50\, r_e$. With $d_☾=0;35,20°$ one finds

$$r_m=\frac{53;50}{120} r_e \cdot \text{crd}\ 0;35,20 \approx 0;16,36\, r_e.$$

Hence Ptolemy could have accepted $r_m \approx 0;17\, r_e$ as a plausible compromise. It is interesting to see that he did not utilize his own data to the fullest extent.

B. Criticism

Ptolemy's values for the mean distance of the moon and for the radius of the epicycle (i.e. for the eccentricity of the orbit) are of the correct order of magnitude. Since the principle of his method for the determination of the apparent diameters is perfectly sound it does not seem surprising that his results deviate only comparatively little from the true values. And, by restricting himself to eclipses, Ptolemy avoided the embarrassing problem of an inadmissibly large apparent lunar diameter which would result from his cinematic model in the case of the quadratures.

The following little table shows the good agreement between Ptolemy's results and modern values which would be the approximate equivalents of the ancient parameters:

at syzygies	Ptolemy	modern
$d_{☾ min}$	0;31,20°	0;29°
$2s_{min}$	1;21,20	1;16
$d_{☾ max}$	0;35,20	0;34
$2s_{max}$	1;32	1;29
$d_{\odot}$	0;31,20	0;32

The explanation given previously for the successful determination of the apparent diameters as resulting from the essentially correct dimensions of the underlying model does not hold up under a more careful investigation of the numerical details of Ptolemy's procedure. In fact it is only the accidental interplay of a great number of different inaccuracies of empirical data and of computations that lead to nearly correct results. In the case of the determination of the geocentric distances of sun and moon (below p. 109 ff.) the same data produce a grossly incorrect value for the distance of the sun, a result which was confirmed over and over again, with only insignificant modifications, by ancient and medieval astronomers, down to the middle of the 17th century.[10]

The chapters on apparent diameters, geocentric distances, and on parallax teach us the very valuable lesson that it is futile to try to establish "the" source of an error in the determination of important parameters of the Ptolemaic theory. The same holds for all astronomers before Flamsteed, Cassini, Halley,

[10] Cf., e.g., the list in Houzeau, Vade-mecum, p. 404 f.

etc. i.e. before the end of the 17th century. In all ancient astronomy direct measurements and theoretical considerations are so inextricably intertwined that every correction at any one point affects in the most complex fashion countless other data, not to mention the ever present numerical inaccuracies and arbitrary roundings which repeatedly have the same order of magnitude as the effects under consideration. In the history of the most causal of all empirical sciences, in astronomy, the search for causes is as fruitless as in all other historical disciplines.

If we denote, as before, by b the distance of the center of the moon from the center of the shadow at the middle of a lunar eclipse of magnitude m, then Ptolemy's method for the determination of $d_{☾}$ is based on the relation

$$\frac{\Delta m}{12} d_{☾} = \Delta b. \tag{1}$$

The magnitudes m come from observations, the distances b are computed from the positions of sun, moon, and node derived from the tables.

Ptolemy does not give the details of his computations of the distances b, except that he refers to the distance $|\omega|$ of the moon from the node and to the value $i=5°$ for the inclination of the orbit. Pappus in his Commentary to Book V and VI of the Almagest computes the b's from these data,[11] using the Menelaos theorems:

	$b_{\mathrm{I}}=0;48,30°$	Ptolemy: 0;48,30°
	$b_{\mathrm{II}}=0;41$	0;40,40
thus	$\Delta b=0;\ 7,30$	0; 7,50.

Accepting the recorded $\frac{\Delta m}{12}=1/4$ Pappus' result would give a $d_{☾}=0;30°$ instead of Ptolemy's 0;31,20°. Assuming also with Ptolemy that under the given circumstances $d_{\odot}=d_{☾}$ we would have to accept for both diameters the value 0;30° if Ptolemy had computed correctly with spherical trigonometry (and making the same roundings as Pappus). But since we do not know how Ptolemy actually did compute one might assume that he simply used plane trigonometry. Or, he may have used the tables of latitudes in Alm. V, 8 in order to solve his right spherical triangles with $i=5°$, thus approximating b by β. Finally one can compute b from $\tan b=\sin\omega\tan 5°$ (and thus check Pappus' results). Hence one has a whole range of values for $d_{☾}=d_{\odot}$ at one's disposal, depending on the method one combines the given data:

	Ptolemy	Pappus (Menelaos)	plane (chords)	latitudes (Alm. V, 8)	$\tan b=$ $\sin\omega\tan 5$
b_{I}	0;48,30°	0;48,30°	0;48,59°	0;49,40°	0;48,46°
b_{II}	0;40,40	0;41, 0	0;40,57	0;41,36	0;40,49
Δb	0; 7,50	0; 7,30	0; 8, 2	0; 8, 4	0; 7,57
$d_{☾}$	0;31,20	0;30, 0	0;32, 8	0;32,16	0;31,48

[11] Ed. Rome, p. 100, 10 to 103, 11 and p. 184, 1 to 187, 7. Cf. also above p. 104, note 5. The value 0;41 for b_{II} is not expressly given by Pappus but results from repeating his computations for case II.

These data, however, are far from reliable. The worst element is the estimate of the maximum obscuration of an eclipse as fraction of the lunar diameter. But this is by no means the only source of error. Two eclipses were observed in Babylon, one at Rhodes, one in Alexandria. Even if the time of the eclipse middle had been correctly determined (which is not the case), Ptolemy had to reduce it to Alexandria local time, a process which in turn requires the determination of geographical longitudes on the basis of other observations of lunar eclipses. An error in the accepted local time of Alexandria influences the computed lunar positions and hence the values of $|\omega|$ which are needed to find the distances b.

How large the discrepancies between data used by Ptolemy (A) and modernly computed data (B)[12] can be is shown in the following table. All hours are reduced to midnight, Alexandria local time.

	I (Babylon)	II (Babylon)	III (Alexandria)	IV (Rhodes)
(A) middle	5^h	$22;10^h$	$2;20^h$	$22;10^h$
magn.	3 from S	6 from N	7 from N	3 from S
Δm	$0;15\, d_\leftmoon$		$0;20\, d_\leftmoon$	
true moon	♎ 27;5	♑ 18;14	♏ 6;16	♌ 5;8
(B) date	−620 Apr. 22	−522 July 16/17	−173 May 1	−140 Jan. 27
middle	$4;17^h$	$22;41^h$	$1;50^h$	$21;22^h$
magn.	2.1 from S	6.1 from N	7.4 from N	2.8 from S
Δm	$0;20\, d_\leftmoon$		$0;24\, d_\leftmoon$	
longit.	♎ 24;50	♑ 16;40	♏ 6;0	♌ 4;40

Only for the eclipses III and IV does the error in time explain the error in longitude ($\Delta_{III}\, t \approx 0;30^h$ $\Delta_{III}\, \lambda \approx 0;15°$ and $\Delta_{IV}\, t \approx 0;50^h$ $\Delta_{IV}\, \lambda \approx 0;30°$). For I and II, however, these errors are unrelated ($\Delta_I t \approx 0;45^h$ but $\Delta_I \lambda \approx 3;15°$ and $\Delta_{II} t \approx -0;30^h$ but $\Delta_{II}\lambda = +1;30°$). In all cases the effect on $|\omega|$ and thus on b would be considerable. If Ptolemy's values for the b's were correct one would obtain from the correct Δm the diameters $d_\leftmoon = 0;23,30°$ $(= d_\odot)$ and $d_\leftmoon = 0;29,28°$, respectively. Thus it is only through the errors in the estimate of eclipse magnitudes that the errors in time and longitudes are so luckily compensated as to produce satisfactory results.

No ancient astronomer had any possibility of analyzing sources of errors in observations made long before his time or at far distant localities. It makes no sense to praise or to condemn the ancients for the accuracy or for the errors in their numerical results. What is really admirable in ancient astronomy is its theoretical structure, erected in spite of the enormous difficulties that beset the attempts to obtain reliable empirical data. Without the cinematic theories of the Almagest it would have been impossible to introduce, on the basis of better observational techniques, those improvements which found their explanation in Newton's celestial mechanics.

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[12] From P. V. Neugebauer, Kanon d. Mondf. The eclipse I was considered invisible by Kepler (Werke 5, p. 270f.) "*luna enim sub terra fuit.*" The cause of Kepler's error lies in the insufficiently known geographical longitudes; he assumes, e.g., that Alexandria lies 2^h to the east of Hven, instead of actually only 1^h (e.g. Werke 3, p. 419, 3). The commentary in Werke 5, p. 453/4 is wrong.

4. Size and Distance of the Sun

A. Hipparchus' Procedure

It is a simple geometrical problem to determine the size and the distance of the sun on the basis of the assumption that the apparent diameter of the sun equals the apparent diameter of the moon when at maximum distance from the earth.[1] Ptolemy credits Hipparchus with the solution, though, of course, using different numerical data.[2] The method itself remained a standard tool of astronomy. The figure from Alm. V, 15 (Heiberg I p. 423; our Fig. 98) is found again, e.g., in al-Battānī[3] and in Copernicus.[4]

The maximum distance of the moon (EM in Fig. 98), measured in earth radii r_e, had been found[5] to be

$$R + r = 59 r_e + 5;10 r_e = 64;10 r_e \tag{1}$$

and from the corresponding apparent radius[6]

$$r_☾ = r_\odot = 0;15,40° \tag{2}$$

the size of the moon was determined[7] to be

$$r_m = 0;17,33 r_e. \tag{3}$$

The radius s of the shadow at the maximum distance of the moon, EM′ = EM, was determined[8] as

$$s = 2\,3/5\, r_☾.$$

For small angles we may replace arcs everywhere by chords and thus assume for the length M′B = σ of the radius of the shadow

$$\sigma = 2\,3/5\, r_m. \tag{4}$$

From EM′ = EM it follows (cf. Fig. 98) that

$$\mathrm{ED} = r_e = 1/2(\mathrm{MC} + \sigma)$$

hence

$$\mathrm{MC} = 2 r_e - \sigma = (2 - 2;36 r_m) r_e$$

and with (3)

$$\mathrm{AC} = \mathrm{MC} - r_m = (2 - 3;36 r_m) r_e = 0;56,49 r_e. \tag{5}$$

From the equality of the apparent diameters one concludes that

$$\frac{r_e}{\mathrm{AC}} = \frac{\mathrm{EF}}{\mathrm{AF}} = \frac{\mathrm{ES}}{\mathrm{MS}} = \frac{\mathrm{ES}}{\mathrm{ES} - \mathrm{EM}}$$

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[1] Above p. 104.
[2] Cf. below I E 5, 4 A.
[3] Opus astron., Nallino I, p. 59.
[4] De revol. IV, 19 (Gesamtausg. II, p. 255 f.).
[5] Above p. 105.
[6] Above p. 104, (1).
[7] Above p. 105, (5).
[8] Above p. 104, (2).

and therefore the distance of the sun (with (1) and (5)):

$$ES = \frac{EM}{r_e - AC} r_e = \frac{64;10}{1 - 0;56,49} r_e \approx 1210 r_e. \quad (6)$$

For the radius of the sun one finds

$$r_s = r_m \frac{ES}{EM} = 0;17,33 \frac{1210}{64;10} r_e \approx 5;30 r_e. \quad (7)$$

Ptolemy does not specify the part of the solar orbit to which the distance (6) refers. Pappus in his Commentary[9] assumes rightly that $1210 r_e$ represents the mean distance of the sun. This can be confirmed as follows. Since we know[10] that the eccentricity of the solar orbit is 1/24 we find for its linear dimension

$$e_{\odot} = \frac{1210}{24} r_e = 50;25 r_e. \quad (8)$$

Hence we obtain for the

$$\begin{aligned} \text{minimum distance: } & 1159;35 r_e \\ \text{maximum distance: } & 1260;25 r_e. \end{aligned} \quad (9)$$

This agrees with a statement by Proclus in his Hypotyposis that the minimum distance of the sun is $1160 r_e$ while he mentions in his Commentary to Plato's Timaeus $1260 r_e$ as the maximum distance.[11]

The value $1160 r_e$ for the minimum distance of the sun is still known to Copernicus[12] whose own values are[13]

$$\begin{aligned} \text{minimum distance: } & 1105 r_e \\ \text{mean distance: } & 1142 r_e \\ \text{maximum distance: } & 1179 r_e. \end{aligned} \quad (10)$$

Actually the distance of the sun is not near $1200 r_e$ but about $24000 r_e$ and the diameter of the sun is not $5\ 1/2\ r_e$ but $111\ 1/2\ r_e$. The formula (6) shows the essential cause of the error: since the distance EM in the numerator is of the right order of magnitude it is the denominator $r_e - AC$ which is too great. One would obtain from (6) the correct result if AC were about $0;59,50 r_e$ instead of $0;56,49 r_e$. This, in turn, would mean that $r_{☾}$ at maximum distance is slightly smaller than $r_{\odot}$ and

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9 Rome p. 107, 10ff.

10 Above p. 58.

11 Hypotyposis, ed. Manitius, p. 222, 4 and p. 224, 13. Comm. Tim. ed. Diehl III, p. 62, 30 trsl. Festugière IV, p. 86 (also Hypotyposis ed. Manitius, p. 131, 1). Both works of Proclus also contain errors: in the Hypotyposis (Manitius, p. 222, 3) $1210 r_e$ is incorrectly called maximum, instead of mean, distance; and in the Comm. Tim. (ed. Diehl III, p. 62, 30 and p. 63, 12) he gives $1076 r_e$ as minimum distance, obviously invented to fit the approximate computation given in the commentary. Thâbit ben Qurra omits this last number, although he otherwise follows Proclus in all the preceding steps, including the use of $1260 r_e$ for the maximum distance of the sun (Thâbit b. Qurra, ed. Carmody, p. 137, De hiis, Nos. 43–45). For Proclus cf. below p. 920.

12 De revol. I, 10 (Gesamtausg. II, pp. 22, 27f.); also $64;10 r_e$ as maximum distance of the moon (l.c. p. 22, 25f.).

13 De revol. IV, 21 (Gesamtausg. II, p. 257).

hence that annular eclipses should not be excluded. It is this very small error in the evaluation of the apparent diameters of sun and moon that introduces an erroneous factor of 1/20 in the determination of the distance of the sun.

As we have mentioned before, the existence of annular eclipses was observed, if not by earlier astronomers, by Sosigenes in A.D. 164,[14] in all probability still in the lifetime of Ptolemy. But neither then nor during the next 1400 years was the obviously necessary modification of the previously described procedure undertaken.

B. Historical Consequences

An error with a factor 20 in the dimensions of the solar system is in itself not of great significance for ancient astronomy. The Greek astronomers had long been used to say that the earth only plays the role of a point with respect to the size of the cosmos. In the practical questions which concern the theory of eclipses, and to a much lesser degree, in the theory of the planetary motions, the exaggerated value of the solar parallax is of little importance compared, e.g., to the effects of refraction and to the errors of measurement of times and angles. But a steady improvement of observational techniques could have slowly enlarged the scale of the planetary system – in much the same way as modern astronomy learned step by step to estimate more accurately the dimensions of our galactic system. In fact, however, nothing of this kind happened before refined observations caused Kepler to accept a reduction to 1/3 of the ancient solar parallax.[1]

A most implausible accident gave the Ptolemaic estimates of the solar and lunar distances a special significance and made them cornerstones for the mediaeval picture of the planetary system. Ptolemy's planetary theory, as derived in the Books IX and X of the Almagest, contains no absolute dimensions for the planetary orbits.[2] For each planet the deferent is given the value $R=60$ and eccentricities and epicycle radii are measured in each case with reference to this R. The only structural hypothesis mentioned in the Almagest (IX, 1) is the assumption that the three planets which can reach opposition to the sun are farther away from us than the sun whereas the two planets of limited elongation are located between the moon and the sun. No consequences, however, are drawn from this hypothetical arrangement. Only the remark is added that the two inner planets must be located so near the sun that their parallax remains below the limits of direct observation because no parallax had been found for either one of these planets.[3] Since the lunar parallax is easily measurable, reaching values of about 1°, it follows from this remark that Ptolemy considered the inner planets far removed from the moon.

In the second book of Ptolemy's "Planetary Hypotheses," however, one finds a very different statement.[4] Here the assumption of an interior position of

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[14] Cf. above p. 104, note 4.

[1] Epitome IV, 1, IV (Werke 7, p. 279).

[2] Cf. below p. 148 f.

[3] This, incidentally, implies that the solar parallax is also considered to be smaller than directly observable; indeed, its values are only computed from the distances found by the Hipparchian method (above p. 109). Cf. also Hipparchus' assumptions about the solar parallax (below I E 5, 4 B).

[4] For this part of the work, preserved only in Arabic, cf. below p. 918.

Venus and Mercury is supported by the argument that the otherwise empty and useless space between moon and sun can be accurately filled by the two planets.[5] How this is to be understood becomes clear from the recently discovered second part of Book I of the "Planetary Hypotheses."[6] First one takes Ptolemy's value for the maximum distance of the moon ($\approx 64 r_e$[7]) as the minimum distance for Mercury. Since all ratios between the dimensions of the cinematic model of this planet are derived in the Almagest one can now also express the maximum geocentric distance of Mercury in units of earth radii. Taking this value again as minimum distance of Venus the same procedure leads to a maximum distance for this planet also. The result, $1190 r_e$, is almost exactly the minimum distance ($1160 r_e$) found for the sun.[8] It is not surprising that no-one believed that such an agreement[9] could have been a pure accident and grossly wrong at its upper limit. Thus the Ptolemaic evaluation of the solar distance, incorrectly assuming the equality of $r_\odot$ and $r_☾$ at maximum distance, became the solid foundation of the planetary shell structure which dominates mediaeval astronomy.

5. The Table for Solar and Lunar Parallax (Alm. V, 18)

The computation of parallaxes presents no theoretical difficulties. The cinematical models allow us to find, for any given moment, the geocentric distance EP of the sun or the moon (cf. Fig. 92, p. 1235) and the tables in Alm. II, 13[1] provide us, for given celestial coordinates, with the zenith distance ζ. With EP and ζ known it is a simple matter of plane trigonometry to compute the desired angle $p=\zeta'-\zeta$.

In the case of the sun, the situation is particularly simple since Ptolemy has found that the eccentricity of the solar orbit has no significant influence on the parallax.[2] Consequently the distance EP can be kept constant ($=1210 r_e$)[3] and a single column suffices for the tabulation of the solar parallax (column 2) as function of the zenith distance (column 2 in Alm. V, 18). The maximum value, the "horizontal parallax," for $\zeta=90°$ is found to be 0;2,51° (cf. also Fig. 99, top), a value about 19 times too great.

The parallax of the moon requires a great deal more computation, not only because of the wide variability of its geocentric distance in Ptolemy's model, but also because of its dependence on three variables (zenith distance ζ, true anomaly α, and mean elongation $\bar{\eta}$). This latter fact necessitates interpolation schemes to make the tables usable.

The underlying idea is exactly the same as in the case of the tables for the equation of the moon (Alm. V, 8).[4] The parallaxes are computed, as function of

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[5] Ptolem., Opera II, p. 118 (Heiberg).

[6] Cf. below V B 7, 6.

[7] Above p. 109, (1).

[8] Above p. 112.

[9] This whole procedure has nothing to do with the Eudoxan-Aristotelian concentric spheres since it is based solely on Ptolemy's model of Mercury and Venus and the parameters given in the Almagest.

[1] Above p. 50.

[2] Cf. above p. 104, (a).

[3] Cf. above p. 110.

[4] Above p. 93 ff.

the zenith distance ζ, for four extremal distances:

$$\left.\begin{array}{r}\text{I: apogee}\\ \text{II: perigee}\end{array}\right\}\text{ of the epicycle at syzygy}$$
$$\left.\begin{array}{r}\text{III: apogee}\\ \text{IV: perigee}\end{array}\right\}\text{ of the epicycle at quadrature.}$$

Then interpolation is used (a) with respect to the position of the moon on the epicycle, i.e. depending on the true anomaly α, (b) with respect to the distance of the epicycle, i.e. depending on the mean elongation $\bar{\eta}$.

The parallaxes for the cases I to IV are computed as function of the zenith distance ζ on the basis of the configuration represented in Fig. 92 (p. 1235) assuming for the distance EP the values[5]

$$\text{I, II: } R \pm r = 59\, r_e \pm 5;10\, r_e = \begin{cases} 64;10\, r_e \\ 53;50\, r_e \end{cases}$$

$$\text{III, IV: } R - 2e \pm r = 38;43\, r_e \pm 5;10\, r_e = \begin{cases} 43;53\, r_e \\ 33;33\, r_e. \end{cases}$$

The resulting parallaxes are tabulated in Alm. V, 18 (as function of the argument ζ in column 1). For case I and III serve the columns 3 and 5 (c_3 and c_5 in Fig. 99). The columns 4 and 6, however, give only the increments of the parallaxes at perigee over the parallaxes at apogee (thus $c_3 + c_4$ is the parallax in case II, $c_5 + c_6$ in case IV).

The next step consists in the computation of coefficients of interpolation with respect to true epicyclic anomaly α, for syzygies (Fig. 100a) denoted here $c_7(\alpha)$, for quadratures (Fig. 100b) $c_8(\alpha)$. These coefficients are found from

$$\left.\begin{array}{l} c_7(\alpha) \\ c_8(\alpha) \end{array}\right\} = \frac{\mathrm{EA}-\mathrm{EP}}{\mathrm{A}\Pi} = \begin{array}{l} \dfrac{R+r-\mathrm{EP}}{2r} = \dfrac{65;15-\mathrm{EP}}{10;30} \\[2ex] \dfrac{R-2e+r-\mathrm{EP}}{2r} = \dfrac{44;37-\mathrm{EP}}{10;30} \end{array} \qquad R = 60 \tag{1}$$

with EP given by

$$\mathrm{EP}^2 = \left(r\cos\alpha + \begin{Bmatrix} R \\ R-2e \end{Bmatrix}\right)^2 + (r\sin|\alpha|)^2 \qquad 0 \leqq |\alpha| \leqq 180 \tag{1a}$$

respectively. These coefficients measure the ratio of the actual approach of the moon toward E to the greatest possible approach $2r$ and it is assumed that the parallax varies in the same ratio. Thus the parallax at syzygies ($\bar{\eta} = 0$) and anomaly α is given by

$$p_S(\zeta, \alpha) = c_3(\zeta) + c_7(\alpha) \cdot c_4(\zeta), \tag{2a}$$

at quadratures ($\bar{\eta} = 90$) and anomaly α by

$$p_Q(\zeta, \alpha) = c_5(\zeta) + c_8(\alpha) \cdot c_6(\zeta). \tag{2b}$$

In looking up $c_7(\alpha)$ and $c_8(\alpha)$ in the tables Alm. V, 18 to given $|\alpha|$ one has to remember that $|\alpha|$ varies between 0° and 180° while the numbers in column 1

[5] Cf. p. 102.

run only to 90. Consequently one has to enter column 1 with the value $|\alpha|/2$ if one wishes to find c_7 or c_8 for the argument $|\alpha|$ (cf. the graph in Fig. 101).

Having found in this way the parallaxes p_S and p_Q, when $2\bar{\eta}=0$ and 180, respectively, a coefficient of interpolation $c_9(\bar{\eta})$ is tabulated in column 9 such that the total parallax of the moon in the most general position is given by

$$p(\zeta,\alpha,\bar{\eta})=p_S(\zeta,\alpha)+c_9(\bar{\eta})\cdot\big(p_Q(\zeta,\alpha)-p_S(\zeta,\alpha)\big). \tag{3}$$

It may be noted that the term p_Q-p_S is always positive because the apogee of the epicycle at quadrature ($R-2e+r=44;37$) is nearer to E than the perigee at a syzygy ($R-r=54;45$), hence $p_Q>p_S$.

The coefficients $c_9(\bar{\eta})$ are determined according to the same principle as the coefficients c_7 and c_8, that is to say $c_9(\bar{\eta})$ is defined by the ratio of the actual approach of the center C of the epicycle toward E to the greatest possible approach ($2e=20;38$) from syzygy to quadrature.

In order to compute the distance EC Ptolemy combines a position of elongation $\bar{\eta}_1$ with a symmetric position $\bar{\eta}_2=\bar{\eta}_1+90$. Because $2\bar{\eta}_2=2\bar{\eta}_1+180$ the three points C_1, E, and C_2 lie on a straight line if one keeps the direction EM fixed (cf. Fig. 102). Then

$$C_1H=HC_2=\sqrt{(R-e)^2-(e\sin 2\bar{\eta}_1)^2}. \tag{4a}$$

The desired distances are

$$\left.\begin{matrix}EC_1\\EC_2\end{matrix}\right\}=C_1H\pm e\cos 2\bar{\eta}_1 \tag{4b}$$

and (since the maximum distance EC is R) hence

$$c_9(\bar{\eta}_1)=\frac{R-EC_1}{2e}\qquad c_9(\bar{\eta}_1+90)=\frac{R-EC_2}{2e}. \tag{4c}$$

The elongations $\bar{\eta}$ range between 0° and 360° but the coefficients $c_9(\bar{\eta})$ depend only on the distance EC which is the same for $\pm\eta'$ and $180\pm\eta'$ where η' belongs to the interval $0°\leqq\eta'\leqq 90°$ (cf. Fig. 103). Hence we define

$$\begin{array}{ll}\text{for}\quad 0\leqq\bar{\eta}\leqq 90: & \eta'=\bar{\eta}\\ \text{for}\quad 90\leqq\bar{\eta}\leqq 180: & \eta'=180-\bar{\eta}\\ \text{for}\quad 180\leqq\bar{\eta}\leqq 270: & \eta'=\bar{\eta}-180\\ \text{for}\quad 270\leqq\bar{\eta}\leqq 360: & \eta'=360-\bar{\eta}.\end{array} \tag{5}$$

With η' as argument one can enter column 1 of the tables Alm. V, 18 and find in column 9 the coefficient $c_9(\bar{\eta})$; cf. also Fig. 101.

Example. We wish to find the parallax of the moon at ♏ 5;20° when M = ♎ 30° is in the meridian at Rome.[6] The difference in right ascension

$$\alpha(☾)-\alpha(M)=212;59-207;50=5;9°\approx 0;20^h$$

shows that the moon is about $1/3^h$ east of the meridian.

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[6] This concerns an observation made by Menelaos in Rome (A.D. 98 Jan. 14). Cf. above p. 96 for $\lambda_☾$ and p. 43, for M. Explicit examples of parallax computations are rare; two are found in Pappus, Comm., ed. Rome, p. 115, 6 to 117, 12 and p. 125, 15 to 126, 10.

If we assume that Rome has the same geographical latitude as the Hellespont[7] we can use Table V in Alm. II, 13 and find there by interpolation for ♏5° and $1/3^h$ the zenith distance of the moon

$$\zeta = 54;40°.$$

For the moment in question we found for the true lunar anomaly and mean elongation[8]

$$\alpha = 294;32 \qquad \bar{\eta} = 277;53.$$

With the argument ζ one finds in V, 18

$$c_3 = 0;44,6 \qquad c_4 = 0;8,33 \qquad c_5 = 1;4,27 \qquad c_6 = 0;21,7.$$

The value of α has to be replaced by $|\alpha| = 360 - 294;32 = 65;28 \approx 65;30$ and as argument one has to use $|\alpha|/2 = 32;45$ in order to find the coefficients of interpolation in anomaly

$$c_7(\alpha) = 0;16,34 \qquad c_8(\alpha) = 0;16,3.$$

Hence

$$p_S = c_3 + c_7 c_4 \approx 0;44,6 + 0;2,22 = 0;46,28$$

$$p_Q = c_5 + c_8 c_6 \approx 1;4,27 + 0;5,39 = 1;10,6.$$

The equivalent of the elongation $\bar{\eta} = 277;53$ is according to (5)

$$\eta' = 360 - \bar{\eta} = 82;7 \approx 82;6.$$

With this η' as argument one finds the coefficient of interpolation in elongation

$$c_9(\bar{\eta}) \approx 0;59.$$

Hence the total parallax

$$p = 0;46,28 + 0;59(1;10,6 - 0;46,28) = 1;9,42° \approx 1;10°.$$

6. The Components of the Parallax

The parallax p of the moon M is a vector MM′ (cf. Fig. 104) which lies in the circle of altitude ZMA. If one wishes to find the ecliptic coordinates of the apparent position M′ of the moon from its true position $M(\lambda, \beta)$ one must split the vector p in its two components p_λ and p_β. Ptolemy discusses this problem in Alm. V, 19 which is of interest for two reasons. First because Ptolemy mentions some details about Hipparchus' approach to the problem – we shall discuss this part later.[1] Secondly it shows a curious primitivity in dealing with a simple problem of spherical trigonometry, to an extent which would lead to serious errors if the parallaxes were not so small.

The basis for the computation of p_λ and p_β are, or course, the tables in Alm. II, 13[2] which allow us to find to the given longitude λ of M the zenith distance $\zeta = ZL$ and the angle γ at L between the ecliptic and the circle of altitude ZLA

[7] Cf. above p. 41.
[8] Above p. 96.
[1] In I E 5, 3.
[2] Cf. p. 50.

(Fig. 104). In II, 13 the angle γ could range between 0° and 180°; for the computation of the components of parallax only the acute angle between ecliptic and ZA′ is needed. Therefore γ means, in the following, either the angle γ' found in II, 13 if $\gamma' \leqq 90$ or $180-\gamma'$ if $90 \leqq \gamma' \leqq 180$.

Ptolemy distinguishes two methods, one "approximate," the other "more accurate." Fortunately he seems to favour the first method; the second one is not only more complicated but less correct than the first one.

First method. The following assumptions are made (cf. Fig. 105):

(a) all triangles are plane
(b) the circles of altitude through L and M are parallel
(c) the parallax p, actually computed for the position of L, remains the same for M.

Under these very reasonable simplifications the components of parallax are given by

$$p_\lambda = p\cos\gamma \qquad p_\beta = p\sin\gamma$$

(of course expressed by Ptolemy in terms of chords and with case distinctions for signs).

Second method. One could easily eliminate the assumption (c) by replacing the zenith distance ζ of L in the computation of the parallax by the approximate zenith distance $\zeta-\beta\sin\gamma$ of M. This, however, is not the modification adopted in Ptolemy's second method. There he retains the assumptions (a) and (c) but drops (b) by assuming that AL and A′L′ meet at Z. Consequently

$$p_\lambda = p\cos\theta \qquad p_\beta = p\sin\theta$$

where the angle θ at L′ or M′ (cf. Fig. 106) is still to be determined. To this end Ptolemy computes

$$\mathrm{MN} = \beta\cos\gamma \qquad \mathrm{NZ} = \zeta-\beta\sin\gamma$$

as a reasonably close approximation to the spherical configuration. But now he finds MZ from

$$\mathrm{MZ} = \sqrt{\mathrm{NZ}^2+\mathrm{MN}^2}$$

and then the angle η at Z from $\sin\eta = \frac{\mathrm{MN}}{\mathrm{MZ}}$, or in his terminology from

$$\operatorname{crd} 2\eta = 120\,\frac{\mathrm{MN}}{\mathrm{MZ}};$$

finally $\theta = \gamma-\eta$.

This procedure is, of course, valueless since it is based on the wrong assumption that large spherical triangles can be approximated by plane triangles. It is surprising to see that Ptolemy considers this method as a refinement of the first one and that Pappus repeats it in his Commentary without criticism[3] when it is geometrically evident that, e.g., any angle η at Z could correspond to right angles both at N and at M.

The chapter on parallax is undoubtedly one of the most unsatisfactory sections in the whole Almagest.

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[3] Pappus, Comm., ed. Rome, p. 166, 16ff.

Example. We apply the first method of approximation (Fig. 105) to the parallax found before (p. 115) for the observation made in Rome by Menelaos. The longitude of the moon was found to be approximately ♏5 when 0;20^h east of the meridian. Substituting again the table "Hellespont" in Alm. II, 13 for Rome[4] one finds the "eastern angle" between ecliptic and circle of altitude $\gamma' \approx 114$, hence $\gamma = 180 - 114 = 66$. For the total parallax we had $p = 1;9,42 \approx 1;10$, hence

$$p_\lambda = +p\cos\gamma = +1;10 \cdot 0;24,24 \approx +0;28^\circ$$
$$p_\beta = -p\sin\gamma = -1;10 \cdot 0;54,49 \approx -1;4^\circ.$$

The coordinates of the true moon are[5] $\lambda =$ ♏5;20 $\beta = +2;5$. Hence we would place the center of the apparent moon in $\lambda' =$ ♏5;48 $\beta' = +1;1$. Ptolemy gives[6] instead $\lambda' =$ ♏5;55 $\beta' = +1;3$[7]; hence he operates with $p_\lambda = 5;55 - 5;20 = 0;35^\circ$ and $p_\beta = 1;3 - 2;6 = -1;3^\circ$. According to the observation the moon occulted the star β Scorpii which in the catalogue Alm. VIII, 1 is given the latitude $+1;20^\circ$, i.e. about $0;17^\circ$ more than the center of the moon. Thus the computed data barely suffice to explain the observation[8].

Supplementary Remarks. The preceding example is the last in a series which all concern the same observation made by Menelaos of an occultation of β Scorpii by the moon (in A.D. 98 Jan. 14). Ptolemy analyses this observation in order to determine the constant of precession and finds it in agreement with Hipparchus' lower estimate of 1° per century (Alm. VII, 3).

Much has been written about this incorrect value but usually without any attempt to investigate the underlying computations. A careful student of this problem must follow the numerical data utilized by Ptolemy step by step. For our specific example these steps are discussed in the following sections:

I B 1, 3 C, p. 60: approximate determination of $\lambda_\odot$
I A 4, 6, p. 41: transformation of seasonal to equinoctial hours in Rome (involving the geographical latitudes of Rome or the Hellespont)
I B 2, 2, p. 64: equation of time
I B 3, 6 C, p. 84 and I B 4, 3, p. 96: determination of $\lambda_☾$ and $\beta_☾$ (involving the geographical longitudes of Rome and Alexandria)
I A 4, 6, p. 43: culminating sign
I B 5, 5, p. 114 and above p. 117: parallax and its components (involving the refined lunar theory).

Each of these steps depends on all the theoretical and observational elements which are embedded in the tables of the Almagest and on the geographical data

[4] Above p. 115.
[5] Cf. p. 96.
[6] Alm. VII, 3 (Man. II, p. 29).
[7] Heiberg's edition (II, p. 33, 19–21) gives for β "$\bar{2}$ and 6′", for β' "$\bar{1}$ and 3′" or (in MS D) "$\bar{1}$ 3′". These numbers must be interpreted as 2;6° and 1;3°, respectively, not as 2 1/6 = 2;10° and 1 1/3 = 1;20° (as in Manitius II, p. 29) because this would give only $p_\beta = -0;50^\circ$.
[8] Ptolemy assigns to β Sco in A.D. 137 the longitude ♏6;20. Thus he assumed for the observation 40 years earlier a longitude of ≈♏5;55. The actual coordinates in A.D. 100 are ♏6;46 and +1;15°. The latitude of the moon was in fact about 0;7° greater than computed by Ptolemy. These two corrections bring the star near the center of the moon.

for Alexandria and Rome. And all this concerns only the reduction of Menelaos' empirical data which in themselves are of a rather complex nature.[9]

It seems to me senseless to single out from this host of data some specific elements which caused specific errors in Ptolemy's final results. And it is obviously absurd to assume that Ptolemy should have tampered with all these computations in order to confirm, for no obvious reason, just the minimum value of Hipparchus' estimate for the constant of precession.

§ 6. Theory of Eclipses

1. Determination of the Mean Syzygies

All tables in the Almagest for the computation of longitudes of the sun, the moon, or the planets are based on the same principle: by means of the tables of mean motions one has first to compute the increment $\Delta\bar{\lambda}$ of mean longitude for the given time interval Δt since epoch in order to find the mean longitude $\bar{\lambda}=\bar{\lambda}_0+\Delta\bar{\lambda}$ at the given moment from the mean longitude $\bar{\lambda}_0$ at epoch. A similar operation is required for the anomalies which are needed for the equations which lead from the mean to the true positions. In later tables, however, in particular in the "Handy Tables," this process is greatly simplified by giving mean positions ready computed for certain equidistant dates, such that one can enter the tables at the date nearest to the given calendar date without going back in each case to the situation at epoch.

In the Almagest only the tables for mean syzygies (VI, 3) are constructed for direct chronological entries. The main tables proceed in steps of 25 years with 45 entries from Nabonassar 1 to Nabonassar 1101 (i.e. from -746 to A.D. 353), supplemented by tables for single years (from 1 to 24) and for months. I shall now describe how these tables were derived.

The first step consists in finding the first mean conjunction and mean opposition after epoch, i.e. after $t_0=$ Nab. 1 Thoth 1. For t_0 the following parameters had been found[1] (cf. Fig. 107):

Moon: mean elongation $\bar{\eta}_0 = 70;37^\circ$
mean anomaly $\bar{\alpha}_0 = 268;49$
argum. of lat. $\bar{\omega}'_0 = 354;15$
Sun: mean anomaly $\bar{\kappa}_0 = 265;15$ (from A = ♊ 5;30).

Since the mean increase of the elongation of the moon from the sun is $12;11,26,41,\ldots^{\circ/d}$ (Alm. IV, 4) it takes

$$70;37/12;11,\ldots \approx 5;47,33^d$$

to accumulate the elongation $\bar{\eta}_0$. The length of the mean synodic month has been found to be $29;31,50^d$. Therefore the first mean conjunction after t_0 has

[9] Cf. Alm. VII, 3 (Manitius II, p. 28).
[1] Cf. above p. 79, p. 82, and p. 60.

from t_0 the distance

$$29;31,50-5;47,33=23;44,17^d.$$

Ptolemy now introduces fractional dates by giving the moment $23;44,17^d$ after Thoth 1 noon the "date" Thoth 24;44,17. Consequently the first entry in the table of mean conjunctions (Alm. VI, 3) is

Nabonassar 1 Thoth 24;44,17.

Half a mean synodic month earlier falls the first mean opposition after epoch:

Nabonassar 1 Thoth 9;58,22

because $1/2\ 29;31,50=14;45,55$ and $24;44,17-14;45,55=9;58,22$.

For each date of a mean syzygy three more elements are needed: the mean eccentric anomaly $\bar{\kappa}$ of the sun, the mean epicyclic anomaly $\bar{\alpha}$ of the moon and the argument of latitude $\bar{\omega}'$, reckoned in the direction of increasing longitudes from the northernmost point of the lunar orbit. Since the values of these parameters are known for t_0 one has only to add the increments which correspond, according to the tables of mean motions, to the time interval from t_0. In this way one finds for the first conjunction

$$\bar{\kappa}=288;38,50 \qquad \bar{\alpha}=218;57,15 \qquad \bar{\omega}'=308;17,21$$

and for the first opposition, one half month earlier:

$$\bar{\kappa}=274;5,38 \qquad \bar{\alpha}=26;2,45 \qquad \bar{\omega}'=112;57,15.$$

These are the first entries in the tables of mean conjunctions and oppositions, respectively (cf. Fig. 107).

Both tables continue in steps of 25 Egyptian years. This step has been chosen because 25 Egyptian years correspond almost exactly to 309 mean synodic months.[2] Indeed

$$\begin{aligned} 25^y &= 25\cdot 6,5^d=2,32,5^d \\ 309^m &= 5,9\cdot 29;31,50,8,20=2,32,4;57,12,55^d \\ &= 25^y-0;2,57,5^d. \end{aligned}$$

This shows that 25-year steps keep the dates of the mean syzygies almost unchanged because

$$\Delta t=309^m=25^y-0;2,57,5^d. \tag{1}$$

For this interval one finds for the corresponding changes in the parameters $\bar{\kappa}$, $\bar{\alpha}$, and $\bar{\omega}'$:

$$\Delta\bar{\kappa}=353;52,34,13^\circ \qquad \Delta\bar{\alpha}=57;21,44,1^\circ \qquad \Delta\bar{\omega}'=117;12,49,54^\circ. \tag{2}$$

Since the tables are only given to two sexagesimal digits the last digit in the differences (1) and (2) appears in the tables only implicitly, recognizable by an occasional change of the differences by 1 in the final place. For example the dates increase ordinarily by $25^y-0;2,57^d$ but every 12th year the difference is $25^y-0;2,58^d$. Similarly $\Delta\bar{\omega}'$ is 9 times $117;12,50^\circ$ followed by one difference

[2] This relation was probably known in Egypt long before hellenistic times; cf., p. 563.

117;12,49°. In each column this change of difference occurs for the first time after half as many lines as correspond to an accumulated error of 1 unit of the last place. Obviously it was assumed that the epoch values were only known to seconds whereas the differences (1) and (2) could be corrected as soon as the accumulated rounding error amounted to 1/2 second. In fact it would have been easy to compute also the epoch values to one more significant figure and to adjust the shifts in the differences accordingly. Such inconsistencies in the level of accuracy in the different components of the same table are common in ancient computations, sometimes causing errors of the same order of magnitude as the effect under consideration.

After the 25-year tables are established in the fashion described so far, one still needs tables for single years and for months. Such tables are easily derived from the tables of mean motions; only in the table for single years one has to distinguish between calendar years which contain 12 syzygies (of the same type) or 13. For the year Nabonassar 1 we have found for the first syzygies dates shortly before Thoth 10 (opposition) and Thoth 25 (conjunction), respectively. In the first case 12 synodic months later give still a date within the year Nabonassar 1, whereas in the second case 12 additional months already lead into year 2. In order to construct a list of the first syzygies for each consecutive year we have to add 13 mean months to the date in year 1 in the first case, but only 12 in the second. If $\bar{m}$ represents the length of the mean synodic month we can express the changes in the dates of the syzygies as follows:

$$\text{case 1: } 13\bar{m} - 1^{y} = 13 \cdot 29;31,50,8,20 - 365^{d} = 18;53,51,48^{d}$$
$$\text{case 2: } 12\bar{m} = 12 \cdot 29;31,50,8,20 = 365^{d} - 10;37,58,20^{d}.$$

If we call "*epact*" the interval

$$\bar{e} = 1^{y} - 12\bar{m} = 10;37,58,20^{d} \tag{3}$$

then we can say that 13 months change the date by

$$13\bar{m} - 1^{y} = \bar{m} - \bar{e} = +18;53,51,48^{d} \tag{4a}$$

whereas 12 months lower the date by

$$\bar{e} = 10;37,58,20^{d}. \tag{4b}$$

Thus we can construct a sequence of 24 single years containing either 12 or 13 mean synodic months (the latter denoted by *) such that their influence on the dates of the syzygies remains between 0 and 30 days:

year 1*	$\bar{m} - \bar{e}$	$= 18;53,51,48^{d}$
2	$\bar{m} - 2\bar{e}$	$= 8;15,53,28$
3*	$2\bar{m} - 3\bar{e}$	$= 27;\ 9,45,16$
4	$2\bar{m} - 4\bar{e}$	$= 16;31,46,56$
5	$2\bar{m} - 5\bar{e}$	$= 5;53,48,36$
6*	$3\bar{m} - 6\bar{e}$	$= 24;47,40,24$ etc.

If we want to have the first syzygy in a year N of the era Nabonassar where $N = (25n + 1) + k$ (n and k integers $\geqq 0$, $k \leqq 24$) we add to the date given for the

year $25n+1$ the days for the year k. If the result is $\leqq 30$ the date concerns the month of Thoth, otherwise the next month (Phaophi).

The values of $\Delta\bar{\kappa}$, $\Delta\bar{\alpha}$, $\Delta\bar{\omega}'$ given in the table for single years are obtained by adding to 0 the mean motions during 13 months in the nine years

$$1^* \quad 3^* \quad 6^* \quad 9^* \quad 12^* \quad 14^* \quad 17^* \quad 20^* \quad 23^*.$$

For all other years mean motions during 12 months are added. For example one finds for the mean motion of the sun

$$\begin{array}{lrl}
\text{year } 1^* & 13\bar{m}: & 0 \qquad\quad + 18;22,59 = 18;22,59^\circ \\
2 & +12\bar{m}: & 18;22,59 + 349;16,37 = \ \ 7;39,36 \\
3^* & +13\bar{m}: & \ \ 7;39,36 + \ \ 18;22,59 = 26;\ 2,35 \\
4 & +12\bar{m}: & 26;\ 2,35 + 349;16,36 = 15;19,11
\end{array}$$

etc., because the tables of mean motions (Alm. III, 2) give

$$\begin{array}{ll}
\text{for } 12\bar{m} = 365^d - \bar{e}: & 349;16,36,16^\circ \\
\text{for } 13\bar{m} = 365^d + \bar{m} - \bar{e}: & 18;22,59,18^\circ.
\end{array}$$

Similar increments are found for the lunar motion during 12 or 13 mean synodic months and finally for single months from 1 to 12. Thus all mean syzygies for the years from Nabonassar 1 to 1126 (A.D. 378) can be found very simply, together with the characteristic parameters for the mean longitudes of sun and moon.

Example. Compute the mean oppositions for the year Nabon. 718. In Alm. VI, 3 we find in the tables for full moons:

Nab. 701:	Thoth	8;40,24
17 single years:		25;57,19
Nab. 718:	II	$4;37,43^d$.

Table 8

No.	Nabon.			Julian		
1	718	I	5; 5,53	−30	Sept.	4; 5,53
2		II	4;37,43		Oct.	3;37,43
3		III	4; 9,33		Nov.	2; 9,33
4		IV	3;41,24		Dec.	1;41,24
5		V	3;13,14		Dec.	31;13,14
6		VI	2;45, 4	−29	Jan.	29;45, 4
7		VII	2;16,54		Feb.	28;16,54
8		VIII	1;48,44		Mar.	29;48,44
9		IX	1;20,34		Apr.	28;20,34
10		IX	30;52,24		May	27;52,24
11		X	30;24,14		June	26;24,14
12		XI	29;56, 5		July	25;56, 5
13		XII	29;27,55		Aug.	24;27,55

In order to find the first opposition in this year we have to subtract $29;31,50^d$ from the above result and obtain as the first date

$$\text{Nabon. } 718 \text{ I } 5;5,53 = -30 \text{ Sept. } 4;5,53$$

reckoning the julian date also from Alexandria noon.

All subsequent mean oppositions are now obtainable by adding step by step $29;31,50^d$, except for the 4th and for the 11th step when we use $29;31,51^d$ as in Ptolemy's table for single months in Alm. VI, 3. The result is shown in Table 8, p. 121. We shall continue this example presently by also computing the true oppositions and by investigating the possibilities for lunar eclipses (cf. below p. 123 and p. 138, respectively).

2. Determination of the True Syzygies

With mean conjunctions and corresponding anomalies known one can easily compute the true longitudes $\lambda_\odot$ and $\lambda_☾$ of sun and moon. But the resulting elongations

$$\eta = \lambda_☾ - \begin{cases} \lambda_\odot & \text{at new moons} \\ \lambda_\odot + 180 & \text{at full moons} \end{cases} \tag{1}$$

will, in general, not be zero. Consequently the time of the true conjunction will differ from the time of the mean conjunction by an amount

$$-\Delta t = \eta / (v_☾ - v_\odot) \tag{2}$$

measured in hours when $v_☾$ and $v_\odot$ represent the hourly velocities of the two luminaries. Since the solar velocity varies very little Ptolemy replaces $v_☾ - v_\odot$ by $v_☾(1 - v_\odot/v_☾)$ and takes for the ratio $v_\odot/v_☾$ the constant value 1/13. Hence he computes the time correction Δt instead of by (2) by means of

$$-\Delta t = 13\eta/12 v_☾. \tag{3}$$

Thus the problem is reduced to finding $v_☾^{\circ/h}$ for given anomaly α.

The true velocity is the same as the mean velocity as long as the equation $c(\alpha)$ remains constant. If, however, the equation changes by

$$\Delta c(\alpha) = c(\alpha + 1) - c(\alpha) \tag{4}$$

when the anomaly changes from α° to $\alpha + 1^\circ$ then the motion of the moon will also experience the increment $\Delta c(\alpha)$. Its amount can be found for given α in the table of the first lunar inequality[1] in Alm. IV, 10. The hourly increment of the lunar anomaly is according to the tables Alm. IV, 4 about 0;32,40°. Therefore the hourly change of the equation is $0;32,40 \cdot \Delta c(\alpha)$. The mean velocity of the moon per hour is about 0;32,56 (Alm. IV, 4) and consequently the hourly velocity to be substituted in (3)

$$v_☾ = 0;32,56 + 0;32,40 \cdot \Delta c(\alpha)^{\circ/h} \tag{5}$$

of course with the proper sign of $\Delta c(\alpha)$.

[1] For syzygies the second inequality is zero.

Ptolemy seems to consider the correction obtained in this way sufficiently accurate for the determination of the moment of the true syzygy. Later works, e.g. of the Byzantine period, would explicitly compute $\lambda_{\odot}$ and $\lambda_{☾}$ for the moment thus obtained and iterate the process until no elongation remains.

Example. Find the true oppositions for the year Nabon. 718. In Alm. VI, 3 we have for the anomaly of the sun ($\bar{\kappa}$) and of the moon (α)

	$\bar{\kappa}$	α
Nab. 701:	102;37,36°	192;11,17°
17 single years:	21;26,58	47;19,30
Nab. 718 II:	124; 4,34	239;30,47.

Note that these values concern the second opposition in the year Nab. 718 as we have seen in computing the moment of the mean opposition (above p. 121). Consequently we must go one step back here also, i.e. subtract 29;6,23° from $\bar{\kappa}$ and 25;49,0° from α. Addition of these values (or of 25;49,1) produces the anomalies for the subsequent months.

Having determined all anomalies we can find the equations $c_{\odot}$ and $c_{☾}$ from Alm. III, 6 and IV, 10, respectively. Since for the mean oppositions $\bar{\lambda}_{\odot} = \bar{\lambda}_{☾} + 180$ we have for the true elongations

$$\eta = \lambda_{☾} + 180 - \lambda_{\odot} = (\bar{\lambda}_{☾} + 180 + c_{☾}) - (\bar{\lambda}_{\odot} + c_{\odot}) = c_{☾} - c_{\odot}.$$

In this way one finds the values given in Table 9.

Next we need the lunar velocity. The coefficients $\Delta c(\alpha)$ can easily be found in the table for the lunar equation (Alm. IV, 10), being the coefficients of interpolation for single degrees; in fact these coefficients had to be used before for the computation of the $c_{☾}(\alpha)$. Hence we tabulate now (cf. Table 10) the values of $\Delta c(\alpha)$, of $0;32,40 \cdot \Delta c(\alpha)$, and of $v_{☾} = 0;32,56 + 0;32,40 \cdot \Delta c(\alpha)$, measured in degrees per hour.

Table 9

No.	Nabon.	$\bar{\kappa}$	$c_{\odot}$	α	$c_{☾}$	η
1	718 I	94;58,11°	−2;23°	213;41,47°	+3; 0°	+5;23°
2	II	124; 4,34	−2; 1	239;30,47	+4;30	+6;31
3	III	153;10,57	−1; 7	265;19,47	+5; 1	+6; 8
4	IV	182;17,20	+0; 6	291; 8,47	+4;31	+4;25
5	V	211;23,43	+1;17	316;57,48	+3;12	+1;55
6	VI	240;30, 6	+2; 7	342;46,48	+1;21	−0;46
7	VII	269;36,29	+2;23	8;35,48	−0;41	−3; 4
8	VIII	298;42,52	+2; 2	34;24,48	−2;37	−4;39
9	IX	327;49,15	+1;13	60;13,48	−4; 9	−5;22
10	IX	356;55,38	+0; 7	86; 2,48	−4;57	−5; 4
11	X	26; 2, 1	−1; 0	111;51,48	−4;48	−3;48
12	XI	55; 8,24	−1;55	137;40,48	−3;36	−1;41
13	XII	84;14,47	−2;21	163;29,49	−1;33	+0;48

In order to find the time difference Δt between mean and true oppositions we tabulate $-13\eta/12=-1;5\eta$ and divide these numbers by $v_☾$. The result is Δt measured in hours. Because the mean oppositions are given with sexagesimal fractions of days we change hours to days by means of the factor 0;2,30.

If we add these values of Δt to the julian dates of the mean oppositions given in Table 8 (p. 121) we obtain the julian dates of the true oppositions with respect to Alexandria noon. True oppositions for Babylon midnight are listed in Goldstine, New and Full Moons, and can easily be changed to Alexandria noon and day fractions. The resulting differences Δ between the Almagest and the modern tables are shown in Table 11 and in Fig. 108.

Table 10

No.	$\Delta c(\alpha)$	$\frac{0;32,40}{\Delta c(\alpha)}$	$v_☾$	$-1;5\cdot\eta$	Δt	Nabon.
1	+0;4,20°	+0;2,21°/h	0;35,17°/h	−5;49,55°	− 9;55, 2h	718 I
2	+0;2,20	+0;1,16	34,12	−7; 3,35	−12;23, 7	II
3	−0;0,20	−0;0,11	32,45	−6;38,40	−12;10,23	III
4	−0;2,20	−0;1,16	31,40	−4;47, 5	− 9; 3,57	IV
5	−0;3,50	−0;2, 5	30,51	−2; 4,35	− 4; 2,18	V
6	−0;4,40	−0;2,32	0;30,24	+0;49,50	+ 1;38,21	VI
7	−0;4,40	−0;2,32	30,24	+3;19,28	+ 6;33,25	VII
8	−0;4,10	−0;2,16	30,40	+5; 2,15	+ 9;51,21	VIII
9	−0;2,40	−0;1,27	31,29	+5;48,50	+11; 4,48	IX
10	−0;0,30	−0;0,16	0;32,40	+5;29,20	+10; 4,54	IX
11	+0;1,40	+0;0,54	33,50	+4; 7, 0	+ 7;18, 1	X
12	+0;3,40	+0;2, 0	34,56	+1;49,25	+ 3; 7,57	XI
13	+0;5,20	+0;2,54	35,50	−0;52, 0	− 1;27, 0	XII

Table 11

No.	Mean Oppos.	Δt	True Oppos. Almag.	True Oppos. Goldst.	Δ=Alm.−G.		Nabon.
1	−30 Sept. 4; 5,53	−0;24,43d	−30 Sept. 3;41,10	3;44,18	−0;3, 8d=	−1;15h	718 I
2	Oct. 3;37,43	−0;30,58	Oct. 3; 6,45	3;10, 0	−0;3,15	−1;18	II
3	Nov. 2; 9,33	−0;30,26	Nov. 1;39, 7	1;41, 5	−0;1,58	−0;48	III
4	Dec. 1;41,24	−0;22,40	Dec. 1;18,44	1;18,23	+0;0,21	+0; 8	IV
5	31;13,14	−0;10, 6	31; 3, 8	31; 1,35	+0;1,33	+0;35	V
6	−29 Jan. 29;45, 4	+0; 4, 6	−29 Jan. 29;49,10	29;48,13	+0;0,57	+0;23	VI
7	Feb. 28;16,54	+0;16,24	Feb. 28;33,18	28;33,58	−0;0,40	−0;16	VII
8	Mar. 29;48,44	+0;24,38	Mar. 30;13,22	30;15, 0	−0;1,38	−0;39	VIII
9	Apr. 28;20,34	+0;27,42	Apr. 28;48,16	28;49,30	−0;1,17	−0;31	IX
10	May 27;52,24	+0;25,12	May 28;17,36	28;18,10	−0;0,34	−0;14	IX
11	June 26;24,14	+0;18;15	June 26;42,29	26;42,28	+0;0, 1	0	X
12	July 25;56, 5	+0; 7,50	July 26; 3,55	26; 4,25	−0;0,30	−0;12	XI
13	Aug. 24;27,55	−0; 3,38	Aug. 24;24,17	24;25,43	−0;1,26	−0;34	XII

3. Eclipse Limits

Because of the large apparent diameters of sun and moon eclipses may occur, in particular partial eclipses, even if the longitude of the syzygy is a considerable amount removed from one of the nodes ($\omega'=90$ or 270). It is of great importance for the prediction of eclipses at what distance from the nodes under given conditions eclipses are possible. These "eclipse limits" are easy to determine for lunar eclipses since the amount of obscuration of the lunar surface by the shadow is independent of the location of the observer. The appearance of solar eclipses, on the contrary, depends very much on the position of the observer in relation to the axis of the shadow cone and the determination of limits for solar eclipses therefore involves the theory of parallaxes.

On the basis of previous discussions we accept as given the following apparent radii for sun, moon, and shadow cone (s) at the distance of the moon[1]:

	apogee	perigee	mean dist.	
$r_{☾}$:	0;15,40°	0;17,40°	0;16,40°	
s:	0;40,40°	0;46°	0;43,20°	(1)
$r_{\odot}$:	always	0;15,40°		

In order to simplify matters configurations on the celestial sphere are replaced by corresponding plane configurations. If β is the latitude of the moon Γ at a distance ω from the node D (Fig. 109), $i=5°$ being the inclination of the orbit, we have from $\beta=\omega \sin i$ approximately

$$\omega=11;30\cdot\beta. \tag{2}$$

Under these assumptions the limits for *lunar eclipses* are easily found. Since at conjunction the center of the shadow is located at A, the center of the moon at Γ, the extremal value of β at which the moon contacts the shadow is given by $r_{☾}+s$ at perigee. Hence

$$|\beta|\leqq 0;17,40+0;46=1;3,40°, \tag{3a}$$

or with (2), the extremal distance of a true conjunction from a node:

$$|\omega|\leqq 12;12,10\approx 12;12°. \tag{3b}$$

Ptolemy wants to express eclipse limits in terms of mean longitudes since these appear as first step in all computations. Therefore we have to increase the above amount $|\omega|$ by the greatest possible distance between a mean and a true syzygy. In order to determine this distance x (Fig. 110) between M and T we have to assume that the maximum equation of the moon ($c_{☾}=5;1$)[2] acts in the opposite direction to the maximum equation of the sun ($c_{\odot}=2;23$).[3] The time τ required for the travel from M to T is for the moon given by $(x+c_{☾})/v_{☾}$ and for the sun by $(x-c_{\odot})/v_{\odot}$. From the equality of these two expressions it follows that

$$x=(c_{\odot}v_{☾}+c_{☾}v_{\odot})/(v_{☾}-v_{\odot}).$$

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[1] Cf. above p. 104, (1) and (2); p. 105, (3) and (4); p. 109, (2).

[2] Above p. 80.

[3] Above p. 59.

If one substitutes for the velocities the round values $v_{☾}=13°$, $v_{\odot}=1°$ one finds for x exactly

$$x=(2;23\cdot 13+5;1)/12=3°. \tag{4}$$

Thus a lunar eclipse is only possible if the mean conjunction has a distance $|\bar{\omega}|\leqq 15;12$ from a node. Since Ptolemy reckons the argument of latitude ω' from the northernmost point of the lunar orbit he finds for the mean moon the following ecliptic limits (cf. Fig. 111a):

$$\left.\begin{array}{ll} 270-15;12\leqq\bar{\omega}'\leqq 270+15;12 & \text{at an ascending node}\\ 90-15;12\leqq\bar{\omega}'\leqq \ \ 90+15;12 & \text{at a descending node}\end{array}\right. \tag{5a}$$

or

$$\left.\begin{array}{ll} 254;48\leqq\bar{\omega}'\leqq 285;12 & \text{at an ascending node}\\ 74;48\leqq\bar{\omega}'\leqq 105;12 & \text{at a descending node.}\end{array}\right. \tag{5b}$$

For *solar eclipses* the obvious condition $|\beta|\leqq r_{\odot}+r_{☾\,\max}$, i.e.

$$|\beta|\leqq 0;15,40+0;17,40=0;33,20° \tag{6}$$

is to be greatly modified by parallax.

Ptolemy restricts himself to localities within the zone of the seven climates, i.e. to latitudes between $\varphi=16\ 1/2°$ (Meroe) and $\varphi=48\ 1/2°$ (Mouth of the Borysthenes). For these boundaries Ptolemy gives, without proof, the following extremal values for the combined effect of lunar and solar parallax[4]

$$\begin{array}{lllll} \text{Meroe:} & p_\beta=0;8° & \text{to N} & p_\lambda=0;30° & \text{for conj. in ♌ and ♊}\\ \text{Borysth.:} & p_\beta=0;58° & \text{to S} & p_\lambda=0;15° & \text{for conj. in ♏ and ♓.}\end{array} \tag{7}$$

Consequently one has to distinguish two extremal cases in which parallax can produce an apparent distance of 0;33,20° between the centers of sun and moon.

Case 1. The moon is south of the ecliptic, the parallax has its extremal northern latitudinal component (cf. Fig. 112: Δ is the true position of the moon, E the apparent one, A the apparent position of the sun). Then we find with (6) and (7) for the latitude of the moon

$$-\beta\approx \mathrm{A}\Gamma=0;33,20+0;8\approx 0;41°$$

as the extremal southern latitude at which parallax can still cause apparent contact between sun and moon.

According to (2), p. 125 the point Γ of latitude β has a nodal distance of $11;30\cdot\beta$, hence in the present case $11;30\cdot 0;41\approx 7;52°$. To this must be added the longitudinal component $p_\lambda=0;30$ in order to find the distance of the true moon Δ from the node. Hence no eclipse is possible if the moon is to the south of the ecliptic and if its true nodal distance is

$$\omega>8;22°. \tag{8}$$

Case 2. The moon is to the north of the ecliptic (Fig. 113). Then we find with (6) and (7) for the latitude at apparent contact

$$\beta\approx \mathrm{A}\Gamma=0;33,20+0;58\approx 1;31°.$$

Hence Γ has the distance $11;30\cdot 1;31\approx 17;26°$ from the node. Therefore $17;26+p_\lambda$ gives as limiting nodal distance of the true moon

$$\omega>17;41° \tag{9}$$

when north of the ecliptic.

[4] Cf. our discussion of these statements below p. 127.

Ptolemy again transposes these limiting conditions into conditions for the mean moon by adding in (8) and (9) the maximum distance of 3° between mean and true conjunction (cf. (4), p. 126). Thus for the mean moon we have as necessary condition for a solar eclipse (cf. Fig. 111 b):

$$\begin{aligned} 0 &\leqq |\bar{\omega}| \leqq 11;22^\circ \quad \text{if} \quad \beta<0 \\ 0 &\leqq |\bar{\omega}| \leqq 20;41^\circ \quad \text{if} \quad \beta>0. \end{aligned} \tag{10}$$

Ptolemy furthermore expresses these conditions in terms of the argument of latitude ω'. At an ascending node ($\omega'=270°$) the latitude of the moon changes from negative to positive values. Thus (10) provides the limits

$$270-11;22 \leqq \bar{\omega}' \leqq 270+20;41.$$

Similarly for a descending node ($\omega'=90$):

$$90-20;41 \leqq \bar{\omega}' \leqq 90+11;22.$$

Thus we have found that solar eclipses within the zone

$$16\ 1/2^\circ \leqq \varphi \leqq 48\ 1/2^\circ$$

are only possible if the argument of latitude $\bar{\omega}'$ of the mean moon lies within the following limits

$$\begin{aligned} 258;38^\circ &\leqq \bar{\omega}' \leqq 290;41^\circ \quad \text{at an ascending node} \\ 69;19 \ &\leqq \bar{\omega}' \leqq 101;22 \quad \text{at a descending node.} \end{aligned} \tag{11}$$

Against Ptolemy's derivation of these limits two objections can be raised. The first is the trivial one that his arbitrary way of rounding made the limits (8) and (9) too small by 0;1,29° and 0;8,25° respectively.

The second is much more serious: Ptolemy is wrong in stating (above p. 126 (7)) that $p_\lambda=0;30°$ and $p_\lambda=0;15°$ are the greatest longitudinal components of the parallax for locations between Meroe and the Borysthenes. It is difficult to explain how he arrived at this result. Pappus in his commentary,[5] as usual, is no help. He only supplies the numerical steps and remarks happily that the results agree with the Handy Tables but he suggests no motivation for the choice of the essential data and he raises no objections.

Ptolemy begins his discussion with the remark that the maximum latitudinal parallax, acting in northerly direction in Meroe, is 0;8° while at the Borysthenes the southerly parallax reaches 0;58°. This is indeed correct; Pappus gives the numerical details: one considers the situation (a) when the sun and the moon are at ♋0° and culminate at Meroe, and (b) when the conjunction occurs at ♑0°, culminating for the latitude of the Borysthenes. In both cases $p=p_\beta$ and $p_\lambda=0$. In the first case the zenith distance is $\zeta=\varphi-\varepsilon=16;27-23;51=-7;24$, in the second $\zeta=\varphi+\varepsilon=48;32+23;51=72;23$. The tables of parallax (Alm. V, 18) then give for the relative parallax $p_☾-p_\odot=p$

$$p=p_\beta=c_3(\zeta)+c_4(\zeta)-p_\odot(\zeta)=\begin{cases}0;\ 7,59\approx 0;8^\circ\\ 0;58,29\approx 0;58^\circ\end{cases} \tag{12}$$

respectively, in agreement with Ptolemy.

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[5] Pappus, Comm., ed. Rome p. 194–197.

The choice of the solstices in culmination gives to the ecliptic extremal positions: ♋0° as much as possible to the north of the zenith thus producing a maximal northerly parallax; ♑0° as low as possible hence causing an extremal southerly parallax. Neither Ptolemy nor Pappus says why this situation is of interest but it seems to be clear from what follows that they knew an important theorem in the theory of parallaxes: for points on (or very near to) the ecliptic the latitudinal component p_β is constant if only the zenith distance of the highest point of the ecliptic remains fixed.[6]

On the basis of this theorem Ptolemy is able to use the values (12) for p_β for any $\lambda_{☾}=\lambda_{\odot}$ as long as ♋0°, respectively ♑0°, are culminating. It must be for this reason that Pappus computes the components of parallax for $\lambda=$♌0° (or ♊0°) 2^h before (or after) noon at Meroe. Since these points have a distance of $\pm 30°$ from ♋0° it is clear that 2^h before (or after) noon ♋0° must be near culmination. Pappus indeed finds that ♋$0\pm 2;4°$ is culminating, which is certainly close enough for the application of the above theorem. Similarly a conjunction at ♓0°, or ♏0°, $\pm 4^h$ from noon brings ♑0° near culmination; Pappus finds that ♑$0\pm 2°$ culminates. Hence Ptolemy's maxima for p_β remain valid for all conjunctions if only the solstitial points are near culmination.

Pappus also computes the values for p_λ for the given longitudes and for moments $\pm 2^h$ and $\pm 4^h$ before noon and confirms Ptolemy's figures within the usual roundings.[7] But there is no reason for selecting just these longitudes. Particularly in the case of Meroe, where the ecliptic stands almost vertical to the horizon, it is evident that one can increase p_λ by going as near as possible to the horizon where p_λ must be almost the same as p_0, the horizontal parallax. And since

$$\omega=(p_\beta+0;33,20)\cdot 11;30+p_\lambda \tag{13}$$

represents the critical nodal distance, an increase in p_λ applies linearly to the ecliptic limits.

The following table shows the numerical results, based on the tables in Alm. II, 13 for the zenith distances and for the angles between ecliptic and circles of altitude and on Alm. V, 18 for the parallaxes.

	λ	before noon	p_β	p_λ	ω	Ptolemy
Meroe	♌0°	2^h	0;7,57	0;28,43	8;23,29	8;22
	♍0	4	0;7,51	0;51,46	8;45,23	
	♎0	6	0;7,52	1; 0,30	8;54,18	
Borys.	♒0°	2^h	0;58,30	0; 7,43	17;43,48	
	♓0	4	0;58,21	0;15, 3	17;49,25	17;41
	♈0	6	0;58, 8	0;18,28	17;50,20	

The column p_β shows how nearly $p_\beta=$const. along the ecliptic. For p_λ, however, there is no good reason to select the first line for Meroe and the second for the

[6] For a proof of this theorem and further discussion cf. Neugebauer, Al-Khwar., p. 122f.

[7] He finds $p_\lambda=0;28,43$ (instead of Ptolemy's 0;30) and 0;15,30 (for Ptolemy's 0;15). Correct would be 0;28,43,56 and 0;15,3(!), respectively.

Borysthenes. In both cases Ptolemy should have taken the position near sunrise (or sunset). Hence his limits for ω are 0;32° and 0;9° in error.[8] In practice this is of very little significance but it is not clear why Ptolemy introduced arbitrary positions which obviously could not represent the extremal conditions he was looking for.

4. Intervals between Eclipses

Very little theoretical insight is required to explain the empirical fact that lunar eclipses do not occur at every full moon but are usually six months apart. During one synodic month the argument of latitude increases by about 30° (tables: Alm. VI, 3); thus a lunar latitude near zero at one syzygy will be followed by a latitude of about 2;30° at the next (tables: Alm. V, 8 col. 7), thus well outside the eclipse limits.[1] On the other hand six synodic months correspond to about 184° increment in the argument of latitude. Thus an eclipse which occurred at one node can be followed six months later by an eclipse at the opposite node.

Ptolemy, in Chap. VI, 6 of the Almagest, gives much more than accurate proofs of these well-known facts. He investigates in detail the question if, and under what conditions, intervals of five, six, or seven months are possible, taking into consideration for solar eclipses the effect of parallax for all geographical latitudes within the "inhabited" part of the northern hemisphere. This is the first step toward the modern method of dealing with solar eclipses, i.e. to discuss their whole paths across the surface of the earth.

We shall now follow Ptolemy in the investigation of the following six cases of possible intervals between consecutive eclipses:

A: 6 months intervals for solar and for lunar eclipses

B and **C**: 5 and 7 months intervals for lunar eclipses

D and **E**: 5 and 7 months intervals for solar eclipses

F: 1 month interval for solar eclipses.

Pliny, in one of the few clear passages in his astronomical chapters (Nat. Hist. II, 57), ascribes to Hipparchus the discovery that lunar eclipses are possible with an interval of 5 months, solar eclipses after 7 months and also with only one month interval, but then not for the same locality. Since these statements imply the essential results of the following investigations we can be sure that we are dealing here with a chapter from Hipparchus' theory of eclipses.

A. Both for solar and lunar eclipses intervals of six synodic months are possible.

The preceding discussion has shown that the ecliptic limits both for solar and lunar eclipses are wider than $|\bar{\omega}| \leq 10°$ (cf. (5) p. 126 and (10) p. 127). If therefore two syzygies are separated by a $\Delta\bar{\omega}'$ which is greater than $180-20°$, but does not exceed $180+20°$, then eclipses are possible at both ends. But the tables in VI, 3 show that the argument of latitude during six mean synodic months increases by $\Delta\bar{\omega}' = 184;1,25°$. Hence both solar and lunar eclipses can occur at six months intervals.

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[8] This includes the rounding errors mentioned on p. 127.

[1] For Ptolemy's much refined investigation of consecutive syzygies cf. below p. 133, F.

B. An interval of five synodic months is possible for lunar eclipses, provided that the total length of these five months is as great as possible.

From the tables in VI, 3 one finds for 5 mean synodic months:

$$\begin{array}{ll} \text{increase of long. of mean sun and moon:} & \Delta\bar{\lambda}=145;32^\circ \\ \text{increase of mean anomaly of the moon:} & \Delta\bar{\alpha}=129;\ 5 \\ \text{increase of argum. of lat. of mean moon:} & \Delta\bar{\omega}'=153;21 \end{array} \qquad (1)$$

The corresponding true synodic months will be as long as possible if the greatest possible solar motion combines with the smallest possible lunar motion. The solar motion will be a maximum if the sun is at its perigee at the midpoint of the interval $\Delta\bar{\lambda}$. Thus we assume that the eccentric anomaly $\bar{\kappa}$ of the sun is at the first syzygy $180-72;46=107;14^\circ$, at the last syzygy $180+72;46=252;46^\circ$. For these anomalies one finds in Alm. III, 6 equations of $\pm 2;19^\circ$, respectively. Thus the total solar travel will be $\Delta\lambda_\odot=\Delta\bar{\lambda}+4;38^\circ$.

The lunar motion during 5 months will be a minimum if the moon is at its perigee at the middle of the 3rd month because in this case the moon will be as near as possible to the apogee at the beginning of the first and at the end of the fifth month (cf. Fig. 114). During this time the epicyclic anomaly of the moon has gained $\Delta\bar{\alpha}=129;5$ over 5 complete rotations. Therefore the moon was at the beginning 64;32° before the apogee, at the end equally much after it. According to Alm. IV, 10 the corresponding equations are about $\pm 4;20^\circ$. Hence the total travel of the moon was $\Delta\lambda_☾=\Delta\bar{\lambda}-8;40^\circ$.

Thus we see that under extremal conditions the sun can gain in 5 months an elongation of $4;38+8;40=13;18^\circ$ over the moon. To cover this distance requires about $13;18/12=1;6^d$ during which time the sun moves about 1;6°. This shows that after 5 months the two true oppositions will be

$$\Delta\bar{\lambda}+4;38+1;6=\Delta\bar{\lambda}+5;44^\circ$$

apart. Practically the same gain of 5;44° will apply to the argument of latitude; hence

$$\Delta\omega'=153;21+5;44=159;5^\circ \qquad (2)$$

is the change of the argument of latitude between the two true oppositions.

As we have seen the moon is at the two endpoints about 65° distant from the apogee of the epicycle. Its distance from the earth is therefore only a little more than the mean distance and we assume for the apparent distance between moon and shadow its mean value $r_☾+s=1^\circ$ (cf. (1) p. 125). Thus eclipses are possible if $|\beta|\leqq 1^\circ$ or for distances from the nodes $|\omega|\leqq 11;30$ (cf. (2) p. 125). Two eclipses are possible under the condition that the increment of the argument of latitude is at least

$$\Delta\omega'=180-2\cdot 11;30=157^\circ.$$

But we have shown that under extremal conditions $\Delta\omega'=159;5$ (cf. (2)). Therefore, under very special conditions (cf. Fig. 115), two lunar eclipses can occur 5 months apart.[2] Ordinarily, however, the intervals will be either 6 months or 11 months.[3]

[2] Example: Oppolzer, Canon, Nos. 2271 and 2272 (A.D. 265 Oct. 12 and 266 March 8).

[3] We ignore here, as always in these discussions, the influence of geographical longitude which can exclude eclipses because of the time of the day.

C. An interval of seven months is excluded for lunar eclipses.

The proof follows the same lines as in the previous case, except for the fact that we now have to select an interval as short as possible.

For 7 synodic months we find in Alm. VI, 3:

$$\Delta\bar{\lambda}=203;45 \qquad \Delta\bar{\alpha}=180;43 \qquad \Delta\bar{\omega}'=214;42. \tag{3}$$

To make the solar motion as slow as possible we have to assume that the midpoint of the arc $\Delta\bar{\lambda}$ coincides with the solar apogee. Then one finds for the equations at $\bar{\kappa}=\pm 101;52$ the values $\mp 2;21$ hence a true motion $\Delta\lambda_{\odot}=\Delta\bar{\lambda}-4;42$. The apogee of the moon should occur at the middle of the fourth month such that the anomalies at the endpoints have a distance of $\pm 1/2\,\Delta\bar{\alpha}=\pm 90;21°$ from the perigee of the epicycle, causing an equation of $\pm 4;59°$. Consequently we have a lunar motion of $\Delta\lambda_{☾}=\Delta\bar{\lambda}+9;58°$ and thus an elongation of $4;42+9;58=14;40°$. The corresponding solar motion is about $14;40/12=1;13°$ such that we find for the distance of the true oppositions

$$\Delta\bar{\lambda}-(4;42+1;13)=\Delta\bar{\lambda}-5;55°.$$

The same change applies to the argument of latitude, thus

$$\Delta\omega'=\Delta\bar{\omega}'-5;55=208;47°. \tag{4}$$

Since the moon at anomaly ± 90 is at mean distance from the earth we have given the eclipse limit of $|\beta|\leqq 1°$ or $|\omega|\leqq 11;30$. Thus the maximum distance for two eclipses 7 months apart would be

$$\Delta\omega'=180+2\cdot 11;30=203°.$$

Our result (4) indicates that even at extremal conditions two syzygies 7 months apart cannot both result in lunar eclipses (cf. Fig. 115).

D. An interval of five months is possible for two solar eclipses.

As in the case of lunar eclipses (cf. **B**) we assume the 5 months to be of greatest possible total length; then the argument of latitude between the two true conjunctions is given by

$$\Delta\omega'=159;5° \tag{2}$$

(p. 130). Since the moon is not far from its mean distance we have for $r_{\odot}+r_{☾}$ (cf. p. 125) the value $0;32,20°$ and therefore for the nodal distance an ecliptic limit of $|\omega|\leqq 0;32,20\cdot 11;30=6;12°$ (because of (2), p. 125). Hence two eclipses, 5 months apart, should be separated by at least $\Delta\omega'=180-2\cdot 6;12=167;36$. Thus our two true conjunctions show a deficit of $167;36-159;5=8;31°$. Only parallax can make up for it.

An arc of $8;31°$ on the lunar orbit near the nodes corresponds to a latitudinal difference $\Delta\beta=8;31/11;30\approx 0;45°$. If the sum of the latitudinal parallaxes at the two conjunctions reaches this amount contact between the lunar disk and the sun is again established.

We have seen (p. 130) that under extremal conditions 5 lunar months can exceed 5 mean lunations by about $1;6^{d}=1^{d}\,2;24^{h}$. Since 5 mean synodic months are about $147;39,10^{d}=147^{d}\,15;40^{h}$ long our interval reaches $148^{d}\,18^{h}$ or $149^{d}-6^{h}$. In order to obtain maximum parallax we assume that the first conjunction occurs near sunset. Then the second conjunction, 5 months later, will occur at noon. We

had to assume (p. 130) that the sun was at its perigee, ♐5;30, at the midpoint between the two syzygies which were separated by an arc $\Delta\lambda_{\odot} = \Delta\bar{\lambda} + 4;38 = 150;10$. Thus the two conjunctions would be located approximately in ♐5 ± 75, i.e. in

$$\lambda_1 = ♍20 \qquad \lambda_2 = ♒20$$

respectively.

The combined latitudinal parallax at these two points should reach 0;45°. Since the interval is considerably shorter than 180° we know that the moon is at both endpoints at the same side of the ecliptic. If the moon is to the south of the ecliptic the parallax should be directed to the north in order to move the moon's apparent position nearer to the sun. Since λ_1 and λ_2 are near to the equator the moon cannot be far from it and we see that under these conditions nowhere on the northern hemisphere is an appreciable northern parallax possible. Thus for the moon south of the ecliptic 5-month intervals for solar eclipses are excluded for the oikoumene.

If the moon is to the north of the ecliptic we require parallaxes of southerly direction. Ptolemy gives, without presenting the computations which lead to them,[4] the following values for the latitudinal parallaxes[5]

	at: $\varphi = 0$	$\varphi = 8;25$
$\lambda_1 = ♍20$ setting:	0;22°	0;27°
$\lambda_2 = ♒20$ culminating:	0;14	0;22.

This shows that the sum of the parallaxes at the equator is only 0;36° and therefore does not suffice to make two eclipses possible. But in the next zone (longest daylight $12\,1/2^h$) the required minimum value of 0;45° is exceeded and this is increasingly so for places farther north. Thus everywhere in the oikoumene solar eclipses 5 months apart are possible.[6]

E. Intervals of seven months are possible for solar eclipses.

Under the same assumptions as in the corresponding case for lunar eclipses (**C**, p. 131) we have for the argument of latitude between the true conjunctions

$$\Delta\omega' = 208;47°. \tag{4}$$

As before (p. 131) eclipses require a distance from the nodes $|\omega| \leq 6;12$ thus a maximum $\Delta\omega' = 180 + 2 \cdot 6;12 = 192;24$. The value (4) exceeds this limit by $208;47 - 192;24 = 16;23°$ to which corresponds a latitudinal difference of $16;23/11;30 \approx 1;25°$. This deficit should be compensated by the sum of the latitudinal parallaxes at both endpoints.

As we have shown (p. 131) the shortest total of 7 lunar months is about $1;13^d \approx 1^d\,5^h$ shorter than 7 mean synodic months, i.e. $206;43^d \approx 206^d\,17^h$. Thus the shortest interval possible amounts to $205^d\,12^h$, and we must assume that the conjunctions occur at sunrise and sunset, respectively. Since the distance between the true conjunctions had been found to be $\Delta\lambda - 5;55 = 197;50$ (p. 131) with the apogee A = ♊5;30 at its midpoint we obtain for the approximate longitudes of the conjunctions

$$\lambda_1 = ♒26 \qquad \lambda_2 = ♍15.$$

[4] Again supplemented in Pappus' Commentary (ed. Rome, p. 226 to 231).

[5] Of course, as always, lunar minus solar parallax.

[6] Example: Oppolzer, Canon, Nos. 5356 and 5357 (A.D. 1049 March 6 and August 1); cf. also p. 133, n. 7.

As before the moon must be at both endpoints at the same side of the ecliptic. This rules out southerly positions since for locations on the northern hemisphere northerly parallaxes totalling 1;25° are excluded, because the greatest northern parallax even for $\varphi = 0°$ is only 0;23°. If, however, the moon is located to the north of the ecliptic the latitudinal parallax to the south reaches for positions near λ_1 and λ_2 at the latitude of Rhodes ($\varphi = 36$) the amount of 0;46° at each endpoint. Consequently the combined effect of 1;32° exceeds the limit of 1;25° and two eclipses 7 months apart become possible for $\varphi = 36°$ and farther north.[7]

F. Solar eclipses one month apart are excluded for localities of the oikoumene.

We assume a combination of the following most favorable conditions even if in fact they cannot exist simultaneously:

(a) the moon is at the perigee of the epicycle, thus showing maximum parallax,

(b) shortest synodic month,

(c) positions in the ecliptic and hours of daytime such that parallaxes are maximal.

For one mean synodic month one has (Alm. VI, 3):

$$\Delta\bar{\lambda} = 29;6 \qquad \Delta\bar{\alpha} = 25;49 \qquad \Delta\bar{\omega}' = 30;40. \tag{5}$$

Because of (b) the solar apogee must be the midpoint of $\Delta\bar{\lambda}$; therefore the anomalies of the endpoints are $\bar{\kappa} = \pm 14;33$ and therefore the equations $c = \pm 0;34$ (Alm. III, 6). Hence $\Delta\lambda_\odot = \Delta\bar{\lambda} - 1;8°$.

Because of assumption (b) the motion of the moon must begin with an epicyclic anomaly $180 - 1/2\,\Delta\bar{\alpha} \approx 180 - 12;55$ and end at $180 + 12;55$ giving rise to equations $\pm 1;14$ (Alm. IV, 10). Hence $\Delta\lambda_{☾} = \Delta\bar{\lambda} + 2;28°$.

In a shortest synodic month the consecutive true conjunctions are therefore $1;8 + 2;28 = 3;36°$ nearer to each other than the mean conjunctions. This corresponds to a time interval of $3;36/12 = 0;18^d$ during which the sun would move about 0;18°. Thus during the shortest possible month the sun moves only $\Delta\bar{\lambda} - 0;18° - 1;8° = \Delta\bar{\lambda} - 1;26°$ and therefore also the argument of latitude will reach only $\Delta\omega' = \Delta\bar{\omega}' - 1;26 = 29;14°$.

If two eclipses should occur this arc would have the node as its midpoint, i.e. the endpoints would have the distance $|\omega| = 14;37°$ from the node and therefore latitudes $|\beta| = 14;37/11;30 \approx 1;16,16$ (p. 125 (2)). Since we assume the moon at perigee we find as eclipse limits from p. 125 (1) $|\beta| \leqq r_\odot + r_{☾} = 0;33,20°$. Thus parallax would have to account for twice the excess in latitude i.e. for $2(1;16 - 0;33) = 1;26°$.

Because the moon is at opposite sides of the ecliptic at the two conjunctions the parallaxes should also be directed in opposite directions in order to eliminate the excess of latitudinal change. For localities on the equator parallaxes in opposite directions do exist but they remain below $\pm 0;25°$ and hence could only compensate about 0;50°. Even if one combines a locality on the equator with a locality on the northern boundary of the oikumene ($\varphi \approx 48\ 1/2$) one finds only a combined effect of 1;25°, still not enough to obtain apparent contact between sun and moon. Thus solar eclipses at consecutive conjunctions are excluded for all localities within the oikoumene.

[7] Example: Oppolzer, Canon, Nos. 4678 and 4681 (A.D. 752 Jan. 21 and August 14). Between these two eclipses, however, occur two more eclipses (Febr. 20 and July 15) such that we have a sequence of four eclipses with intervals 1 month, 5 months, 1 month, respectively.

Ptolemy also compares places on different hemispheres which shows that he knew that for the earth as a whole, solar eclipses one month apart are possible.[8] On the other hand lunar eclipses can never be only one month apart since the eclipse limits, even for the moon at perigee, are $|\beta| \leqq r_{☾} + s = 1;3,36°$. But we have just seen that under most favorable conditions the latitude of the moon at the two syzygies is $\pm 1;16°$ thus well outside the eclipse limits.

5. Tables (VI, 8)

In Almagest VI, 8 we find two main tables (here called I and II) which allow us to find for a given argument of latitude ω' the magnitude m and the duration of an eclipse, Table I for solar eclipses, Table II for lunar eclipses. Each one of these tables distinguishes two cases: a) moon at apogee, b) at perigee of its epicycle. Table III gives the necessary coefficients of interpolation for values of the epicyclic anomaly α which lie between 0 and 180. A Table IV has for argument *linear digits*, i.e. twelfths of the apparent diameter of the eclipsed body. Tabulated are the corresponding *area digits* (for sun and moon, respectively) i.e. twelfths of the area of the apparent disk of the eclipsed body.[1]

If one wants to find the circumstances of an eclipse (magnitude and duration) one can assume the true argument of latitude ω' known (corrected for parallax in a solar eclipse) and the fact established that ω' lies within the ecliptic limits.

With ω' as argument one finds in column 3 of Table I or II, respectively the magnitudes m_a and m_p for apogee ($\alpha = 0$) and perigee ($\alpha = 180$). In Table III can be found the coefficient $q(\alpha)$ which belongs to the given epicyclic anomaly of the moon.[2] Then

$$m = m_a + (m_p - m_a)\, q(\alpha) \tag{1}$$

is the eclipse magnitude, expressed in linear digits.

A similar procedure applies to the numbers which determine the duration of the phases. With the same coefficient $q(\alpha)$ one forms

$$\eta = \eta_a + (\eta_p - \eta_a)\, q(\alpha). \tag{2}$$

In the case of solar eclipses η, measured in degrees, represents the elongation of the moon at first contact with respect to the middle of the eclipse.[3] In the case of lunar eclipses one may have two values for η. One (column 4) for all values of ω', within the eclipse limits, with the same meaning as for solar eclipses. A second (column 5) for values of ω' near the nodes such that the eclipse is total; in this case η gives the elongation of the moon at the beginning of totality, again with respect to the middle of the eclipse.

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[8] Examples passim in Oppolzer, Canon; e.g. Nos. 6201 and 6202, or 6501 and 6502. Cf. also p. 133, n. 7.

[1] Cf. below p. 140.

[2] Exactly the same numerical values are found as coefficients of interpolation in the table of parallaxes (Alm. V, 18) column 7 though associated with $\alpha/2$ instead of α; cf. above p. 113/114.

[3] For certain values of ω' near the ecliptic limits one will not find entries for m_p or η_p. For example for solar eclipses the values of m_a begin at $\omega' = 84;0$, of m_p at $\omega' = 83;36$ (in both cases $m = 0$). Thus, e.g. for $\omega' = 83;45$, one has to substitute in (1) or (2) the values $m_p = 0$, $\eta_p = 0$.

Since η is the elongation of the moon from the center of the sun at the moment of the beginning of a (partial or total) phase one estimates that the moon has to travel until the middle of the eclipse (elongation zero) a longitudinal arc $\Delta\lambda = \frac{13}{12}\eta$. Therefore the time from the beginning of a phase to the middle of the eclipse can be found from η by means of

$$t = 13\eta/12 v_{☾} \tag{3}$$

where $v_{☾}$ is the hourly velocity of the moon under the given circumstances i.e. for given anomaly α. How $v_{☾}$ can be computed has been explained in connection with the determination of the moments of the true syzygies.[4] The same time t is assumed for the duration of the corresponding phase from the middle of the eclipse to its end. The time of the middle of the eclipse is considered to be known from the time of the true conjunction (including parallax for solar eclipses); by adding and subtracting t one obtains the time for each phase. This completes the computation of the circumstances of an eclipse and we can turn to the discussion of the structure of the tables.

Magnitudes. If β_{max} is the extremal (apparent) latitude of the moon at which the disk of the moon touches the disk of the sun or the circle of the shadow then the corresponding nodal distance is, as usual,[5] assumed to be

$$|\omega| = \beta_{max} \cdot 11;30 \tag{1}$$

to which belong as limits for the argument of latitude ω' the values

$$\omega' = \left.\begin{matrix} 90 \\ 270 \end{matrix}\right\} \pm |\omega|. \tag{2}$$

The magnitude $m = 12$ is reached when the lunar latitude has decreased from β_{max} by the amount $d = d_{\odot}$ in the case of a solar eclipse, or $d = d_{☾}$ for a lunar eclipse. A change in latitude of $d/12$ causes a change of 1 digit for the eclipse magnitude. The corresponding change in the argument of latitude is

$$\Delta\omega = 11;30 \cdot d/12 = d \cdot 0;57,30 \qquad d = \begin{cases} d_{\odot} & \text{solar ecl.} \\ d_{☾} & \text{lunar ecl.} \end{cases} \tag{3}$$

Consequently the greatest possible magnitude is

$$m_{max} = |\omega|/\Delta\omega = 12\beta_{max}/d \tag{4}$$

where $|\omega|$ is defined by (1).

If we call "*immersion*" the angular distance μ from the rim of the eclipsing circle (cf. Figs. 116 and 117) to the point nearest to the center at the middle of the eclipse then the "*magnitude*" m measured in linear digits is related to μ by

$$m = 12\mu/d = 6\mu/r \qquad d = \begin{cases} d_{\odot} \\ d_{☾} \end{cases} \quad r = \begin{cases} r_{\odot} & \text{solar ecl.} \\ r_{☾} & \text{lunar ecl.} \end{cases} \tag{5}$$

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[4] Above p. 122.

[5] Above p. 125, (2).

The parameters obtained in this way for the two types of eclipses and for the two extremal positions of the moon on its epicycle are shown in our Table 12.[6]

Table 12

	$\odot$-eclipses		☾-eclipses		
	apogee	perigee	apogee	perigee	
$\beta_{max}=r_{\odot}+r_{☾}$	0;31,20°	0;33,20°	0;56,24°	1;3,36°	$\beta_{max}=r_{☾}+s$
$\lvert\omega\rvert=11;30\cdot\beta_{max}$	6°	6;24°	10;48°	12;12°	$\lvert\omega\rvert=11;30\cdot\beta_{max}$
limits for ω'	$\left.\begin{matrix}90\\270\end{matrix}\right\}\pm 6°$	$\left.\begin{matrix}90\\270\end{matrix}\right\}\pm 6;24°$	$\left.\begin{matrix}90\\270\end{matrix}\right\}\pm 10;48°$	$\left.\begin{matrix}90\\270\end{matrix}\right\}\pm 12;12°$	limits for ω'
$m=12{:}\ \Delta\beta=d_{\odot}$	0;31,20°			0;35,20°	$m=12{:}\ \Delta\beta=d_{☾}$
$m=1{:}\ \Delta\omega=11;30\cdot d_{\odot}/12$	0;30°			0;34°	$m=1{:}\ \Delta\omega=11;30\cdot d_{☾}/12$
$m_{max}=\lvert\omega\rvert/\Delta\omega$	12	12;48	21;36	21;32	$m_{max}=\lvert\omega\rvert/\Delta\omega$

The two main tables in Alm. VI, 8 show for the arguments ω' exactly the limits given in our Table 12. For solar eclipses and for lunar eclipses at apogee the arguments proceed in steps of 0;30°, for lunar eclipses at perigee in steps of 0;34°. The result of this arrangement is that the tabulated magnitudes proceed in all cases in steps of 1. In other words the independent variable in the Tables I and II of Alm. VI, 8 is in fact not the argument of latitude but the eclipse magnitude in linear digits. This is by no means convenient because ω' is the primarily given quantity and not m; consequently Ptolemy's arrangement leads to more numerous and more complicated interpolations than necessary.

Phases. In order to describe conveniently the determination of the elongations from the beginning of a partial or total eclipse to its middle we use the following notation

$$\begin{array}{lll} & \text{solar ecl.} & \text{lunar ecl.} \\ a: & r_{\odot} & s \\ b: & r_{☾} & r_{☾} \\ m=12: & 2r_{\odot} & 2r_{☾} \\ \text{immersion } \mu: & mr_{\odot}/6 & mr_{☾}/6 \qquad \text{cf. (5).} \end{array} \tag{6}$$

Then we find from Fig. 116 for the elongation c at the beginning of the partial phase

$$c=\sqrt{\Sigma A^2-\Sigma B^2} \qquad \Sigma A=a+b \qquad \Sigma B=a+b-\mu. \tag{7}$$

If we substitute in (6) for m the consecutive integers from 0 to 12 for solar eclipses, from 0 to 21 for lunar eclipses[7] we obtain the values of μ needed in (7). For a and b one has to take once the values for the radii at apogee, once at perigee, listed as β_{max} in Table 12. The resulting values of c are tabulated in column 4 of the two first tables in Alm. VI, 8.

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[6] Using the values from (1), p. 125 but with small roundings in the results.

[7] Cf. the values of m_{max} given in Table 12.

Since $r_\odot = r_☾$ at apogee Ptolemy excludes annular solar eclipses. At perigee one has only $r_☾ - r_\odot = 0;2°$ thus no special column for the total phase is given.

For total lunar eclipses one sees from Fig. 117 that the elongation τ from the beginning of totality to the middle of the eclipse is given by

$$\tau = \Gamma\mathrm{B} = \sqrt{\Sigma\Gamma^2 - \Sigma\mathrm{B}^2}$$

and for the partial phase ("incidence"[8])

$$\tau' = \mathrm{A}\Gamma = \mathrm{AB} - \Gamma\mathrm{B} = \sqrt{\Sigma\mathrm{A}^2 - \Sigma\mathrm{B}^2} - \tau.$$

All these quantities can be computed since

$$\Sigma\mathrm{A} = r_☾ + s \qquad \Sigma\Gamma = s - r_☾ \qquad \Sigma\mathrm{B} = s - (\mu - r_☾)$$

and μ is again found from (5): $\mu = m r_☾/6.$

The resulting values of τ' and τ are tabulated in columns 4 and 5, respectively, of the Table II in Alm. VI, 8.

Remark. The elongations τ' and τ of the partial and of the total phase, respectively, can be found very simply by a graphical method (Fig. 118) which is useful for the checking of eclipse tables in general.

Let $s = \mathrm{AB}$ be the radius of the circle of the shadow,[9] $r = \mathrm{BC} = \mathrm{BD}$ the radius of the moon and DE, perpendicular to DA, the direction of the lunar orbit. The circle of radius $\mathrm{AD} = s + r$ is the locus for the center of the moon at first contact whereas the circle with radius $\mathrm{AC} = s - r$ contains the center of the moon at beginning of totality.

If, for a specific eclipse, the lunar orbit has the position FGH then GH defines the duration of the partial phase.[10] It reaches its maximum for the position KIC of the orbit. Thereafter the partial phase decreases to a relative minimum $\mathrm{OP} = 2r$ while the total phase increases from zero at C to its maximum $\mathrm{PA} = s - r$ at centrality.

We now choose coordinates (cf. Fig. 119) such that the eclipse limit $\omega = \mathrm{DE}$ in Fig. 118 is represented as abscissa by the same length as $\mathrm{AD} = r + s$ at the ordinates. This can be done graphically by inclining DE in Fig. 118 at 45° to the ecliptic AE. Then the curve which represents the duration of the partial phase is represented (cf. Fig. 119) by a circular arc DK with center P, where $\mathrm{DC} = 2r$, followed by an arc KO whose points L have ordinates LM directly measurable in Fig. 118. The duration of the total phase is represented as function of the nodal distance ω by a quarter of a circle of radius $s - r$ and center P. The magnitudes m are a linear function of ω, zero at D, 12 at C. If we therefore draw a straight line through the points D and K then $\mathrm{CK} = 12$ gives the scale for linear digits as function of ω.

Fig. 120 shows the graphs which correspond to the values of r and s at apogee and at perigee, respectively. They are the exact representation of the values tabulated by Ptolemy for lunar eclipses in Alm. VI, 8, Table II columns 4 and 5.

[8] Greek ἔμπτωσις.

[9] For solar eclipses one has to replace s by $r_\odot$.

[10] "Duration" is here always meant in the sense of the elongations tabulated in the Almagest.

Example. Investigate the lunar eclipses which occur in the year Nabon. 718. We first find from Alm. VI, 3 the mean argument of latitude $\bar{\omega}'$ for the beginning of the given year:

Nab. 701:	154;56,32°
17 single years:	351;29,44
Nab. 718 II:	146;26,16 .

That this number refers to the second month of the years follows from the date of the mean opposition (above p. 121).

We now have to look in the last table of Alm. VI, 3 for a monthly increment such that the total value comes near to 270° or 90°. This is the case for two months

Nab. 718 II:	146;26,16		146;26,16
4 months:	122;40,57	10 months:	306;42,21
Nab. 718 VI:	269; 7,13	XI:	93; 8,37.

Obviously both values, six months apart, lie within eclipse limits and exhaust all possibilities for the year under consideration.

Table 10 (p. 124) tells us that the true conjunctions fell

$$\Delta t\colon\ 1;38,21^{\mathrm{h}} \quad \text{and} \quad 3;7,57^{\mathrm{h}}$$

respectively, later than the mean conjunctions. Hence we have to add to the above given values of $\bar{\omega}'$ the motions during Δt and thus find at the moments of the true conjunctions

$$\bar{\omega}'\colon\ 270;1,26^\circ \quad \text{and} \quad 94;52,13^\circ.$$

The increments Δt also increase the anomalies by a small amount beyond the values given in Table 9 (p. 123) to

$$\alpha\colon\ 343;40^\circ \quad \text{and} \quad 139;23^\circ.$$

For these values we find from Alm. IV, 10 the equations

$$c_{☾}\colon\ +1;17^\circ \quad \text{and} \quad -3;29^\circ$$

and therefore

$$\omega'\colon\ 271;18^\circ \quad \text{and} \quad 91;23^\circ$$

as arguments of latitude at the true oppositions.

The values of α lead, according to Alm. VI, 8, to coefficients of interpolation

$$q(\alpha)\colon\ 0;1,25 \quad \text{and} \quad 0;52,9.$$

For ω' one finds (also in Alm. VI, 8) the magnitudes

$$m_{\mathrm{a}} = 19;0 \quad m_{\mathrm{p}} = 19;14 \quad \text{and} \quad m_{\mathrm{a}} = 18;50 \quad m_{\mathrm{p}} = 19;7$$

respectively. Hence, interpolating with $q(\alpha)$

$$m\colon\ 19;0 + 0;1,25 \cdot 0;14 \approx 19;0 \quad \text{and} \quad 18;50 + 0;52,9 \cdot 0;17 \approx 19;5$$

for the magnitudes.

For the elongations (cf. Fig. 117, p. 1242) one has with Alm. VI, 8

$$\tau'_a = 0;31,51^\circ \quad \tau'_p = 0;35,50^\circ \quad \text{and} \quad \tau'_a = 0;31,56^\circ \quad \tau'_p = 0;35,53^\circ$$
$$\tau_a = 0;24,\ 8 \quad \tau_p = 0;27,20 \qquad\qquad \tau_a = 0;23,59 \quad \tau_p = 0;27,16\ .$$

Thus, interpolating again with $q(\alpha)$

$$\tau' = 0;31,51 + 0;1,25 \cdot 0;3,59 \quad \text{and} \quad \tau' = 0;31,56 + 0;52,9 \cdot 0;3,57$$
$$\approx 0;31,57^\circ \qquad\qquad \approx 0;35,22^\circ$$
$$\tau = 0;24,\ 8 + 0;1,25 \cdot 0;3,12 \qquad \tau = 0;23,59 + 0;52,9 \cdot 0;3,17$$
$$\approx 0;24,13^\circ \qquad\qquad \approx 0;26,50^\circ.$$

The corresponding solar motion increases these intervals by a factor 13/12, i.e. to

$$0;34,37^\circ \quad 0;26,14^\circ \quad \text{and} \quad 0;38,19^\circ \quad 0;29,4^\circ.$$

For the lunar velocity we may use the values given in Table 10 (p. 124) at the mean conjunctions

$$0;30,24^{\circ/h} \quad \text{and} \quad 0;34,56^{\circ/h}$$

which gives us for the previously listed elongations the durations

partial phase:	$1;\ 8^h$	and	$1;\ 6^h$
total phase:	0;52		0;50 .

Table 8 (p. 121) gave us the moments of the mean oppositions to which we have to add Δt in order to find the eclipse middles. Converting to midnight Alexandria one then obtains for the two eclipse middles

$$-29 \text{ Jan. } 30\ 7;40^h \quad \text{and} \quad -29 \text{ July } 26\ 13;34^h$$

Alexandria civil time.

We can compare these results with modern data. For the first eclipse we have:

	Ptolemy	Oppol.	P.V. Ngb.	Goldst.
middle	$7;40^h$	$7;23^h$	$6;53^h$	$7;17^h$
partial	$1;\ 8^h$	$1;\ 1^h$	$1;12^h$	
total	$0;52^h$	$0;50^h$	$0;50^h$	
m	19; 0	19.8	19.5	

Sunrise at Alexandria was, according to the tables in Alm. II, 8, at about $6;45^h$ since the solar longitude was about $\approx 8^\circ$. Thus, according to Ptolemy's data, the eclipse began about $5;40^h$ in the morning and reached totality at $6;48^h$, i.e. just at moon set.

For the second eclipse we have

	Ptolemy	Oppol.	Goldst.
middle	$13;34^h$	$13;39^h$	$13;46^h$
partial	$1;\ 6^h$	$1;\ 1^h$	
total	$0;50^h$	$0;50^h$	
m	19; 5	19.9	

Because of daytime the eclipse was invisible at Alexandria.

6. Area-Eclipse-Magnitudes

Ptolemy tells us that "the majority of those who observe eclipse prognostications"[1] do not measure eclipse magnitudes in fractions of the diameter of the eclipsed disk but by estimating the ratio of the darkened area to the illuminated area. For this reason Ptolemy gives a little table (Alm. VI, 8, Table IV) which allows one to find for given linear magnitude, ranging from $m=1$ to $m=12$, the corresponding area digits. Since the areas under consideration do not only depend on the depth of the immersion but also on the different radii one has to distinguish between area-eclipse-magnitudes for solar eclipses and for lunar eclipses.

Whether Ptolemy was the first to compute such a table we do not know. His procedure is straightforward. Let $a=6$ denote the radius of the eclipsed body, b the radius of the eclipsing circle, measured in the same units, i.e. in linear eclipse magnitudes or digits. Thus

$$b=6r_{☾}/r_{\odot} \quad \text{for solar ecl.,} \qquad b=6s/r_{☾} \quad \text{for lunar ecl.}$$

The variation in the values of $r_{☾}$ and s are so small that it suffices to compute b only for the mean values of these radii.[2] Thus we can consider a, b, and m to be given.

It follows from Fig. 121 that

$$\mathrm{AE}^2=a^2-c^2 \qquad \mathrm{EB}^2=b^2-c^2$$

therefore

$$a^2-b^2=(\mathrm{AE}-\mathrm{EB})\,\mathrm{AB}.$$

Since

$$\mathrm{AE}+\mathrm{EB}=a+b-m \tag{1}$$

is known we can find AE and EB separately from (1) and

$$\mathrm{AE}-\mathrm{EB}=(a^2-b^2)/(a+b-m). \tag{2}$$

With AE found one has also

$$c=\sqrt{a^2-\mathrm{AE}^2}$$

and therefore the common chord $\Delta\Gamma=2c$. Hence one can find the angle α and β which subtend $\Delta\Gamma$ at A and B, respectively. This allows us to compute the following areas[3]

$$\text{sector A}=\frac{\alpha}{360}a^2\pi \qquad \text{sector B}=\frac{\beta}{360}b^2\pi$$

$$\text{triangle A}\Gamma\Delta=c\cdot\mathrm{AE} \qquad \text{triangle B}\Gamma\Delta=c\cdot\mathrm{EB}.$$

The area L of the eclipsed part is then known from

$$L=\text{sector A}+\text{sector B}-(\mathrm{A}\Gamma\Delta+\mathrm{B}\Gamma\Delta). \tag{3}$$

[1] Alm. VI, 7 Heib., p. 512, 8.

[2] One finds 6;23 and 15;36, respectively. For the first value Ptolemy gives incorrectly 6;10, an error discussed by Pappus (Rome CA I, p. 261 ff.).

[3] Ptolemy approximates π by 3;8,30, a value which he motivates (Heib., p. 513, 2–5) as mean value between the Archimedean approximations 3 1/7 (=3;8,34,...) and 3 10/71 (=3;8,27,...).

If we define *area digits* by giving 12 such digits to the area $a^2\pi$ of the eclipsed body we finally have for the area-eclipse magnitude

$$m' = 12L/a^2\pi \tag{4}$$

with L computed from (3) as function of a, b, and m. The resulting values, for integer values of m, as listed in Alm. VI, 8, are represented in our graph Fig. 122.

Conversion tables between linear and area digits and discussions of their computation are frequently found in Islamic, Byzantine, and western medieval treatises, including Copernicus (Revol. IV, 32). I can see no practical reason for this stubbornly maintained tradition. We know through many recorded estimates of linear magnitudes of eclipses how difficult it is to correctly evaluate the obscured fraction of the diameter of the moon (not to mention partial solar eclipses). But to estimate the ratio of parts of circular areas is sheer guesswork and without any astronomical significance.

7. Angles of Inclination

The Almagest contains several sections which were only added by Ptolemy in order not to omit from his works topics which traditionally belonged to mathematical astronomy. The transformation of linear digits to area-eclipse-magnitudes, discussed in the preceding section, is one example. Another one is found at the end of the planetary theory (Alm. XIII, 7 to 10): the determination of first and last visibility of the planets.[1] In the same way also the lunar theory is terminated by a chapter (VI, 11) which is of little astronomical significance. It deals with the determination of certain angles which were considered, apparently by a very old tradition, of importance for weather prognostication.

The technical term connected with this problem is "prosneusis," the same word which Ptolemy introduced in the lunar theory in connection with the second anomaly.[2] The present usage has nothing to do with lunar theory but is developed from the original meaning of the verb νεύειν to nod, to incline the head, etc. According to the terminology of hellenistic astrology the planets or the moon can, e.g., give their consent by "inclining" toward a certain position, i.e. by being found in a favorable configuration. Directly observable is the "inclination" of the tail of a comet or of the eclipsed part of the moon toward a section of the horizon and thus held indicative for certain geographical regions.[3] Equally direct are the prognostications of the weather from "inclinations." The moon's sickle or the moon's latitude are considered significant for storms and weather according to the part of the horizon toward which they are "inclined"[4] since the segments of the horizon are naturally associated with the winds and the weather they bring.

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[1] Unfortunately from the viewpoint of our historical interests, Ptolemy completely ignored the problems of first and last visibility of the moon.

[2] Above p. 88.

[3] E.g. Tetrabiblos II, 10 (p. 92, 2, 13 Boll-Boer).

[4] E.g. Tetrabiblos II, 13, 14 (pp. 99, 3f.; 100, 8f.; 102, 3 Boll-Boer). Cf. also the "wind" toward which points the latitudinal component of the lunar parallax (below p. 999, n. 29).

Not only for the sake of precision in our own terminology is it of importance to distinguish sharply between these early forms of meteorological theory and the abstract doctrines of hellenistic-Roman astrology. Actually one is dealing with two totally different strata of cultural development which still exist side by side in late antiquity and provide us with material which is centuries older than the astrological doctrines developed in hellenistic Alexandria.

It is in connection with weather prognostication[5] that Ptolemy rationalizes the concept of "inclinations" for eclipses. Without telling us how the angles under consideration are eventually to be used he takes it for granted that in the case of eclipses two types of angles are of importance: first the rising and setting amplitudes of the ecliptic and secondly the angles made between the ecliptic and the great circle which connects the center of the moon with the center of the sun or with the center of the shadow, respectively. Both angles change continually during the course of an eclipse and, under certain conditions, these changes can reach considerable amounts in a short time. Ptolemy does not consider it worth the great computational effort required if one wished to describe in detail these variations in direction for the whole duration of an eclipse. He therefore restricts himself to the determination of the said angles only for the characteristic phases: beginning and end of the partial phase and beginning and end of totality. Even so he introduces finally such "simplifications" that the results lose all astronomical meaning and degenerate to some sort of empty game which associates non-existent "angles of inclination" with eclipses, simply in order to produce some numerical result when one gets tired of computing correctly, although there exists no theoretical difficulty in obtaining the exact result in all cases.

The first angle, the rising or setting amplitude of the ecliptic, is undoubtedly computed correctly for the data under consideration, i.e. for the geographical latitudes of the seven climata and the endpoints of the zodiacal signs rising or setting. The details of the computations are not given but the method is certainly the same as we described for the special case of the solstitial points (cf. above p. 37). The results are presented in the form of a circular diagram which we discussed in connection with Fig. 32, p. 1216. Thus we can consider to be known the azimuth η of the rising point H or of the setting Δ of the ecliptic, reckoned from the points E or W of the horizon (cf. Fig. 123).

The second type of angles, between the ecliptic and the circle which connects the centers of eclipsed and eclipsing body, is correctly determined if one accepts a small modification of the problem. In our Figs. 116 to 118 we have considerably exaggerated the angle between ecliptic and lunar orbit which in fact amounts to only 5°. For the computation of the angles which are defined for the characteristic phases of an eclipse it is convenient to ignore this small inclination of the lunar orbit and to assume that the latitude of the moon remains unchanged during the whole duration of the eclipse (cf. Fig. 124).

If σ denotes the desired angle (restricted for all phases to the interval $0 \leq \sigma \leq 90$), i.e.

$$\sigma = \Sigma \mathrm{AB} \quad \text{or} \quad \Sigma \Gamma \mathrm{B}$$

[5] Heiberg I, pp. 512, 9; 536, 21; 537, 8; 545, 3, 4. In fact this holds for all the above-mentioned appendices: cf. for area digits Heiberg I, p. 512, 9, for heliacal phenomena II, p. 204, 7.

respectively, then one has

$$\sin\sigma = \Sigma\mathrm{B}/\Sigma\mathrm{A} \quad \text{or} \quad \Sigma\mathrm{B}/\Sigma\Gamma.$$

Using the same notation as on p. 136 we have

$$\Sigma\mathrm{A} = a+b \qquad \Sigma\Gamma = a-b \qquad \Sigma\mathrm{B} = a+b-\mu$$

with the immersion μ known through (5) p. 135 for an eclipse of given linear magnitude m. Assuming for the radii a and b the values for mean distances one can compute the angles σ as function of m. The result is the table in Alm. VI, 12, represented graphically in our Fig. 125.

With the values of η given in the circular diagram and the values of σ tabulated in VI, 12 we can consider as correctly known the azimuths of the points H and Δ of the ecliptic and the angles σ at the center of the sun or the shadow for each characteristic phase. The "inclination" of each phase should now be determined by the azimuth of the point P at which the great circle through the centers of eclipsed and eclipsing body intersects the horizon. The determination of P obviously requires the solution of the spherical triangle ΣHP except for the one trivial case of an exactly central eclipse because then P coincides with H or Δ.

For an eclipse which is not exactly central there exists one situation in which it is trivial to locate the point of intersection of horizon and connecting circle: that is the case when the eclipse occurs in the zenith because the angle σ is then measured in the horizon itself. But in all other cases Ptolemy introduces a "simplification" which makes the answer practically meaningless: he identifies the arc in the horizon between P and H (or Δ) with the angle σ at the center of the eclipse. Since an eclipse can occur at practically any distance from the zenith (e.g. near the horizon) Ptolemy's rule can lead to grotesquely wrong results. This fact was, of course, known in antiquity[6]; Ptolemy himself introduced a correcting factor in the computation of the arc PH (or PΔ) for his Handy Tables[7] but the only reasonable attitude would have been not to "simplify" the problem but to ignore it.

Book VI of the Almagest concludes with a set of rules how the arc σ should be reckoned in the horizon. In order to reproduce these rules in a more condensed form we denote the characteristic phase of a lunar eclipse by the numbers 1 to 5 as follows:

1 ... first contact
2 ... beginning of totality
3 ... middle of eclipse
4 ... end of totality
5 ... last contact.

For a solar eclipse only the phase 1, 3, and 5 are of interest since in a total eclipse the phases 2, 3, and 4 practically coincide.

Now the following associations are made: the phases 1 and 4 of a lunar eclipse are referred to the eastern horizon because the first darkening as well as the first brightening appear on the eastern rim of the moon. Similarly the phases 2 and 5

[6] E.g. to Pappus; cf. his Commentary to Alm. VI ed. Rome, p. 309.
[7] Cf. p. 997f.

are related to the western horizon because the last light and the last darkness disappear on the western rim. For a solar eclipse the phase 1 points west because the first darkening occurs on the western rim, while phase 5 points east where the last darkening vanishes. These associations determine whether the angle σ is reckoned from the rising point (H) or from the setting point (Δ) of the ecliptic. Fig. 126 shows schematically the resulting pattern for lunar eclipses, placing the center of the shadow in the zenith Z and looking perpendicularly from above onto the horizon. The given angles σ are then represented by the arcs HP_1, HP_4, and ΔP_2, ΔP_5, assuming a northern lunar latitude. For negative latitudes P_1 and P_4 change sides with respect to H, P_2 and P_5 with respect to Δ. For $\beta_{☾}=0$ one has, of course $P_1=P_4=H$, $P_2=P_5=\Delta$.

For solar eclipses the σ of phase 1 is reckoned from Δ, for phase 5 from H (cf. Fig. 127). No rule is given in the Almagest for any phase 3, but this is done in the Handy Tables.[8]

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[8] Cf. below p. 998.

C. Planetary Theory

§ 1. Introduction

1. General

The theory of the lunar inequalities and of planetary motion, both in longitude and in latitude, are the great achievements which secured the Almagest an unsurpassed position for the next fourteen centuries. The conceptual elegance of Ptolemy's cinematic models and the logical consistency of the derivation of the fundamental parameters from carefully selected observations made it extremely difficult to introduce more than insignificant modifications of the basic theory. Even the glaring inadequacy of Ptolemy's model for the second lunar anomaly with respect to distances found no proper modification before the later Islamic period.[1] Ptolemy's planetary theory shows no such obvious discrepancies between predicted and observable phenomena though here also systematic observations would have revealed differences which should have called for minor revisions of the basic parameters. Ptolemy himself corrected some of the elements of the theory as first presented in the Almagest[2] and later centuries could easily have further improved on the dimensions of Ptolemy's models and should have increased the numerical accuracy of those tables for which Ptolemy had resorted to mere approximations when higher accuracy implied too heavy a burden of numerical computations. He himself was fully aware of these shortcomings and in particular of the fact that for the planets he had far less reliable data at his disposal than for sun and moon, a fact which we can now fully appreciate through our knowledge of the Babylonian planetary ephemerides[3] with their emphasis on visibility phenomena, least suited for Ptolemy's needs and furthermore subject to the greatest observational inaccuracy.

Muslim astronomers, in spite of much boasting, restricted themselves by and large to the most elementary parts of Greek astronomy: refinements in the parameters of the solar motion, and increased accuracy in the determination of the obliquity of the ecliptic and the constant of precession. But every attempt at a revision of the foundations of the planetary theory must have appeared, rightly, as a gigantic task, not lightly to be undertaken in view of the consistency of the structure erected in the Almagest.

In the following we shall more or less maintain the arrangement of Ptolemy's presentation. In particular we shall also separate the theory of the motion in

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[1] Cf. Roberts [1957]; Neugebauer [1968, 2].

[2] Cf. below V C 4, 5 B and 5 C.

[3] Cf. p. 386ff.

longitude from the computation of latitudes, a procedure which considerably simplifies matters. It is clear that the smallness of the inclinations of the orbital planes causes only negligible errors; Ptolemy proves this explicitly[4] by computing for specific situations the resulting differences, which never exceed a few minutes of arc.

To the modern reader the question of accuracy of Ptolemy's results stands naturally in the foreground. However, I am not in a position to answer such questions properly. Only a systematic analytic representation of Ptolemy's models could do this, clearly isolating the influence of each characteristic parameter. I do not think, however, that the results would contribute much to our understanding of the methodology upon which the ancient theory rests. The type of deviations one might expect seems to me sufficiently shown by some examples computed for characteristic situations of planetary motion (cf., e.g., below Fig. 228, p. 1283 and Fig. 233, p. 1285). Besides the trivial discrepancies in longitude, caused by the incorrect constant of precession, one can find deviations of up to 2° in latitude, in spite of the good representation of the general shape of the apparent orbits. Later observations could have considerably reduced these deviations, without changing Ptolemy's procedures, by determining the parameters of his models more accurately on the basis of a larger and better selected observational material.

2. Distances and Eccentricities

Before entering upon a description of Ptolemy's planetary theory it seems useful to anticipate summarily some of his results and to relate them to our present knowledge. I have in mind (a) the values found in the Almagest for the radii of the planetary epicycles in their relation to the mean distances of the planets in the solar system, and (b) the direction of the planetary apsidal lines as determined by Ptolemy in comparison with their modern counterparts.

The first point is quickly settled. We know since Copernicus that the size of a planetary epicycle is a measure for the heliocentric distance of the planet, as is immediately evident if one transforms a heliocentric circular motion to geocentric coordinates (cf. Fig. 128). If a is the semi-major axis of an elliptic orbit, measured in units such that $a=1$ is the mean heliocentric distance of the earth, r the radius of the epicycle, R of the deferent, then we should have for an inner planet $a=r/R$, for an outer planet $a=R/r$. Hence with Ptolemy's norm $R=60$: for an inner planet $r=60a$, for an outer planet $r=60/a$. This leads to the following numerical comparison:

	modern		Alm.
	a	r	r
☿	0.3871	23;14	22;30
♀	0.7233	43;24	43;10
♂	1.5237	39;22	39;30
♃	5.2028	11;32	11;30
♄	9.5388	6;17	6;30

[4] In Almagest XIII, 4.

One sees that the Ptolemaic theory excellently, however unknowingly, represents the order of magnitude of the actual dimensions of the planetary system.

The second point needs a more extensive discussion. The simple schematic tranformation used for Fig. 128 requires an obvious modification as soon as eccentricities are taken into consideration. For the system earth/sun it is, of course, irrelevant whether one applies the eccentricity to the orbit of the earth or of the sun. For the planetary orbits, however, it is essential to refer the eccentricities to the sun in order to understand their geocentric effects. Thus one has to start (cf. Fig. 129) with two independent heliocentric eccentricities e_0 and e_1 for the earth and for the planet, respectively. For the geocentric observer these two eccentricities produce a resultant eccentricity e which is the vector sum of e_0 and e_1. Ptolemy, who did not realize this situation, ascribed the resultant eccentricity e to the deferent of the planetary orbit.

In the following we give a comparison of these eccentricities computed from modern data[5] and under the simplifying assumption that all orbital planes coincide with the ecliptic. Some modifications due to the actual inclinations of the orbits will be discussed when we are dealing with the planetary latitudes.[6] One element, however, that introduces a much greater amount of incertitude is the replacement of elliptic orbits by eccentric circles. We know that directions are best represented when the center of the eccenter is placed into the second focus of the elliptic orbit, i.e. when the circular eccentricity is twice the Kepler eccentricity.[7] This holds for the optimal representation of the solar longitudes; but for the planetary orbits, one may argue, the heliocentric distances also are of importance for the geocentric longitudes of the planet. Distances, however, are better represented if one identifies the midpoint of the elliptic orbit with the center of the deferent. In order to avoid a detailed analytical investigation of the interplay of these different sources of inaccuracies we simply compare three typical cases with Ptolemy's data: we compute the resultant eccentricity under one of the three following assumptions

(1) both e_0 and e_1 are of the amount of the elliptic eccentricity

(2) only the planetary eccentricity e_1 is elliptic whereas e_0 has twice the elliptic value

(3) both e_0 and e_1 are twice the respective elliptic eccentricity.

In this way we obtain the following table in which A indicates the geocentric longitude of the apogee, e the amount of the resultant eccentricity.

	case	☿	♀	♂	♃	♄
A	(1) and (3)	♏16	♉17	♋23;47*	♌23*	♏19;30*
	(2)	♏14*	♊ 1*	♋18	♌12	♏ 8
	Ptolemy[8]	♎ 9;40	♉24;40	♋25;10	♍10;40	♏22;40
e	(1)	11;25	0;40	6;10*	2;50*	2;40*
	(2)	10;26*	1;40	6;56	3;0	1;48
	(3)	22;50	1;20*	12;20	5;40	5;20
	Ptolemy	9	1;15	6	2;45	3;25

[5] Cf. Appendix VI B 7, 2.

[6] Cf. below p. 208 and p. 212.

[7] Cf. below p. 1101.

[8] Computed for A.D. 100 from Almagest XI, 11 using Ptolemy's constant of precession.

The asterisks indicate the computed values which come nearest to Ptolemy's values. This suggests that one can usually expect best agreement with Ptolemaic planetary models if one identifies the midpoints of the elliptic and of the circular orbits,[9] at least for the earth and for the outer planets.

3. Ptolemy's Introduction to Almagest IX

Absolute distances are no part of Ptolemy's planetary theory. All dimensions of the cinematic model for each planet are expressed in units such that the radius R of the deferent obtains the value 60. It is only in the introductory chapter of Book IX that Ptolemy makes some remarks about the physical arrangement of the planetary orbits. He says that practically all astronomers agree in assuming that Saturn, Jupiter, and Mars are nearer to us than the fixed stars but farther away than the sun. There existed, however, differences of opinion concerning Venus and Mercury which were placed by earlier astronomers between moon and sun whereas more recent astronomers argued that this arrangement would imply, contrary to experience, the occurrence of transits; therefore they placed all five planets beyond the solar orbit. Ptolemy does not share this opinion because he finds it more likely that the planets with limited elongation are separated by the sun from those planets which reach opposition. Besides this purely aesthetic argument he also points to the motion in latitude which can exclude transits, exactly as only a few conjunctions of the moon produce solar eclipses. Ptolemy does not show that this explanation is a consequence of his theory of planetary latitudes and in fact it is easy enough to show that it is not. We shall return to this problem in connection with the computation of latitudes.[10]

Ptolemy also remarks that no theory should place a planet so near to us that an observable parallax would result. One has to remember in this context that the solar parallax had not been determined directly-as in the case of the moon-but only indirectly from the lunar parallax by means of eclipses.[11] Thus the possibility of computing the absolute distance of the sun does not contradict the fact that no parallax for Venus and Mercury can be detected, even if their orbits are nearer to us than the sun.

Ptolemy explicitly states that the problem of planetary distances could only be solved by the measurements of parallaxes and thus has to remain unanswered. It is evident from this discussion that he did not yet think of an arrangement of contiguous planetary spheres, a hypothesis which he later on developed in his "Planetary Hypotheses"[12]. What is more important is the fact that he did not see that the identification of the center of the epicycles of Venus and Mercury with the sun would lead, at least for these two planets, to an estimate of their absolute distances. One must admit, however, that the existence of both solar and planetary anomalies makes a direct identification of the sun with the center

[9] Case (1); for the elliptic approximation of the deferent of Mercury cf. below p. 168.
[10] Below p. 227ff.
[11] Above p. 109ff.
[12] Cf. p. 111.

of the epicycles difficult and may have prevented Ptolemy from following this road.

The second chapter of Book IX of the Almagest formulates the problem: to develop a planetary theory on the basis of the philosophical postulate that the observed irregularities of planetary motion are the result of the combination of uniform circular motions. Ptolemy claims, no doubt rightly, that no one before had reached this goal.[13]

Ptolemy distinguishes two "anomalies" of planetary motions. One "with respect to the ecliptic," i.e. concerning the general progress of the planet in longitude; the other "with respect to the sun," observable, e.g., in the retrogressions of planetary motions. In Chaps. 5 and 6 of Book IX Ptolemy discusses these two inequalities in general terms and tries to demonstrate that only an epicyclic model, with the planet moving on the epicycle in the same sense as the center of the epicycle moves on the deferent (cf. Fig. 130), can explain the empirical data. Ignoring latitudes, as always in this first part of the theory, Ptolemy argues as follows. In order to investigate the anomaly with respect to the ecliptic we have to consider positions in which the planet is always in the same situation with respect to the sun, e.g. at opposition or at a stationary point. Then the observations show that the spacing between such points is greater in one part of the ecliptic than in another[14] and that the time from the smallest velocity to mean motion exceeds the time from mean motion to most rapid progress. Such behavior, Ptolemy says correctly, can be represented by an epicycle as well as by an eccenter since the mean motion appears for the first model at the point of tangency of the line of sight from the observer to the epicycle, in the second model at the quadrature with respect to the apsidal line (cf. Fig. 131), while slowest progress occurs at the apogees, fastest at the perigees. Ptolemy considers it more "natural" to assume for the progress in the ecliptic the eccentric model, obviously because otherwise he would have been led to epicycle-epicycles in consequence of his description of the anomaly with respect to the sun by an epicycle.

In order to study the anomaly with respect to the sun one has to observe the sequence of synodic phenomena and compare it with the progress of the planet in longitude. Then one finds that the time between mean progress and minimum (negative) progress is always shorter than from maximum velocity to the mean (cf. the schematic Fig. 132). This, Ptolemy asserts, excludes an eccenter because then the arc from minimum velocity (at the apogee in Fig. 131) to the mean velocity (at quadrature) exceeds the arc from mean to maximum velocity (at perigee), contrary to experience. This observed asymmetry can be explained, however, by an epicycle with rotations as indicated in Fig. 133, where the maximum velocity correctly belongs to the greater arc. Obviously this is decisive for the determination of the sense of rotation on the epicycle, once it is agreed upon that an epicyclic model should be used. But it is not correct to say that the same motion could not be obtained with an eccenter. In fact in Ptolemy's description of Apollonius' theory of planetary stations use is made of exactly this possibility.[15]

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[13] For more historical details cf. below p. 270f.

[14] As we shall see (below p. 421) the Babylonian planetary computations make use of precisely this fact.

[15] Cf. below I D 3, 1.

All that is needed is a movable apsidal line which proceeds with the sum[16] of the mean motion in longitude and in anomaly (cf. Fig. 134), thus keeping the apsidal line OM always parallel to CP. (This is, of course, nothing else but the transformation of heliocentric to geocentric coordinates in the case of an outer planet.) Ptolemy's implicit assumption of a fixed apsidal line is probably suggested by the observation that the apsidal line of the deferent is not movable.

In the final section of these introductory chapters (IX, 2) Ptolemy excuses himself with the complexity of the phenomena of planetary motion for not presenting step by step the arguments which led him to the model finally adopted. Indeed, there are fewer observational details reported in the Books IX to XIII on the planets than in the Books III to VI which concern sun and moon. On the other hand Ptolemy has clearly described the cinematic model which is common to all planets, excepting Mercury for which special modifications had to be adopted (Book IX, 6). For all planets the apsidal lines of the deferents participate in the motion of precession (1° per century); thus the apsidal lines are sidereally fixed in contrast to the solar theory which assumes a fixed tropical longitude for the apogee (♊ 5;30[17]). As a final assumption is mentioned an inclination of the planes of the deferents with respect to the ecliptic and an inclination of the planes of the epicycles with respect to the deferents, assumptions made in order to explain the planetary latitudes.[18]

4. Parameters of Mean Motion

The two components of planetary motion in Ptolemy's theory are represented by the motion of the center of the epicycle on the deferent and by the motion of the planet itself on the circumference of the epicycle. The first one describes the mean progress in longitude; the second one, the "anomaly," causes the "synodic" phenomena, in particular the retrogradations which occur in a fixed relation to the sun.

Long before the invention of these geometric representations Babylonian astronomers had discovered numerical identities which regulate the periodic recurrence of the synodic phenomena. It is from these identities that Ptolemy derived the first approximations for the mean values which he had to assign to the angular velocities of the center of the epicycle and to the motions in anomaly.

Let N be an integer number of tropical years, R an integer number of rotations of the planet in longitude (longitudes counted, as always, with respect to the vernal point, consistent with Ptolemy's definition of "year" as tropical year[19]), A an integer number of synodic periods. Then, for the outer planets, holds

$$N = R + A. \tag{1}$$

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[16] In the lunar theory also the apsidal line is movable, but proceeds with the difference velocity (cf. above p. 68).

[17] Cf. above p. 58.

[18] Details to be discussed later; cf. below p. 207.

[19] Cf. above p. 54.

As good numerical approximations were known in an early phase of Babylonian astronomy[20] the following values

$$\begin{array}{lll} ♄: N=59 & ♃: N=71 & ♂: N=79 \\ \quad R=\ 2 & \quad R=\ 6 & \quad R=42 \\ \quad A=57 & \quad A=65 & \quad A=37. \end{array} \tag{2}$$

Because the two inner planets have a bounded elongation from the sun one has

$$N=R \tag{3}$$

and A is independent. Again from Babylonian astronomy the following integer values are known:

$$\begin{array}{ll} ♀: N=R=8 & ☿: N=R=46 \\ \quad A=5 & \quad A=145. \end{array} \tag{4}$$

Ptolemy, on the basis of his refined theory of planetary motion was able to introduce some corrections to these relations and his tables for the mean motions (IX, 4) have incorporated these improvements though their deviation is postponed to later chapters; the same holds for the epoch values which are listed at the beginning of the tables.[21]

The improved relations are supposed to correspond in each case to an exact number of completions of the anomaly:

$$\begin{array}{lll} ♄: 59^y+1;45^d= & 2\text{ rot.}+1;43° & A=\ 57 \\ ♃: 71^y-4;54^d= & 6\text{ rot.}-4;50° & A=\ 65 \\ ♂: 79^y+3;13^d= & 42\text{ rot.}+3;10° & A=\ 37 \\ ♀: \ 8^y-2;18^d= & 8\text{ rot.}-2;15° & A=\ \ 5 \\ ☿: 46^y+1;2^d\ = & 46\text{ rot.}+1° & A=145. \end{array} \tag{5}$$

In these relations "years" are to be understood to be tropical years, $365;14,48^d$ in length.[22]

In principle one can derive from (5) the parameters of mean motion in anomaly and in longitude that underlie the mean motion tables in Alm. IX, 4. For example in the case of Saturn (5) implies that $59^y+1;45^d$, i.e. $5,59,11;18,12^d$ correspond to $57 \cdot 6,0=5,42,0°$ motion in anomaly. Consequently, Ptolemy states that one finds for the daily motion in anomaly

$$5,42,0/5,59,11;18=0;57,7,43,41,43,40^{°/d} \tag{6}$$

which is the number found in the tables. If one subtracts this value, because of (1), from the mean motion of the sun[23]

$$0;59,8,17,13,12,30^{°/d},$$

one obtains as mean motion in longitude for Saturn

$$0;2,0,33,31,28,51^{°/d}. \tag{7}$$

[20] In the so-called "Goal-year-texts"; cf. below p. 351 and p. 554.

[21] Cf. for the outer planets below p. 180f. and p. 182; for the inner planets p. 157 and p. 167f.

[22] Cf. above p. 54.

[23] Cf. above p. 55.

From (6) and (7) are then derived all values tabulated for hours, for 30 days, and for single and 18 Egyptian years (Alm. IX, 4). For the inner planets the mean motions in longitude are, of course, identical with the mean motions of the sun (as tabulated in Alm. III, 2).

The above given derivation of the mean motion in anomaly is, however, historically misleading. Looking at the relations (5) one could assume that they were the direct result of observations and hence (6) the logical consequence of empirical data.

That this is not so is stated, however, in the Almagest itself where Ptolemy gives for each planet the observations by means of which he determined the change of anomaly during some interval of time depending on extant observational records and unrelated to complete periods of anomaly.[24] Hence the relations (5) are not the basis for the determination of the mean motion in anomaly but on the contrary the mean motions in anomaly had to be derived first from observations on the basis of the geometrical models of planetary motion[25]; only at the end could the relations (5) be derived from the tables of mean motions for nearly integer multiples of A and the small deviations from integer multiples of tropical years (N) and rotations in longitude (R) derived by roundings from the tables in IX, 4.

The tables also give epoch values which refer, as usual, to Nabonassar 1, Thoth 1, Alexandria noon. Listed are the "mean longitude" $\bar{\lambda}_0$ of the center of the epicycle, counted from the vernal point, the longitude λ_A of the apogee of the deferent (changing with precession), and the position $\bar{\alpha}_0$ of the anomaly, reckoned from the apogee of the epicycle (with respect to the equant[26]). The values listed are:

♄:	$\bar{\lambda}_0 =$ ♑ 26;43	$\lambda_A =$ ♏ 14;10	$\bar{\alpha}_0 =$ 34;2	
♃:	♎ 4;41	♍ 2;9	146;4	
♂:	♈ 3;32	♋ 16;40	327;13	(8)
♀:	♓ 0;45[27]	♉ 16;10	71;7	
☿:		♎ 1;10	21;55.	

§ 2. Venus

1. Eccentricity and Equant

The first problem consists in determining the position of the apsidal line for the eccentric deferent which carries the center of the epicycle. Since Ptolemy had no proper ancient observations of Venus at his disposal he used three observations

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[24] Cf. for the outer planets below p. 182, for Venus p. 157, for Mercury p. 167.

[25] This is confirmed by the fact that, e.g., the quotient (6) is not accurate since one would obtain ... 41,33, ... instead of ... 41,43,40. Furthermore in this division $0;0,12^d$ is disregarded in the denominator. Hence the accurate result would be only ... 39,46, ... instead of ... 41,43,40.

[26] Cf. below p. 155; p. 171.

[27] Identical with the mean longitude of the sun at epoch (cf. p. 60).

by (his teacher?) Theon of maximum elongations, supplemented by three similar observations made by himself. The idea of the method is simple (cf. Fig. 135). Theon observed Venus as evening star at maximum elongation (in A.D. 132 March 8) in ♉ 1;30 while the mean sun was in ♓ 14;15, hence at an elongation of 47;15°. Ptolemy found the same maximum elongation of 47;15° for the morning star (in A.D. 140 July 30), the mean sun being in ♌ 5;45, Venus in ♊ 18;30. The equality of the elongations indicates symmetry of the positions of the epicycle with respect to the apsidal line. Hence the latter bisects the angles between the two positions of Venus, ♉ 1;30 and ♊ 18;30 and therefore passes through the points $A=$ ♉ 25 and $A'=$ ♏ 25. This result was confirmed by a second pair of equal greatest elongations (now 47;32°) observed by Theon in A.D. 127 and by Ptolemy in A.D. 136.

In order to decide whether A or A′ is the apogee two observations of maximum elongations are utilized, while the center of the epicycle, i.e. the mean sun, lies in the apsidal line (cf. Fig. 136). Theon found (in A.D. 129) an elongation of 44;48°, the mean sun being in A. Ptolemy, seven years later,[1] found 47;20° with the sun in A′. Since the second angle exceeds the first it is clear that $A'=\Pi$ is the perigee.

These data also allow us to determine the eccentricity $OM=e$ (cf. Fig. 137) which causes the radius r of the epicycle to be seen under two different angles $\alpha=44;48$ and $\beta=47;20$ when located in A or Π, respectively. Obviously

$$r=(R+e)\sin\alpha=(R-e)\sin\beta \tag{1}$$

and hence

$$e=R\,\frac{\sin\beta-\sin\alpha}{\sin\alpha+\sin\beta}. \tag{2}$$

[1] A.D. 136. The date of this observation of Venus in "maximum elongation" as evening star is only 37 days earlier than the one used just before: Hadrian 21 Tybi 2/3 (A.D. 136 Nov. 18; cf. Fig. 136) and Mechir 9/10 (Dec. 25; cf. Fig. 135), respectively. Obviously Ptolemy uses here the term "maximum elongation" only in a vague sense. In fact the actual maximum elongation occurs about midway between the two dates as Table 13 shows. Ptolemy had to select different dates in order to obtain elongations symmetric to Theon's observations. Incidentally: the latter also were made somewhat later than the accurate moment of the greatest elongation.

Table 13

	Sun	Venus	$\Delta\lambda$	
136 Nov. 14	232.1	277.8	45.2	
19	237.2	283.4	46.2	Nov. 18: 46.1
24	242.3	288.9	46.6	
29	247.4	294.4	47.0	
Dec. 4	252.5	299.7	47.2	
9	257.6	304.8	47.2	
14	262.7	309.8	47.1	
19	267.8	314.6	46.8	
24	272.9	319.2	46.3	Dec. 25: 46.1
29	278.0	323.4	45.4	

Ptolemy's procedure is unnecessarily complicated.[2] Starting from

$$r=\frac{R+e}{120}\,\mathrm{crd}\,2\alpha=\frac{R-e}{120}\,\mathrm{crd}\,2\beta \tag{3}$$

he introduces units (here indicated by primes $'$) such that $R'+e'=120$. Thus he has the following two equations:

$$\begin{aligned} R'+e'&=120\\ R'-e'&=120\,\frac{\mathrm{crd}\,2\alpha}{\mathrm{crd}\,2\beta}\end{aligned} \tag{4}$$

from which one obtains

$$R'=60\left(1+\frac{\mathrm{crd}\,2\alpha}{\mathrm{crd}\,2\beta}\right) \qquad e'=60\left(1-\frac{\mathrm{crd}\,2\alpha}{\mathrm{crd}\,2\beta}\right). \tag{5}$$

Now one has to change the units such that $R=60$; hence

$$e=60\left(1-\frac{\mathrm{crd}\,2\alpha}{\mathrm{crd}\,2\beta}\right)\cdot\frac{60}{60\left(1+\dfrac{\mathrm{crd}\,2\alpha}{\mathrm{crd}\,2\beta}\right)}. \tag{6}$$

This is, of course, the equivalent of (2) because it can be written in the form

$$e=R\,\frac{\mathrm{crd}\,2\beta-\mathrm{crd}\,2\alpha}{\mathrm{crd}\,2\alpha+\mathrm{crd}\,2\beta} \tag{7}$$

which could have been obtained directly from (3). Moreover, the use of the form (6) introduces unnecessary inaccuracies in the numerical execution. Ptolemy finds from (6) and (3)

$$e=1;15 \qquad r=43;10 \qquad (R=60)^3. \tag{8}$$

More accurate computation would lead to $e=1;16,48$ and $r=43;10,48$.[4] The differences are, of course, of no practical consequence for the computation of positions of Venus.

The most important discovery, however, not only for the theory of the motion of Venus but for all planets, resulted from Ptolemy's determination of the position of the center of the epicycle at 90° distance from the apogee or the perigee. At apogee (♉25) and at perigee (♏25) the direction from O to the mean sun and to the center of the epicycle coincide. Ninety degrees of mean motion brings the mean sun to ♒25. Also the center C of the epicycle (Fig. 138) travels 90° about a center E which one cannot expect to coincide with O since the observer is located eccentrically to the deferent. But apparently all pre-Ptolemaic planetary theories

[2] In the case of Mercury, however, the same procedure has its advantages; cf. below p. 161. The same geometrical problem occurs once more in the derivation of (9), p. 155. The absence of algebraic notations obscures such parallelisms.

[3] The resulting extremal geocentric distances of Venus are $M=R+e+r=104;25$, $m=R-e-r=15;35$ hence $M/m\approx 6;42$.

[4] Ptolemy's own roundings would give as final result $e=1;16,23$ and $r=43;10,18$.

took it for granted that E must be the center M of the deferent, i.e., that OE $= e$, the eccentricity which we have just determined. Ptolemy, however, asked the question in a general form: what is the distance $d=$ OE between the observer O and the center E of uniform motion.[5] He found that d is twice the value of e, thus as far removed from M toward A as the observer O toward Π. This "*center of uniform motion*" (later called "*equant*") is probably Ptolemy's most important discovery in the theory of planetary motion. To philosophical minds it appeared to be the major blemish of the Ptolemaic system. Copernicus tried to achieve the same practical results by means of secondary epicycles; but Kepler not only reintroduced the equant for the planetary motion but applied it also for better representation of the earth's motion, eventually to become the second focus of the elliptic orbit.

In order to solve the problem of determining $d=$ OE, Ptolemy had to use observations of Venus at maximum elongation at times when the mean sun was at or near ♒25 (cf. Fig. 139). One such observation (at A.D. 134) showed Venus as morning star in ♑11;55; another one, six years later, as evening star in ♈13;50 (cf. Fig. 140). The mean sun was in both cases in ♒25;30, thus the elongations were

$$\alpha = 43;35 \qquad \beta = 48;20$$

respectively. Because of the identity of the solar positions we can combine the two cases in one figure (cf. Fig. 138) which shows that

$$\gamma = 1/2(\alpha+\beta) \qquad \delta = 1/2(\beta-\alpha).$$

Obviously $r = \rho \sin\gamma$ thus

$$d = \rho \sin\delta = r\frac{\sin 1/2(\beta-\alpha)}{\sin 1/2(\alpha+\beta)}. \tag{9}$$

Ptolemy finds in this way correctly

$$d = 2;29,36 \approx 2;30 = 2e. \tag{10}$$

Thus it was on the basis of systematic observations and a simple mathematical analysis that Ptolemy was led to a fundamental innovation in mathematical astronomy: the factual abolition of the dogma of uniform circular motion in the strict sense.

It seems likely that it was this experience with the motion of Venus from which the concept of a separate center of uniform motion originated. Ptolemy was able to confirm the existence of an eccentrically located equant also in the case of Mercury[6] but for the outer planets the radius of the epicycle is not directly observable. I assume that it was on the basis of uniformity that Ptolemy introduced the same generalization also into the theory of the outer planets and that he convinced himself of the correctness of this hypothesis by the good agreement

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[5] Again it is only Brahe and Kepler who returned to Ptolemy's attitude and required that the distances OM and ME should be determined empirically. In applying this principle to the orbits of Mars and of the earth Kepler went far beyond Ptolemy. Cf. Kepler, Werke 3, Astronomia Nova, Chaps. 16 and 23.

[6] Cf. below p. 161. In the final presentation of the theory in the Almagest Mercury precedes Venus (to be followed by Mars, Jupiter, Saturn). For the chronology of the observations cf. Fig. 16 (p. 1375).

of the consequences with the observed longitudes. We shall describe later on[7] the empirical data which Ptolemy mentions in the case of Mars in order to show that the equant occupies also for an outer planet a symmetric position to the observer with respect to the center of the deferent.

2. Mean Motion in Anomaly. Epoch

Using the new model for the motion of Venus Ptolemy was able to derive from sufficiently far distant observations a more accurate value for the mean motion in anomaly than that obtainable from the Babylonian rule that 5 anomalistic periods correspond to 8 years.[1]

Ptolemy combines a reliable observation by Timocharis (occultation of η Virg by Venus) with one of his own (Venus in relation to Spica and to the moon near β Scorp). The moment t_1 of the first observation corresponds to Nabonassar 476 Mesore 17/18 (= −271 Oct. 12), the moment t_2 of the second to Nabonassar 886 Tybi 29/30 (= 138 Dec. 16). Thus the interval is

$$\Delta t = 409 \text{ eg.y.} + 167^{d}.$$

This is slightly less than 4 1/10 centuries; hence Ptolemy assigns to it a motion of 4;5° in precession. Since the perigee at his time was found to be in ♏25[2] it was in ♏20;55 at t_1. From his solar tables he finds for the mean sun at t_1 the longitude ♎17;3 thus a position $\kappa_1 = 33;52°$ before the perigee, at t_2 in ♐22;9 or $\kappa_2 = 27;9°$ beyond the perigee. Venus itself was observed at t_1 in ♍4;10, at t_2 in ♏6;30. We wish to determine for both of these moments the epicyclic anomaly of the planet. In the following we reproduce Ptolemy's procedure except for the numerical values (and, as usual, making the trivial substitution of trigonometric functions for Ptolemy's chords). Since the pattern is the same for t_1 and for t_2 we omit the indices 1 and 2 in the formulae but represent both situations in our Figs. 141 and 142.[3]

Since the angle κ represents the distance of the mean sun from the perigee Π it also gives the angular distance in mean motion of the center C of the epicycle from Π and is therefore to be reckoned with the equant E as vertex (cf. Fig. 141).

Step 1. Determine the angles β and γ at C and O, respectively. Since $e = \mathrm{EM}$ has been found (in terms of $\mathrm{MC} = R = 60$) we also know

$$\mathrm{ML} = e \sin\kappa \qquad \mathrm{EL} = e\cos\kappa \qquad \mathrm{ON} = 2e\sin\kappa$$

and therefore

$$\mathrm{CL} = \sqrt{R^2 - e^2\sin^2\kappa} \qquad \mathrm{CN} = \mathrm{CL} - e\cos\kappa.$$

Hence

$$\rho = \sqrt{\mathrm{CN}^2 + \mathrm{ON}^2} \qquad \beta = \arcsin\frac{\mathrm{ON}}{\rho} \qquad \gamma = \kappa + \beta$$

can be found.

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[7] Below p. 172ff.

[1] Above p. 151.

[2] Above p. 153.

[3] For the sake of clarity the eccentricity is exaggerated in these figures but the angles at O are drawn essentially correctly.

Step 2. We now consider the position of the planet P on the epicycle of radius r and center C (cf. Fig. 142). From observation is known the angle δ under which P appears at O with respect to the perigee;[4] since we have just found the distance γ of C from the perigee we also know the angle ζ under which r appears from O. Since ρ is also known we can determine ε from

$$CD = r \sin \varepsilon = \rho \sin \zeta.$$

Step 3. The *mean anomaly* $\bar{\alpha}$ of P is reckoned from the *mean apogee* F of the epicycle, i.e. from the apogee with respect to the equant E (cf. Fig. 142). This angle can now be found as sum or difference of the angle GCP (reckoned in the sense of the motion of the planet on the epicycle from the "*true apogee*" G, i.e. the apogee with respect to the observer O) and the angle β found in Step 1.

Here one has to remark that the angle GCP is not uniquely determined by the observation of the longitude of the planet because in general two points, P and Q, appear from O under the same longitude. Consequently Ptolemy underlines the fact that his observation (at t_2) was made after Venus had been at greatest elongation as morning star. Thus P_2 is located beyond the point of tangency to the epicycle and we have under the given circumstances (cf. Fig. 142)

$$\bar{\alpha}_2 = 360 - (\zeta_2 + \varepsilon_2 - \beta_2).$$

For the early observation Ptolemy had at his disposal another observation by Timocharis, made four days after t_1, showing that the elongation of Venus from the mean sun had decreased by 0;44°. Thus Venus was again past maximum elongation and its anomaly can be found from

$$\bar{\alpha}_1 = 360 - (\zeta_1 + \varepsilon_1 + \beta_1).$$

Step 4. The numerical results obtained in this way are

$$\bar{\alpha}_1 = 360 - 107;53 \qquad \bar{\alpha}_2 = 360 - 129;28. \tag{1}$$

This shows that the planet has moved during the time interval $\Delta t = 409$ eg.y. $+167^d$ in anomaly $-21;35° = 338;25°$ plus an integer number of complete revolutions. Since 8 years correspond to 5 synodic periods[5] the number of rotations of the planet on the epicycle in 408 years is $408 \cdot \frac{5}{8} = 255$. The remaining time interval of 1 eg.y. $+167^d = 532^d$ is shorter than one synodic period which is approximately $365 \cdot \frac{8}{5} = 584^d$ long. To this time interval belongs the motion of $\bar{\alpha}_2 - \bar{\alpha}_1 = 338;25°$ as derived from the observations. Consequently, the mean motion in anomaly is given by

$$\frac{255 \cdot 360 + 338;25°}{409 \cdot 365 + 167^d} = 0;36,59,25,53,11,28^{\circ/d}. \tag{2}$$

This is the amount on which Ptolemy based his tables in Almagest IX, 4.[6]

[4] We have for the distance of P from Π: $\delta_1 =$ ♏ 20;55 − ♍ 4;10 = 76;45 and $\delta_2 =$ ♏ 25 − ♏ 6;30 = 18;30.
[5] Cf. p. 151.
[6] The approximate period relation leads to about $0;36,58^{\circ/d}$.

Finally it is easy to compute the elements for the beginning of the era Nabonassar. The mean longitude is the same as for the mean sun, i.e. ♓ 0;45.[7] The mean anomaly can be computed backwards from either t_1 or t_2, using (1) and (2); one finds $\bar{\alpha}_0 = 71;7$. And precession yields ♉ 16;10 as longitude of the apogee of the eccenter at Nabonassar 1.

3. The Observational Data

If one remembers that Venus requires 8 years for only 5 synodic periods one will realize that it is by no means simple to select observational data for such specific situations as needed for the determination of the eccentricity and the apsidal line according to the methods described in the preceding pages. One has only to look at the dates quoted by Ptolemy to see that he continued a systematic observational program, begun by Theon whose observations of Venus belong to the years A.D. 127, 129, 132. Ptolemy's observations date from 134, 136, 138, and 140.[1] These are, of course, only the few selected observations, suitable for a specific purpose and therefore mostly maximum elongations. But it is obvious that many more carefully recorded observations are required in order to establish the fact that maximum elongation is reached, or bypassed, or still to come and to provide the material from which, e.g., maximum elongations of opposite but equal amount can be selected. Exactly as Kepler could not have determined the parameters for the orbit of Mars, irrespective of the particular cinematic hypothesis under consideration, without Tycho Brahe's enormous observational material so Ptolemy could never have developed his planetary theory as documented by the Books IX to XIII of the Almagest without a similar observational program of which the greater part was undoubtedly executed by himself though begun by Theon some years earlier.

§ 3. Mercury

The great eccentricity of Mercury, the "thankless planet,"[1] is the hidden cause of a serious complication in Ptolemy's planetary theory which is for all other planets of such beautiful simplicity. But the troubles did not begin with Ptolemy; already the arithmetical procedures of the Babylonian astronomers of the hellenistic period had greatly to be modified in order to deal properly with Mercury's shifting appearances and disappearances.[2] Ptolemy's cinematic description of Mercury's motion remained the guiding principle during the Islamic period as well as for Copernicus, only insignificantly modified by aṭ-Ṭūsī's

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[7] Above p. 60, (5).

[1] Cf. also Fig. 16 (p. 1375).

[1] Leverrier: "Nulle planète n'a demandé aux astronomes plus de soins et de peines que Mercure, et ne leur a donné en recompense tant d'inquiétudes, tant de contrariétés" (Annales de l'observ. de Paris 5, p. 1, quoted by Tisserand [1880], p. 35).

[2] Cf. below II A 5, I C and II A 7, 6.

mechanism.[3] And we need not underline here the important rôle the fickle planet (though responsible for things "that depend upon intelligence"[4]) played for the modern theory of gravitation by its disobedience to the laws of Newtonian dynamics.

1. Apogee

The determination of the characteristic parameters for the cinematic model of Mercury follows exactly the same ideas which we described in detail for the case of Venus. From two pairs of observations of greatest elongations of equal amount (cf. Fig. 143) Ptolemy concluded that the apsidal line for Mercury had at his time a position defined by the points A = ♎ 10° and A′ = ♈ 10°. Two additional observations[5] (cf. Fig. 144) at which the mean sun was located in A or A′, respectively gave a smaller maximum elongation at A = ♎ 10° which is therefore the apogee.

The observations symmetric to the apsidal line (Fig. 143) were chosen in such a way that the position of the sun is the same for each of the two observations on the same side of the apsidal line (note the identical julian dates in each pair), one for maximum elongation as morning star, one as evening star. The total of the two elongations, 21;15 + 26;30 = 47;45° represents therefore the angle under which the epicycle of Mercury is seen when the mean sun is at ♊ 10° or at ♒ 10°, i.e. 120° distant from A. The observations represented in Fig. 144 indicate that at the perigee Π the epicycle appears only under an angle of 2 · 23;15 = 46;30° which is less than 47;45° found in ♊ and ♒. This implies the discovery of the fact, decisive for Ptolemy's theory of Mercury, that the epicycle is nearer to the observer at 120° distance from the apogee A than at the "perigee" Π diametrically opposite A. We shall return to this point in the next section.[6]

Ptolemy also had access to a group of older observations (from the third century B.C.) which he used skilfully for the determination of the secular motion of the apsidal line although they were not directly suitable for his purpose. These observations are of interest also for the history of ancient chronology. Out of six observations cited by Ptolemy, four use an otherwise unknown astronomical "Era Dionysius" with months named after the zodiacal signs; the two remaining observations are dated in the "Chaldean Era" with Macedonian month names. This latter era is the well-known Seleucid era, Syrian style, beginning in October[7] 311 B.C. The Dionysian era counts from a year 1 which corresponds to 285/4 B.C.[8]; at least one of these observations was also utilized by Hipparchus who determined (from what data and for what purpose we are not told by Ptolemy) the distance of Mercury from Spica.

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[3] Cf. Neugebauer [1968, 2].

[4] Tetrabiblos IV, 3 (Robbins, p. 381).

[5] These observations by Ptolemy belong to the years A.D. 132, 134, 135, 138, and 141; cf. above p. 158 and Fig. 16 (p. 1375).

[6] Cf. p. 161 and I C 3, 5.

[7] Not with Nisan (April) as in the Mesopotamian version of the Seleucid era which is used in the cuneiform texts.

[8] Cf. for details p. 1066.

The first triple of maximum elongations, recorded by Ptolemy, is the following one:

(1) Dionys. 23 Hydron 21: (= −261 Febr. 12)	Mercury	♑22;20	mean sun	♒18;10
(2) Dionys. 23 Tauron 4: (= −261 Apr. 25)		♉23;40		♈29;30
(3) Dionys. 28 Didym. 7: (= −256 May 28)		♊29;20		♊ 2;50.

In case (1) Mercury was 25;50 behind the mean sun. In the second and third case the planet was 24;10 and 26;30, respectively ahead of the sun. The desired symmetric value, 25;50, lies between the two figures. Assuming that the maximum elongations vary linearly with the longitude of the mean sun one finds that the maximum elongation would be 25;50° when the mean sun is at ♉23;30.

Similarly in the second triple:

(4) Dionys. 24 Leonton 28: (= −261 Aug. 23)	Mercury	♍19;30	mean sun	♌27;50	elong.	21;40
(5) Chald. 75 Dios 14: (= −236 Oct. 30)		♎14;10		♏ 5;10		−21; 0
(6) Chald. 67 Apellaios 5: (= −244 Nov. 19)		♏ 2;20		♏24;50		−22;30.

The cases (5) and (6) straddle the required elongation of −21;40° which would be symmetric to (4). Linear interpolation gives for the mean sun ♏14;10.

According to the first triple the apsidal line would bisect the arc from ♒18;10 to ♉23;30, i.e. go through ♈5;50. In the second triple the arc in question extends from ♌27;50 to ♏14;10. It is bisected by ♎6. Hence Ptolemy concludes that 400 years before his time the apsidal line of Mercury was going through ♎6/♈6 as compared with ♎10/♈10 at his time. This increase of longitude by 4° in 400 years corresponds exactly to Ptolemy's constant of precession. Thus he concluded that the apsidal line of Mercury, and hence probably of all planets, is sidereally fixed. One may well imagine that this agreement contributed to strengthen his confidence in the value of 1° per century for the constant of precession.

In fact it is a rash procedure to extrapolate a result obtained for one planet, and in particular for the most irregular planet, to all the other ones. But we have seen that Ptolemy had no old data for Venus, thus no possibility for checking this planet. In the case of the outer planets he determined the position of the apsidal lines for his time from three observations for each planet.[9] He had furthermore at his disposal some accurate observations from the third century B.C. which he used for the refinement of the parameters of mean motion under the assumption that the apsidal lines participate exactly in the motion of precession.[10] Thus the six old observations of maximum elongation of Mercury are the sole foundation for the theory of a precessional motion of the planetary apsidal lines. It is a fortunate accident that precession constitutes by far the most important term

[9] Cf. below p. 173ff.

[10] Cf. below p. 180ff.

in the secular displacement of the planetary apsidal lines, such that the continued use of Ptolemy's hypothesis did not constitute a serious source of error in planetary theory.[11]

2. Eccentricity and Equant

Adopting the same method as in the case of Venus the two observations represented in Fig. 144, p. 1252 can be used for the determination of the distance OM (cf. Fig. 137, p. 1249), with M bisecting the distance $A\Pi$. The algebraic procedure is also the same, based on units such that OA = 120. Then, as on p. 154 (3) to (5), one has

$$\begin{aligned} \mathrm{OA} &= R' + e' = 120 \\ r' &= \mathrm{crd}\, 2\alpha \\ \mathrm{O}\Pi &= R' - e' = 120\,\frac{\mathrm{crd}\, 2\alpha}{\mathrm{crd}\, 2\beta}. \end{aligned} \tag{1}$$

Substituting the values

$$\alpha = 19;3 \qquad \beta = 23;15 \tag{2}$$

found by observation (Fig. 144), Ptolemy obtains[1]

$$r' = 39;9 \qquad R' - e' = 99;9 \qquad e' = 10;25. \tag{3}$$

One would now expect (as in the case of Venus) a change of units such that MA becomes 60. Ptolemy does not do this because it would imply that M is the center of the deferent. This, however, is excluded by the observation, mentioned before (p. 159), that the epicycle does not appear largest, seen from O, when its center C is in Π but that greater angles were measured when C is in ♒ or ♊, 60° before or after Π. Thus, for C in Π, the center of the deferent must be nearer to O than M.

The situation is similar to the case of the second lunar anomaly[2]: parameters determined from observations in the syzygies do not explain increased effects due to the epicyclic anomaly in positions of the epicycle different from the syzygies. Ptolemy's explanation for this phenomenon consists in the assumption that the epicycle is nearer to the observer when the effect of the anomaly appears magnified. Such a change of distance, however, requires a movable center of the deferent. Ptolemy applied exactly the same remedy in the case of Mercury. The center N of the deferent is made movable in such a fashion that the center C of the epicycle is at the proper distance OA and $\mathrm{O}\Pi$ when its orbit crosses the apsidal line, but comes nearer to O than in Π when C is in ♒ or ♊. Circular motions being taken for granted, the center N of the eccenter is made to rotate on a small circle the dimensions of which must now be determined.

The next step, therefore, consists in following the same procedure by which the position of the equant for Venus had been found (cf. above p. 154 f.). Ptolemy takes two observations of maximum elongations, the mean sun being very near

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[11] Kepler seems to have been the first to recognize an independent displacement of the planetary apsides (Houzeau, Vade-Mecum, p. 384). Real insight came, of course, only with Newton.

[1] Using Ptolemy's own tables of chords accurately one finds 39;10,10, 99;13,20, and 10;23,20 respectively, very close to the values obtainable by modern tables.

[2] Above p. 85.

♋ 10°, i.e. in a direction perpendicular to the apsidal line, one observation made by Theon, one by himself.[3] The observed angles, denoted by α and β in Fig. 138 (p. 1249) are

$$\alpha = 26;15 \qquad \beta = 20;15. \tag{4}$$

Continuing as in the case of Venus one finds for the distance $d' = \mathrm{OE}$ (with $\mathrm{OA} = 120$)

$$d' = 5;12. \tag{5}$$

Since, according to (3), $\mathrm{OM} = e' = 10;25$ Ptolemy accepts

$$\mathrm{OE} = d' = 1/2\, e' = 1/2\, \mathrm{OM}. \tag{6}$$

Thus the equant E for the motion of the epicycle of Mercury is the midpoint of the distance OM while for all other planets M is the midpoint of OE.

The position of the epicycle which served for the determination of the equant can be called quadrature with respect to the apsidal line. The angle under which the epicycle is seen from O is given (cf. (4)) by

$$\alpha + \beta = 46;30.$$

Exactly the same angle $2 \cdot 23;15 = 46;30$ was found for the apparent size of the epicycle when located at Π (cf. p. 159 and Fig. 144, p. 1252). In other words: the epicycle is at the same distance from O when it is at the perigee Π and in quadrature.

The mean motion of the center C of the epicycle has the equant E as its center. The same mean angular velocity must be given to the motion of the center of the deferent such that the original configuration is restored each time C is in the apsidal line, i.e. at A and at Π. At these two moments the radius s of the small circle on which the center N of the deferent rotates, as well as the radius R of the deferent, must coincide with the apsidal line. Consequently, M must be the center of the small circle of radius s and $R + s = \mathrm{AM} = \mathrm{M}\Pi$ (cf. Fig. 145). But $\mathrm{AM} = \mathrm{M}\Pi$ is the distance which we previously denoted by R' (cf. (1), p. 161); hence

$$R + s = R'. \tag{7}$$

We now return to the position of C in quadrature. C has moved 90° from the apsidal line, therefore also the center N of the deferent must have moved 90° on its small circle with center M, but such that C has been pulled nearer to the apsidal line because we have found that $\mathrm{OC} = \mathrm{O}\Pi$ and we know that $\mathrm{O}\Pi < \mathrm{M}\Pi$ (p. 161). Hence we obtain a configuration as represented in Fig. 146. The distance of C from the apsidal line is approximately given by $R - s$ and, since all eccentricities are small, we may also say that $\mathrm{OC} \approx R - s$. But $\mathrm{OC} = \mathrm{O}\Pi$ and, because

[3] The dates given are Hadrian 14 Mesore 18 (= 130 July 4) and Antoninus 2 Mesore 24 (= 139 July 8). The solar longitude is only correct for the first date; the second one must be changed to Mesore 21 (= July 5) in order to bring the sun to the same place as in the first observation. The error is of a very common type (interchange of A and Δ) but must be old since it is common to all manuscripts. The whole passage is obviously corrupt; it should read *Μεσορὴ [κ'] εἰς τὴν κα' ὄρϑρου.*

Gerard of Cremona gave the correct reading in his translation of the Almagest (Venice 1515, p. 106) while George of Trapezunt had missed it (ed. 1451, p. 101). Copernicus, De revol. V, 27 (Gesamtausgabe, p. 343, 10) adopted the correct date (as noted by Menzzer note 444, only to be again overlooked in the Gesamtausgabe).

of (6), $O\Pi = M\Pi - OM = R' - 2d'$. Therefore

$$R - s = R' - 2d'. \tag{8}$$

Combining this with (7) one finds

$$s = d'. \tag{9}$$

Thus we have obtained, guided step by step by observational data, the following model for the motion of Mercury (cf. Fig. 147): the center N of the deferent moves retrograde with the same angular velocity about the midpoint M of $A\Pi$ as the center C of the epicycle is seen from the equant E to move forward.[4] The radius of the small circle is given by

$$d' = OE = EM = MN \tag{10}$$

such that N coincides with E when C coincides with Π.

All that remains to be done is to change the norm of the units such that $R = NC$ becomes 60. We had so far assumed that $R' + e' = R' + 2d' = 120$. Thus from (7), (9), and (5)

$$R = R' - d' = 120 - 3d' = 120 - 15;36 = 1,44;24.$$

Consequently, if we adopt $R = 60$, we have to divide the value of r' in (3) and of d' in (5) by 1;44,24. This results in

$$r = 22;30 \qquad d = 2;59,19 \approx 3 \qquad (R = 60). \tag{11}$$

From now on we shall denote the second of these parameters by e and call it the "*eccentricity*" of Mercury. Hence

$$e = OE = EM = MN = 3 \tag{11a}$$

such that

$$O\Pi = R - e = 57 \qquad OA = R + 3e = 69 \tag{12}$$

are the distances of O from the endpoints of the apsidal line.

3. Perigees

As shown in the previous discussion[1] observations of maximum elongations of Mercury convinced Ptolemy that the center C of the epicycle has a smaller geocentric distance when located in ♒ 10° or ♊ 10° than in the "perigee" Π in ♈ 10°. No proof is given, however, that ♒ 10° and ♊ 10° are actually the points of minimum distance $\rho = OC$, although this is in fact very nearly correct as we shall show presently.[2] In other words Ptolemy is quite right in calling ♒ 10° and ♊ 10° the "*perigees*" of the deferent although a rigorous proof is lacking.

[4] The velocity of this motion is, of course, equal to the mean motion of the sun.

[1] Above p. 159.

[2] Below p. 168f.

Ptolemy does compute, however, on the basis of the adopted cinematic model the distance ρ for these perigees and shows explicitly that it is less than $O\Pi = R-e=57$. He also determines the angle 2γ under which the epicycle appears from O when it is located in one of the perigees; he finds good agreement with the observed greatest elongations, not very surprisingly since these observations had been used to establish the dimensions of the model. Because some of the numerical data will be needed later on we give here the single steps of Ptolemy's computation.

We assume (cf. Fig. 148) that C appears, seen from E, at 120° from the apsidal line $A\Pi$. Consequently MN makes the same angle with OM and MNE is an equilateral triangle; thus $\eta=60°$. Hence, for $e=3$:

$$\mathrm{EL}=\frac{e}{120}\operatorname{crd} 2(90-\eta)=1;30$$

$$\mathrm{OL}=\frac{e}{120}\operatorname{crd} 2\eta=2;36 \qquad \text{(accurately: 2;35,53)}$$

and therefore

$$\begin{aligned}\mathrm{CL}&=R-(\mathrm{EL}+e)=55;30\\ \rho&=\sqrt{\mathrm{CL}^2+\mathrm{OL}^2}=55;34.\end{aligned} \tag{1}$$

This distance ρ is indeed smaller than $O\Pi=R-e=57$.

We now draw the tangent OT to the epicycle of radius $r=22;30$. Then one finds for the angle γ at O

$$\operatorname{crd} 2\gamma=\frac{120r}{\rho}=48;35 \qquad \text{(accurately: 48;35,25)}$$

and from the table of chords

$$2\gamma=47;46 \qquad \text{(accurately: 47;46,20).}$$

This is close enough to the observed maximum elongation (cf. Fig. 143, p. 1251)

$$26;30+21;15=47;45$$

the mean sun being in ♒ 10° or ♊ 10°.

Combining (11) and (12) of p. 163 with (1) we can say that according to Ptolemy's theory Mercury can be as far as

$$D=R+3e+r=91;30 \qquad (R=60) \tag{2}$$

from the earth and come as near to it as

$$d=\rho-r=33;4. \tag{3}$$

For the ratio of these extremal distances one finds

$$\frac{D}{d}=\frac{1,31;30}{33;4}\approx 2;46. \tag{4}$$

4. Mean Motion in Anomaly. Epoch

Ptolemy applies his model for the motion of Mercury to establish as accurately as possible the mean motion in anomaly of the planet. The method is exactly the same as in the case of Venus. He compares an observation of Mercury as evening star in the year Nabonassar 886 Epiphi 2 (=A.D. 139 May 17) with an earlier observation as morning star in Nabonassar 484 Thoth 19 (= −264 Nov. 15); in both instances the planet had not yet reached maximum elongation.

Then, in each case, two angles are known: the longitudinal difference δ of the planet from the apsidal line and the longitude of the mean sun, identified with the longitude of the center C of the epicycle, at the moment of observation. Therefore the distance $\bar{\kappa}$ of C from the apsidal line is known. From the two angles, δ and $\bar{\kappa}$, and from the given eccentricity e and radius r of the epicycle the position of the planet P on the epicycle can be determined as follows.

In Fig. 149 we know $\bar{\kappa}$ and e (and $R=60$), O representing the observer, E the equant, M the center of the circle of radius e on which the center N of the deferent moves the same amount backwards by which EC moves forward (cf. Fig. 147, p. 1252). Then

$$\mathrm{EN}=2e\cos\frac{\bar{\kappa}}{2} \qquad \mathrm{LN}=\mathrm{EN}\sin\frac{3\bar{\kappa}}{2} \qquad \mathrm{LE}=\mathrm{EN}\cos\frac{3\bar{\kappa}}{2}$$

$$\mathrm{CL}=\sqrt{R^2-\mathrm{LN}^2} \qquad \mathrm{CE}=\mathrm{CL}+\mathrm{LE}.$$

In order to find the angle β (which is the equation of center of C) Ptolemy computes $\rho=\mathrm{OC}$ from

$$\mathrm{OQ}=e\sin\bar{\kappa} \qquad \mathrm{EQ}=e\cos\bar{\kappa} \qquad \mathrm{CQ}=\mathrm{CE}+\mathrm{EQ}$$

$$\rho=\sqrt{\mathrm{OQ}^2+\mathrm{CQ}^2}$$

and finally β from

$$\sin\beta=\frac{\mathrm{OQ}}{\rho}.$$

We now consider the epicycle (cf. Fig. 150) on which the planet is seen at an angle δ with respect to the apsidal line OE. Its position on the circle of radius r is uniquely determined since we know that the maximum elongation has not yet been reached. From the given $\bar{\kappa}$ and from β just found we know

$$\gamma=\bar{\kappa}-\beta.$$

The angle δ is given by observation, therefore also

$$\zeta=\gamma-\delta=\bar{\kappa}-\beta-\delta$$

is known and the angle ε at P can be found from

$$\sin\varepsilon=\frac{\rho}{r}\sin\zeta.$$

This allows us to find the mean anomaly $\bar{\alpha}$ of P, counted from the "*mean apogee*" F of the epicycle:

$$\bar{\alpha}=180-\beta+\varepsilon-\zeta=180+\varepsilon+\delta-\bar{\kappa}. \tag{1}$$

Combining the values of $\bar{\alpha}$ obtained in this fashion for the two observations and dividing the result by the corresponding time difference gives the desired value for the mean motion in anomaly.

Before substituting numerical values we must discuss some features of the basic observations. The earlier observation (from the year Nabonassar 484) was originally dated in the era Dionysius, year 21, month Scorpion, day 22. Thus it belongs to the same group of records which Ptolemy had used before for the determination of the motion of the apsidal line of the planet.[1] Ptolemy describes the position of Mercury according to this observation as follows. With reference to a straight line connecting the stars β and δ Sco Mercury was one lunar diameter to the east of this line and two lunar diameters to the north of β Sco.

At that time the coordinates of these two stars were[2]

$$\beta\,\text{Sco}:\ \lambda=\text{♏}2;20 \qquad \beta=+1;20$$
$$\delta\,\text{Sco}:\ \lambda=\text{♏}1;40 \qquad \beta=-1;40.$$

Hence, so Ptolemy concludes, Mercury had the longitude ♏3;20.

In order to find out how Ptolemy reached this result it is best to proceed graphically (cf. Fig. 151). One lunar diameter is certainly close to 1/2°, "east" must mean to the left of the line connecting the two stars, and "north" cannot much differ from the direction of positive latitudes. Obviously the line $\lambda=$♏3;20 cannot contain a point within the area satisfying these general conditions. If, however, we determine the point of latitude 1;20 + 1;0° on the parallel 0;30° to the east of the line connecting the two stars we find for its longitude about ♏3;3. This could suggest the emendation of the passage *μοίρας* $\bar{\gamma}$ *καὶ* γ'[3] to ... *καὶ λεπτὰ* γ.[4] The subsequent computations, however, are based on the value 3;20°.

In order to show that the planet had not yet reached maximum elongation from the sun (i.e. to show that it was located, according to Fig. 150, at the point P and not at Q) Ptolemy adduces a second observation, made 4 days later. At that time the planet was found 1 1/2 lunar diameters east of the line defined by δ and β Sco. Thus the planet had gained in 4 days only 1/2 lunar diameter or 0;15° while the sun had moved about 4°. Obviously the planet was very near a stationary point and therefore still before maximum elongation.

These data can be compared with modern computations.[5] Fig. 152 represents the motion of Mercury during the period in question in 5-day intervals. The position No. 10 corresponds to −264 Nov. 14, thus one day before Ptolemy's first observation; No. 11, for Nov. 19, coincides with the second observation.

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[1] Cf. above p. 159f.

[2] Obtained from the catalogue of stars in Alm. VIII, 1 by subtracting 4° of longitude for precession during 400 years (instead of 5;30°). This is one of the cases which illustrate the intricate interplay between constant of precession and values of specific parameters, here mean motion in anomaly.

[3] Heiberg, p. 288, 20/289, 1.

[4] Copernicus, De revol. V, 29 (Gesamtausg., p. 346) simply inverted the data: 2 lunar diameters east, 1 north. But the continuation of the text excludes this remedy, even if it were not too drastic.

[5] Using Tuckerman, Tables.

The positions of the two stars at that time were[6]

$$\beta\,\text{Sco}: \lambda = 211.7 \quad \beta = +1.3$$
$$\delta\,\text{Sco}: \lambda = 211.1 \quad \beta = -1.7.$$

With respect to the line defined by these two stars the earlier position would correspond to the point a in Fig. 152, the second to the point b which should coincide with the point No. 11.

The agreement in longitudes would be almost perfect if we would use for the two fixed stars the coordinates given to them by Ptolemy on the basis of his constant of precession because this would move the line of reference, defined by the two stars, 1/2° to the left. The latitudes, however, would remain about 1/2° too low.

We now turn to Ptolemy's evaluation of the two observations as he had received them. The moments in question are

t_1 = Nabon. 484 Thoth 19 6;0 a.m. (= −264 Nov. 15 = j.d. 1624 951)
t_2 = Nabon. 886 Epiphi 2 7;30 p.m. (= +139 May 17 = j.d. 1771 964)

hence

$$\Delta t = 402 \text{ eg.y. } 283^d\ 13;30^h.$$

At t_1 Mercury was seen in ♏ 3;20, its apogee was in ♎ 6°,[7] therefore $\delta_1 = 27;20$. The mean sun at this date was, according to Ptolemy's tables, in ♏ 20;50, hence $\bar{\kappa}_1 = 44;50$.

Similarly for t_2: Mercury was observed in ♊ 17;30, the apogee was located in ♎ 10,[8] hence $\delta_2 = 180 - 112;30 = 67;30$. The mean sun was in ♉ 22;34, hence $|\bar{\kappa}_2| = 180 - 42;34 = 137;26$.

Computing with these quantities one finds by the steps outlined above

$$\beta_1 = 1;54 \qquad \beta_2 = 2;5,30$$
$$\rho_1 = 64;7 \qquad \rho_2 = 55;51$$
$$\varepsilon_1 = 50;4 \qquad \varepsilon_2 = 74;31$$

and finally

$$\bar{\alpha}_1 = 212;34 \qquad \bar{\alpha}_2 = 99;27.$$

Hence the gain in anomaly during Δt is given by

$$\Delta\bar{\alpha} = 360 - \bar{\alpha}_1 + \bar{\alpha}_2 = 246;53.$$

In order to find the total motion in anomaly during Δt we have to determine the number of complete synodic periods, i.e. rotations of the planet on the epicycle. From the relation[9] 46 years = 145 synodic periods it follows that one synodic period amounts to nearly 1,55;50 days. Hence 63 synodic periods are close to $2,1,37^d \approx 2,1,40^d = 20$ eg.y.; thus 400 eg.y. contain $20 \cdot 63 = 1260$ synodic periods. The remaining $2^y\ 283^d = 1013^d$ contain 8 additional synodic periods (totalling about 927^d). Thus Δt contains a total of 1268 complete rotations in

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[6] According to Schoch, Planetentafeln.
[7] Cf. above p. 160.
[8] Cf. above p. 159.
[9] Cf. p. 151.

anomaly to which we have to add $\Delta\bar{\alpha}$. In this way Ptolemy finds for the mean motion in anomaly

$$\frac{1268 \cdot 360 + 246;53^\circ}{402 \cdot 365 + 283^d\, 13;30^h} = 3;6,24,6,59,35,50^{\circ/d}. \tag{2}$$

This is the value used in the tables Alm. IX, 4.[10]

Finally one finds for Nabonassar 1 Thoth 1 the following elements for Mercury: mean longitude ♓ 0;45, apogee of eccenter at ♎ 1;10, mean anomaly $\bar{\alpha}_0 = 21;55^\circ$ (using (2) and the observations at t_1 or t_2).

5. Minimum Distance and Motion of the Center of the Epicycle

W. Hartner investigated[1] the whole family of curves which originate by the crank mechanism which Ptolemy had adopted in the cinematic model for the motion of the center C of the epicycle of Mercury. Hartner showed that for the Ptolemaic parameters, $R = 60$ and $e = 3$, the curve in question is nearly an ellipse and that Ptolemy's assertion that the minimum distance of C from the observer O is found at $\bar{\kappa} = 120^\circ$ ($\bar{\kappa}$ being the angle of mean motion with the equant E as vertex and reckoned from the apogee A) is very nearly correct since it actually occurs at about $\bar{\kappa} = 120\,1/2^\circ$. The following is a summary of Hartner's discussion.

It is easy to find the equation of the curve described by C in polar coordinates $y = \mathrm{EC}$ and $\bar{\kappa}$ with the equant E as center (cf. Fig. 149, p. 1253). Since $\mathrm{EN} = 2e \cos\frac{\bar{\kappa}}{2}$ one has

$$R^2 = y^2 + 4e^2 \cos^2\frac{\bar{\kappa}}{2} - 4ey \cos\frac{\bar{\kappa}}{2} \cos\frac{3\bar{\kappa}}{2} \tag{1}$$

or, by simple trigonometric transformations, the following quadratic equation for y:

$$y^2 - 2e(\cos\bar{\kappa} + \cos 2\bar{\kappa})\, y - R^2 + 2e^2(1 + \cos\bar{\kappa}) = 0$$

from which follows

$$y = e(\cos\bar{\kappa} + \cos 2\bar{\kappa}) + \sqrt{R^2 - e^2(\sin\bar{\kappa} + \sin 2\bar{\kappa})^2}. \tag{2}$$

Hartner then showed by numerical computation that the points of this curve lie very close to the points of an ellipse of midpoint M and half axes $a = R + e$, $b = R - e$. The approximation is so close that the same results are obtained for the equation of center, within the accuracy of the tables in Alm. XI, 11, whether one uses the points of Ptolemy's curve or of the approximating ellipse.

The geometrical principle upon which Hartner's computation rests makes it easy to see that the two curves will be very near to each other. A point K (cf. Fig. 153) describes an ellipse of center M and half-axes $a = R + e$, $b = R - e$ when the angle ψ of KN ($= R$) with the major axis (MA) is always the same as the angle which the radius MN ($= e$) of a circle with center M makes with MA.[2] In Ptolemy's model the position of C is found (cf. Fig. 154) by making the angle AEC (E being the equant) equal $\psi = \bar{\kappa}$ and the distance NC equal R. Thus the point C of Ptolemy's

[10] In checking this division I find for the last three digits only 58,39,48.

[1] Hartner [1955], p. 109 to 117.

[2] Obviously the cartesian coordinates of K are $\mathrm{ML} = (R + e)\cos\psi$ and $\mathrm{LK} = (R - e)\sin\psi$, respectively.

curve and the point K of the ellipse are both located on a circle of radius R and center N; one can say that K lies always on the circle of instantaneous motion of the point C which corresponds to K in Ptolemy's mechanism. Thus it is plausible to assume that K and C trace very similar curves as $\psi=\bar{\kappa}$ varies from 0 to 180; this is fully confirmed by the numerical computation. The points C and K coincide exactly not only for $\psi=\bar{\kappa}=0$ and 180 but also for $\psi=\bar{\kappa}=\pm 120°$ which gives, according to Ptolemy, the two perigees of the orbit of Mercury.

Hartner also computed the geocentric distances $\rho=\mathrm{OC}$. His table for the quantity $s=\rho/e$ is unfortunately marred by a misprint at the decisive point and should read as follows[3]

$$\begin{array}{lll} \psi=119° & s=18.5211 & \\ 120 & 18.5203 & \\ 121 & 18.5203 & \text{(not 18.5176)} \\ 122 & 18.5212. & \end{array}$$

Thus Ptolemy's statement is nearly correct since the accurate minimum lies not quite 1/2° beyond 120°.

The question how Ptolemy could assert that the minimum distance OC occurs at 120° of mean anomaly $\bar{\kappa}$ seems to me to allow of only one, rather obvious, answer: by direct numerical computation of the distances OC which he needed in any case in order to build up the table of equations in Alm. XI, 11. Computing in steps of 3° the smallest value found would occur at $\bar{\kappa}=120$ and it would have been of no practical interest to establish the fact that the accurate minimum belongs to a value between 120° and 121°. The following Table 14 gives the distances $\rho=\mathrm{OC}$ as function of $|\bar{\kappa}|$, computed with Ptolemy's method. I consider deviations from Ptolemy's table of more than $\pm 0;1$ as unlikely.

Table 14

$\lvert\bar{\kappa}\rvert$	ρ	$\lvert\bar{\kappa}\rvert$	ρ	$\lvert\bar{\kappa}\rvert$	ρ	$\lvert\bar{\kappa}\rvert$	ρ
0	69; 0, 0						
3	68;58,20	48	63;30,50	93	56;43,10	138	55;52,50
6	68;53,40	51	62;57,50	96	56;28,10	141	55;59,10
9	68;45,50	54	62;24,50	99	56;15, 0	144	56; 5,30
12	68;35,10	57	61;52,10	102	56; 3,50	147	56;12, 0
15	68;21,20	60	61;19,50	105	55;54,30	150	56;18,50
18	68; 4,50	63	60;48, 0	108	55;46,50	153	56;25,30
21	67;45,50	66	60;17,20	111	55;41, 0	156	56;31,50
24	67;24, 0	69	59;47,40	114	55;37, 0	159	56;37,50
27	67; 0,10	72	59;19, 0	117	55;34,30	162	56;43,20
30	66;34,30	75	58;51,50	120	55;33,40	165	56;48,10
33	66; 6,40	78	58;26,10	123	55;34,10	168	56;52,20
36	65;37,40	81	58; 2, 0	126	55;35,50	171	56;55,40
39	65; 7,10	84	57;39,40	129	55;38,40	174	56;58, 0
42	64;35,40	87	57;19, 0	132	55;42,40	177	56;59,30
45	64; 3,20	90	57; 0,20	135	55;47,30	180	57; 0, 0

[3] Hartner [1965], p. 268, note 25.

§ 4. The Ptolemaic Theory of the Motion of an Outer Planet

1. The Basic Ideas

Any theory of the motion of the outer planets must satisfy two conditions: (a) it must explain the characteristic phenomena, e.g. retrogradations, stations, etc.; (b) it must maintain a fundamental numerical identity between the number of occurrences of the same phenomenon (e.g. opposition to the sun) and the number of sidereal rotations of the planet and of the sun. This relation, well known in Babylonian astronomy,[1] states that

Number of occurrences + number of sid. rot. of the planet = number of years. (1)

It is easy to see that a sequence of observed positions of an outer planet suggests an epicyclic motion (cf. Figs. 155 and 156).[1a] The mean motion of the center C of the epicycle (cf. Fig. 157) in the direction of increasing longitudes accounts for the general progress of the planet in the ecliptic. When the motion of the planet P on the epicycle has the same direction as the motion of C then the planet proceeds faster than the mean motion; but if the motion of P on the epicycle is sufficiently fast in the opposite direction a retrograde motion will be the result.

That the motion of C proceeds from west to east is directly shown by observation. It is, however, not at all obvious in what sense the planet rotates on its epicycle since retrogradation can result not only from a rotation as indicated in Fig. 157 but also with the opposite sense of rotation of P. We know, e.g. from Pliny, that also this second possibility had been proposed in planetary theories and we shall discuss presently its consequences.[2] By the time of Ptolemy, however, it had long become clear that the sense of rotation on the epicycle must be the same as the sense of rotation of the center of the epicycle. In other words it had been established that the epicyclic motion of a planet proceeds in a direction opposite to the motion of the moon and of the sun on their respective epicycles.

It is also readily recognizable that the fundamental relation (1) can be satisfied by an epicyclic motion. All one has to do is to regulate the motion of the planet on the epicycle in such a fashion that the epicycle radius CP is always parallel to the direction from the observer O to the sun (Fig. 158). The number of occurrences of identical phases is obviously counted by the rotations of the angle α; for example retrogradations occur each time when α is near 180°. The angle κ, however, measures the mean progress of the planet, i.e. the number of sidereal rotations. Finally γ counts the number of sidereal rotations of the sun, i.e. years. The parallelism of CP and $O\odot$ is then expressed by

$$\alpha + \kappa = \gamma \tag{2}$$

and this is the equivalent of the relation (1).

[1] Cf. below p. 389; also above p. 150f., (1) and (2).

[1a] The small circles give the position of Saturn in 20-day intervals, for Jupiter in 10-day steps. The degrees of latitude are represented in units twice as large as the longitudes.

The planet is visible from Γ to Ω, invisible from Ω to Γ. The graph shows clearly the motion of an epicycle along an inclined deferent and the return to a loop of similar shape and position after 30 years and after 12 years, respectively. This periodicity would be still more outspoken after a period of 59 years.

[2] Cf. below V A 1, 4.

The parallelism of the direction from C to the planet and from the observer to the sun also explains the fact that the planet becomes invisible for a stretch near $\alpha=0$ because it is then too near to the line of sight to the sun.

Thus we have shown that an epicycle model of the type of Fig. 158 accounts at least qualitatively for the main empirical features of planetary motion. Yet, it is evident that this model is too simple because it would follow from it that the planetary phenomena occur in exactly the same fashion in every part of the zodiac; for example the length of the retrograde arc would be the same everywhere because the epicycle is always at the same distance from O. Experience, however, shows that the size of the retrograde arcs and the shape of the loops described by the planet depend on the location in the zodiac (cf. Figs. 155 and 156). The obvious answer to this observation is the assumption of an eccentric deferent which causes a variation in the distance of the epicycle from the observer. The consequences of this modification must now be investigated.

2. Refinement of the Model

We consider a model in which the observer O is no longer located at the center M of the deferent (Fig. 159). In order to satisfy the basic relation (1) we require again that

$$\alpha+\kappa=\gamma. \tag{2}$$

It is clear that this is equivalent, also in the new arrangement, to the statement that the direction CP is always parallel to the direction from O to the sun, since (Fig. 160) it follows from (2) and $\kappa=\varepsilon+\delta$ that $\varepsilon+\alpha=\gamma-\delta$.

Both in the simple model, Fig. 158, and in the refined model, Fig. 159, the basic assumption of Greek astronomy is implied, namely that the motion of C as well of P is uniform, that is to say that both κ and α increase proportionally with time. Consequently the same holds for γ and we could have replaced in all previous statements the word "sun" by the more accurate term "mean sun." Thus it is the postulate of the uniformity of circular motions that introduces into planetary theory the mean sun as the fundamental point of reference, a procedure which is still fully maintained in the Copernican theory. It was Kepler who first referred all empirical data to the true sun instead of the mean sun.

An equally important step in the history of the theory of planetary motion is due to Ptolemy, a step which was eliminated by philosophical reasons in Copernicus' theory but again fully recognized in its importance by Kepler. Ptolemy's generalization is based on the fact that the preservation of the relation (2), or of the equivalent construction in Fig. 160, does not require that the distance CM remains constant. In other words it is not necessary to assume that the center of the uniform motion of C coincides with the center of the deferent. Thus it becomes a matter of empirical investigation where the center E of uniform motion[3] of C is located with respect to the center M of the circular deferent on which C moves. And indeed, as we have seen, it did turn out that for Venus and Mercury E and M do not coincide.

[3] For the corresponding term "*equant*" cf. below p. 1102.

Having once admitted the possibility of a differentiation between an equant E and the center M of the deferent the problem arises to determine the relative position not only of E with respect to M but also with respect to O. From the introduction to the theory of Mars (Almagest X, 6) one can deduce what type of empirical data led Ptolemy to his final model in which M is the midpoint of OE. First he succeeded in determining a circular orbit (with center M) which accounted for the observed "anomaly with respect to the ecliptic," i.e. for the anomalistic motion of the center C of the epicycle. This point C itself is, of course, not observable but at opposition the planet is seen in the same direction as C because when O lies on the straight line which connects the sun with the planet then also the radius CP must coincide with this line (cf. Fig. 160). In the same fashion in which the size and position of the epicycle which causes the lunar anomaly can be found from three lunar eclipses, it is possible to find from three planetary oppositions the size and position of OE.

Since the eccentricity of the deferent determines the distance of the epicycle from the observer and since the arc of retrogradation decreases with increasing distance one can conversely derive the eccentricity of the deferent from the observed retrogradations. Though Ptolemy does not tell the details of his procedure he states that he found that the observed arcs of retrogradation require a deferent of only about half the eccentricity found before from the motion of C.[4] Consequently he assumed for the outer planets a model of the type of Fig. 161 in which the direction ECF moves with constant angular velocity about the equant E whereas the circular deferent which carries C has its center at M, halfways between E and O. The planet P on the epicycle moves uniformly such that CP is always parallel to the direction from O to the mean sun.

The close resemblance of this arrangement to the model obtained for Venus from much more direct observational data certainly contributed to the acceptance of exact symmetry of E and O with respect to M. Kepler, though operating very much along the same lines as Ptolemy, at first abandoned the assumption of equality between the two eccentricities when he tried to explain the observed oppositions of Mars by means of a circular orbit. In a totally unexpected fashion he finally restored symmetry by realizing that O and E are the foci of an elliptic orbit.

3. Determination of the Eccentricity and Apogee

Ptolemy does not show us in detail the steps which led him to his final model. Except for the above mentioned introductory remarks (in Alm. X, 6) concerning the different eccentricities obtained (a) from the anomaly with respect to the ecliptic and (b) with respect to the sun, we are not told how he originally tried to

[4] The observations are specified as made at extremal distances of the epicycle, i.e. when C is a point of the straight line OE. In this case the determination of the eccentricity OM which accounts for the observed retrograde arc causes no difficulty (whereas a general position of C leads to a rather complicated computation). By reason of symmetry it is clear that also M must lie on OE. This is no proof, however, that OME are always on one line but I do not know whether such a proof has ever been attempted.

determine the elements for the orbits of the outer planets. In the long computation of these parameters (Alm. X, 7 to XI, 8) the symmetric arrangement of observer, center of deferent, and equant is already taken for granted. As justification is used the agreement with observed positions of the longitudes computed according to this model. The successful description of empirical facts is indeed a most valid argument in favor of a theory; but it does not help our attempts to reconstruct the history of the problem. Whatever we may say about the motivation of certain steps in Ptolemy's procedures remains therefore largely conjectural.

A. Eccentricity from Oppositions

It is our first goal to determine the eccentricity and the direction of the apsidal line of the circular deferent which carries the center C of the epicycle. Since a circle is determined by three points we consider to be given by observations three positions C_1, C_2, C_3 of C, of course always ignoring latitudes. Thus we know two longitudinal differences, δ_1 and δ_2, between C_1 and C_2 and between C_2 and C_3, respectively, corresponding to time intervals $\Delta_1 t$ and $\Delta_2 t$ between the observations. Since the mean motions of the planets are known with sufficient accuracy we can also determine the angles $\bar{\delta}_1$ and $\bar{\delta}_2$ which separate C_2 from C_1 and C_3 from C_2 as seen from the center E of mean motion.

Thus one should solve the following problem: three points appear from O under the angles δ_1, δ_2, from E under $\bar{\delta}_1$, $\bar{\delta}_2$. Find the position of E with respect to O.

Unfortunately the problem is not determined in this form as is easy to see. Assume that C_1, C_2, C_3 are located as shown in Fig. 162 on their proper rays. Then E lies on one of the intersections of the circles with given angles $2\bar{\delta}_1$, $2\bar{\delta}_2$ at their centers over the chords C_1C_2 and C_2C_3, respectively. Obviously the position of E is a function of the arbitrarily chosen distances of C_1, C_2, C_3, from O.[1]

In order to make the problem definite Ptolemy assumes at first that E is the center of the circle on which C_1, C_2, C_3 are located. That the problem is solvable in this form is known from the solar and from the lunar theory. The Hipparchian determination of the eccentricity of the solar orbit is of this type: from three observations of the sun and the corresponding angles of mean motions at the center of the orbit the eccentricity of the orbit can be determined.[2] In Hipparchus' procedure both angles δ_1 and δ_2 were right angles which makes the problem very simple.[3] But in the lunar theory the same problem occurred in full generality in the determination of the eccentricity from three lunar eclipses[4] and it is exactly the same method which Ptolemy follows now for the outer planets.

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[1] Hill [1900] has shown that the problem can be made definite by requiring that the center M of the circle through C_1, C_2, C_3 is the midpoint of OE. In this form the problem leads to an algebraic equation of the 8th degree and one of its 6 real roots corresponds to Ptolemy's solution which he obtained by an iteration process; cf. below p. 178, note 6. I owe the reference to Hill's paper to Mr. Stephen Gross.

[2] Cf. above p. 57ff.

[3] Determinations of the solar eccentricity from differently located observations occur in Islamic astronomy (e.g. Bīrūnī, Chronol., p. 167), also by Copernicus (De revol. III, 16 = Gesamtausg. II, p. 190f.), and Brahe (Progymn. I = Opera II, p. 19ff.); cf. Neugebauer [1962, 2], p. 274f.

[4] Above p. 73ff.

This simplification of the problem implies that the equant and the center of the deferent coincide; thus a return to the more primitive model of Fig. 159. One may perhaps assume that this step was not only induced by mathematical necessity. It seem the most natural procedure to operate first with a model of the type of Fig. 159 and find the corresponding eccentricity OM. Thereafter Ptolemy must have discovered that the resulting retrogradations are not confirmed by observations which indicate that the eccentricity of the deferent should be made considerably smaller. Having nevertheless resolved to keep the equant E at essentially the eccentricity found before he had to face the problem of investigating the modifications required by the separation of E from M.

Whether or not this sequence of considerations reflects the historical events cannot be proved but the procedures actually followed by Ptolemy show exactly this order.

B. Approximative Solution

Ptolemy begins his determination of the eccentricities of Mars, Jupiter, and Saturn with the assumption that the center of the deferent serves also as equant. Thus we consider to be given two angles, $\bar{\delta}_1$ and $\bar{\delta}_2$, of mean motions with center E, and two angles, δ_1 and δ_2, with vertex O, the observer, leading to three points D_1, D_2, D_3 on the deferent of radius R and center E (Fig. 163). The latter angles are obtained from observations of oppositions and represent therefore, at least in principle, positions of the center of the epicycle. "In principle" means that we must keep in mind that eventually the center of the epicycle will not be located on a circle with center E but with center M, halfway between E and O. Consequently D_1, D_2, D_3 can only be considered as approximations of the positions of the center of the epicycle. It is for this reason that we have replaced the previous letters C_1, C_2, C_3 by D_1, D_2, D_3.

Under the assumptions represented in Fig. 163 Ptolemy proceeds as follows.

Step 1. Fig. 164: extend D_3O to Δ. In the triangle $O\Delta D_2$ all three angles are known, thus the ratio a/x can be found.

Step 2. Fig. 165: in the triangle $O\Delta D_1$ all three angles are known, thus b/x can be found.

Step 3. Fig. 166: since $c^2 = D_2\Gamma^2 + D_1\Gamma^2$ and

$$D_2\Gamma = a - b\cos(\bar{\delta}_1/2) \qquad D_1\Gamma = b\sin(\bar{\delta}_1/2)$$

one has

$$\left(\frac{c}{x}\right)^2 = \left(\frac{a}{x} - \frac{b}{x}\cos\frac{\bar{\delta}_1}{2}\right)^2 + \left(\frac{b}{x}\sin\frac{\bar{\delta}_1}{2}\right)^2. \tag{1}$$

Since a/x and b/x have been found (Steps 1 and 2) the right-hand side in (1) is known and therefore also c/x.

Step 4. In the circle of radius R one can find c alone from

$$c = 2R\sin(\bar{\delta}_1/2) \tag{2}$$

and therefore x will be known from (1). This, in turn, leads to the value of a and b in terms of the units of R (in Ptolemy's norm $R = 60$).

With b known one can find β (Fig. 167) from

$$b = 2R \sin(\beta/2)$$

and finally also

$$\gamma = 360 - (\bar{\delta}_1 + \bar{\delta}_2 + \beta).$$

Consequently

$$d = 2R \sin(\gamma/2)$$

will be known.

Step 5. Fig. 168: anticipating later steps we denote the distance OE as "*double eccentricity*" $2e$. Since

$$(R+2e)(R-2e) = x(d-x)$$

we can find $2e$ from

$$4e^2 = R^2 - x(d-x).$$

Ptolemy operates, of course, throughout with the specific numerical values furnished by his observations. The resulting double eccentricities $2e$ are, respectively,

$$♂\text{: } 13;7 \qquad ♃\text{: } 5;23 \qquad ♄\text{: } 7;8. \tag{3}$$

Step 6. Fig. 169: all that remains to be found is the direction of the apsidal line with respect to the given observations.

From $\Delta F = d/2$ and $\Delta O = x$, both known from Step 4, one finds

$$OF = x - d/2.$$

With $2e$, found in Step 5, one can determine the angle ε from

$$\sin \varepsilon = OF/2e = (x - d/2)/2e.$$

Thus the distance of D_3 from the perigee Π is given by

$$\measuredangle D_3 E \Pi = \gamma/2 - \varepsilon$$

where γ is known from Step 4 (Fig. 167).

Finally

$$\measuredangle D_2 EA = 180 - (\gamma/2 - \varepsilon + \bar{\delta}_2)$$
$$\measuredangle D_1 EA = \bar{\delta}_1 - \measuredangle D_2 EA$$

give the distances of D_1 and D_2 from the apogee A.

The numerical results, obtained in this fashion, are:

♂	♃	♄	
$D_1EA = 36;31°$	$D_1EA = 79;30°$	$D_1EA = -55;52°$	
$D_2EA = 45;13$	$D_2E\Pi = -0;35$	$D_2EA = 19;51$	(4)
$D_3E\Pi = 39;19$	$D_3E\Pi = 32;51$	$D_3EA = 57;43.$	

Were it not for the separation of deferent and equant we would have reached at this point our goal, the determination of the parameters of the model. In fact, however, the most difficult part of Ptolemy's procedure is yet to come.

C. Separation of Equant and Deferent

We now turn to the definitive model according to which the center C of the epicycle moves on a deferent with center M halfway between O and E whereas

the angles $\bar{\delta}_1$ and $\bar{\delta}_2$ of mean motion have their vertices in E (Fig. 170). Our previous computation was carried out under the assumption that the arcs D_1D_2 and D_2D_3 of a circle of radius R and center E are seen from O under angles δ_1 and δ_2, respectively. Actually, however, these directions lead to different points, C_1, C_2, C_3, located on a circle of radius R and center M. It is therefore necessary to evaluate the error committed by the identification of the points C and D.

Ptolemy proceeds as follows. Considering the relative smallness of the eccentricities it is plausible to assume that the points D are not very far removed from the correct positions C. Suppose the points D were exactly correct. Then the correct points C would be points of the straight line ED (cf. Fig. 171) and we could find the angles ε under which CD is seen from O. In the preceding computations we found the parameters of the model under the assumption that D_1D_2 and D_2D_3 appear from O under the observed angles δ_1 and δ_2. Now we know that we should have rather taken at O angles δ_1' and δ_2' which differ from δ_1 and δ_2 by the proper correction ε shown in Fig. 171. Hence we can now ask for the parameters of a model which satisfies improved data: the arcs D_1D_2 and D_2D_3 should appear from E under the angles $\bar{\delta}_1$ and $\bar{\delta}_2$ but from O under the angles δ_1' and δ_2'. This, then, requires a new double eccentricity $2e'$ and a new apsidal line $A'\Pi'$.

Repeating the above argument we can again compute corrections ε' between the directions OD and OC′ (cf. Fig. 172), C′ being the points of E′D on the new deferent with center M′. This leads to new angles, δ_1'' and δ_2'', under which D_1D_2 and D_2D_3 should appear from O and hence to a new model with a new double eccentricity $2e''$ and a new apsidal line OM″E″. This process can be iterated until (one may hope) no new values for the eccentricity and for the position of the apsidal line are obtained. Before discussing the question of convergence, however, we have to return to the actual computation of the corrections ε and the subsequent steps.

Step 7. Resuming the description of Ptolemy's procedure where we left it with Step 6 on p. 175 we now determine the angle η in Fig. 173. From Step 6 we know the angle $\mathrm{DEA}=\bar{\kappa}$ and from Step 5 the double eccentricity $2e$. Because

$$\sin\eta=\frac{2e}{\mathrm{OC}}\sin\bar{\kappa}$$

we have to find OC in order to determine η. But

$$\begin{aligned}\mathrm{OC}^2&=\mathrm{OG}^2+\mathrm{GC}^2=4e^2\sin^2\bar{\kappa}+(\mathrm{GF}+\mathrm{FC})^2\\&=4e^2\sin^2\bar{\kappa}+\left(e\cos\bar{\kappa}+\sqrt{R^2-e^2\sin^2\bar{\kappa}}\right)^2\end{aligned}$$

is known.

Step 8. Fig. 173: find the corrections ε from

$$\varepsilon=\eta-\zeta$$

where η is known from Step 7 and ζ can be found from

$$\sin\zeta=\frac{\mathrm{OG}}{\mathrm{OD}}=\frac{2e\sin\bar{\kappa}}{\mathrm{OD}}$$

since OD is known from

$$OD^2 = OG^2 + GD^2 = (2e \sin \bar{\kappa})^2 + (GE + ED)^2$$
$$= (2e \sin \bar{\kappa})^2 + (2e \cos \bar{\kappa} + R)^2.$$

Step 9. Iteration (cf. Fig. 174). We know now that the points D_1, D_2, D_3 of the circle of radius R and center E are seen from E under the angles $\bar{\delta}_1$ and $\bar{\delta}_2$ but appear from O under angles which are known to be approximately given by

$$\begin{aligned} \delta_1' &= \varepsilon_1 + \delta_1 + \varepsilon_2 \\ \delta_2' &= -\varepsilon_2 + \delta_2 - \varepsilon_3. \end{aligned} \tag{5}$$

By the process described in Steps 1 to 6 the distance E′O and the three angles DE′A′ can be found. If the previously obtained values had been correct one would obtain now once more the same values. If not we use the new angles $DE'A' = \bar{\kappa}'$ to find (as in Step 8) corrections ε' (Fig. 172) which we combine with the angles δ' in the same fashion as before in (5) to new angles δ'' which represent improved angles under which the arcs D_1D_2 and D_2D_3 appear from O. Thus we have a new set of angles $\bar{\delta}$ and δ'' which can be used once more in Steps 1 to 6 to find an eccentricity and an apsidal line. And so on.

D. Results

In reality this iteration process is not carried very far. For Mars, Ptolemy computes three approximations, for Jupiter and Saturn only two. Obviously the corrections seemed to him small enough to expect no significant improvements from additional sets. Instead he showed directly that the accepted values which resulted from the last approximation lead to a satisfactory agreement with the observational data. Of course he made no attempt to prove the convergence of the iteration process as such.

The numerical data obtained by Ptolemy are:

	Approximation			
Mars	1st	2nd	3rd	
2e	13;7	11;50	12;0	$R = 60$
D_1EA	−36;31°	−42;45°	−41;33°	
D_2EA	45;13	38;59	40;11	
$D_3E\Pi$	−39;19	−45;33	−44;21	

	Approximation	
Jupiter	1st	2nd
$2e$	5;23	5;30
D_1EA	79;30°	77;15°
$D_2E\Pi$	−0;35	−2;50
$D_3E\Pi$	32;51	30;36

	Approximation		
Saturn	1st	2nd	
$2e$	7;8	6;50	$R = 60$
D_1EA	−55;52°	−57; 5°	
D_2EA	−19;51	−18;38	
D_3EA	57;43	56;30	

If one computes in the case of Mars one more approximation[5] one finds $2e = 11;59$. For the apsidal line the changes of direction are

from the 1st to the 2nd approximation: $+6;14°$
from the 2nd to the 3rd approximation: $-1;12$
from the 3rd to the 4th approximation: $-0;\ 3$.

The corresponding changes in the correcting angles are

	ε_1	ε_2
Δ_{12}	$+0;4°$	$-0;0,30°$
Δ_{23}	$-0;5$	$-0;0,20$
Δ_{34}	$+0;10$	$+0;1,16.$

It is obvious that one more step would no longer influence the minutes of the parameters of the model and that Ptolemy's values are indeed sufficiently accurate.[6]

Step 10. Check with observations. As a result of the iteration process we have found for each planet the double eccentricity $2e = \mathrm{OM}$ and the direction of the apsidal line $\mathrm{A}\Pi$ defined by its angles with the radii ED (cf. Fig. 170, p. 1259). According to our model the points C must be located on the intersection of the radii ED with the deferent of radius R and center M which is at a distance $e = \mathrm{OM} = \mathrm{ME}$ from the observer O and the equant E.

Through this model and through the numerical values found for its parameters the angles $\mathrm{C_1OC_2} = \delta_1$ and $\mathrm{C_2OC_3} = \delta_2$ are determined. If the dimensions of the model are correct these values δ_1 and δ_2 should be identical with the longitudinal differences found in the initial observations for $\mathrm{C_1C_2}$ and $\mathrm{C_2C_3}$. This is the proof which Ptolemy gives for the validity of his parameters.

In order to derive the values of δ_1 and δ_2 which result from the model we have to find the longitudinal distance κ for each one of the points C with respect to the apsidal line. For this we use (cf. Fig. 175) the angle $\bar{\kappa}$ of the point D (Step 6) and by the method of Step 7 we then can determine the angle η. Thus

$$\kappa = \bar{\kappa} - \eta$$

is known.

Having determined the three angles κ which the lines of sight OC make with the apsidal line we have for a configuration as shown in Fig. 170, p. 1259

$$\delta_1 = \kappa_1 + \kappa_2 \qquad \delta_2 = \kappa_3 - \kappa_2.$$

The numerical values found in this way agree in all cases for the three outer planets with the initially observed values δ_1 and δ_2. This constitutes the desired

[5] Computed by Mr. E. S. Ginsberg.

[6] Hill [1900] has shown, for the case of Mars, that the modern solution agrees excellently with Ptolemy's results, i.e. with the third approximation. Hill finds $e = 60 \cdot 0.1000026$ (instead of Ptolemy's $60 \cdot 0.1$) and for the apogee ♋25;29,33.01 (instead of Ptolemy's ♋25;30; cf. below p. 179, (6)). But the above given fourth approximation shows slightly larger deviations, a fact that underlines the accidental character of purely numerical comparisons.

proof for the correctness of the parameters found by means of the iteration process.

Step 11. Any one of the angles κ, found in the previous step, combined with the observed longitude of the corresponding opposition C, gives the longitude of the apogee at the time of observation. Ptolemy found in this way for his observations

Apogee of Mars:	♋25;30	(A.D. 139)	
of Jupiter:	♍11;0	(A.D. 136)	(6)
of Saturn:	♏23;0	(A.D. 136).	

Step 12. The last element deducible from the three observations of oppositions is the value of the mean epicyclic anomaly of the planet at the moment of opposition. The situation which prevails at that time is described in Fig. 176. The mean motion of the planet is measured by the angle $\bar{\kappa}$ with respect to the apsidal line, the (mean) epicyclic anomaly is the angle $\bar{\alpha}$ which increases linearly with time, reckoned from the "mean epicyclic apogee" F, determined by the direction EC. That it is the uniform increase of this angle that determined the motion of the planet P on its epicycle follows from the fact that only under this condition does the direction CP remain always parallel to the direction from O to the mean sun (cf. Fig. 159, p. 1257).

Obviously $\bar{\alpha}$ can be found from

$$\bar{\alpha}=\kappa-\bar{\kappa}+180$$

where κ has been determined in Step 10 and $\bar{\kappa}$ in Step 6. The values found by Ptolemy for his observations are

Mars:	$\bar{\alpha}=171;25°$	(A.D. 139, May 27)	
Jupiter:	182;47	(A.D. 137, Oct. 8)	(7)
Saturn:	174;44	(A.D. 136, July 8).	

These are the elements from which later on the epoch values for the beginning of the era Nabonassar are computed (cf. p. 182).

4. The Size of the Epicycle

Ptolemy now proceeds to the determination of the radius r of the epicycle in terms of the radius $R=60$ of the deferent. This he achieves by using an observation of the planet made shortly after one of the oppositions which had been used in the preceding procedure for the determination of the eccentricity and of the apsidal line of the deferent. This choice has the advantage of guaranteeing agreement with the model just established since a few days' motion cannot introduce essential errors for the new position.

In the case of Mars, for example, Ptolemy uses the third opposition at which the center C_3 of the epicycle had been found to be located at $\bar{\mu}=44;21°$ before

the perigee $\Pi = ♑ 25;30$ while the mean epicyclic anomaly was $\bar{\alpha} = 171;25°$.[9] The new observation took place about 3 days minus 1 hour after opposition, Mars being seen at a longitude of ♐ 1;36, i.e. at an angle $\mu = 53;54°$ before Π (cf. Fig. 177). Mean motions are known accurately enough to say that Mars has moved since the opposition 1;21° in anomaly and 1;32° in mean longitude.[10] Thus $\bar{\mu}$ is now $44;21 - 1;32 = 42;49$ and the mean anomaly is $\bar{\alpha} = 171;25 + 1;21 = 172;46°$. From these data one can compute $CP = r$.

As in Step 7 (p. 176) we find first (cf. Fig. 178) the distances

$$OQ = 2e \sin \bar{\mu} \qquad OC = \sqrt{OQ^2 + CQ^2}$$

with $CQ = \sqrt{R^2 - (e \sin \bar{\mu})^2} - e \cos \bar{\mu}$ and then the angle η from[11]

$$\sin \eta = \frac{OQ}{OC}.$$

With η and $\bar{\mu}$ is also known the third angle $\nu = \eta + \bar{\mu}$ (Fig. 179). The angle μ is given by the observation, therefore we know also the angle $\mu - \nu$ under which CP appears from O. Having found previously OC we have also

$$C\Gamma = OC \sin (\mu - \nu).$$

On the other hand the angle δ at C is given by

$$\delta = \eta - (180 - \bar{\alpha}).$$

Therefore we know also the third angle

$$\theta = \delta + (\mu - \nu) = \bar{\alpha} - 180 + \mu - \bar{\mu}.$$

Since

$$C\Gamma = r \sin \theta$$

we have

$$r = \frac{OC \sin (\mu - \nu)}{\sin \theta}.$$

The numerical values obtained in this way for the radii of the epicycles are

$$\begin{array}{rl} \text{for Mars:} & r = 39;30 \\ \text{Jupiter:} & 11;30 \\ \text{Saturn:} & 6;30. \end{array} \tag{8}$$

5. Mean Motion in Anomaly

Ptolemy uses his refined model of planetary motion and a combination of observations of his own with the oldest available reliable observations for an

[9] Cf. above p. 177, (6) and (7), p. 179.

[10] Actually the tables of the Almagest (IX, 4) would give 1;22° and 1;33°, respectively. This is one of the many cases where unnecessarily inaccurate data are used for a computation of seemingly higher accuracy. For the final result the present deviations are without effect.

[11] The determination of OC is required by the absence of the tangent function as well as by the following steps.

improvement of the Babylonian parameters, in particular of the mean motion in anomaly.[12] These corrected values are then used for the tables of mean motions given in IX, 4 of the Almagest and for the computation of the epoch values at Nabonassar 1 Thoth 1.

First we must find the mean epicyclic anomaly $\bar{\alpha}$ which corresponds to a given observation of the planet P (cf. Fig. 180). Let us assume that the planet has been observed at a given moment at the true longitude κ reckoned from the apogee A.[13] We know the eccentricity e and the radius r of the epicycle, and, from the solar tables, the longitude of the mean sun, hence the angle $\bar{\lambda}'$ in our figure. Since CP is parallel to the direction from O to the mean sun we can find the angle γ at O and at P from

$$\gamma = 180 - \kappa + \bar{\lambda}'.$$

Hence one knows

$$CG = r \sin \gamma.$$

If MQGH is a rectangle we have $HG = MQ = e \sin \kappa$ and thus

$$CH = CG - HG = R \sin \delta.$$

Thus δ is known and with it also

$$\kappa_0 = \kappa - \delta.$$

Since $EN = e \sin \kappa_0$ and $CN = R - e \cos \kappa_0$ we also know

$$EC = \sqrt{CN^2 + EN^2} \quad \text{and} \quad \sin \eta = \frac{EN}{EC}.$$

With η found we also have the mean longitude of the planet

$$\bar{\kappa} = \kappa_0 + \eta.$$

Thus from

$$\bar{\alpha} + \bar{\kappa} = 180 + \bar{\lambda}'$$

one can find the mean epicyclic anomaly $\bar{\alpha}$ at the moment of the observation.

The number of complete epicyclic rotations of the planet between two observations separated by a given number of years is, of course, known from the fundamental period relations. If for each of the two observations the anomaly $\bar{\alpha}$ has been found in the fashion just described one has at one's disposal the total motion in anomaly of the planet during the given time interval. The quotient $\Delta\bar{\alpha}/\Delta t$ gives the desired mean motion.

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[12] Ptolemy does not determine corrections for the mean motion $\bar{\lambda}$ in longitude since, for an outer planet, $\bar{\lambda}$ is the difference $\bar{\lambda}_\odot - \bar{\alpha}$ of the known mean motion $\bar{\lambda}_\odot$ of the sun and of the mean motion $\bar{\alpha}$ in anomaly of the planet.

From a strictly logical viewpoint this procedure appears to be circular since Ptolemy determined the parameters of his model by means of angles $\bar{\delta}_1$ and $\bar{\delta}_2$ (cf. above p. 174) which require the knowledge of $\bar{\lambda}$; hence $\bar{\alpha}$ is no longer free. In fact, however, no high accuracy of $\bar{\lambda}$ is required for $\bar{\delta}_1$ and $\bar{\delta}_2$; hence it is legitimate to determine in a second step $\bar{\alpha}$ as accurately as possible and then correct $\bar{\lambda}$ accordingly such that $\bar{\alpha} + \bar{\lambda} = \bar{\lambda}_\odot$ is exactly satisfied, as is the case in the tables of Alm. IX, 4.

[13] The position of A is known under the assumption, made by Ptolemy for all planets (cf. above p. 160 and below p. 182), that the apsidal line participates with all fixed stars in the motion of precession.

The intervals used by Ptolemy in this process are:

	Old observations	Ptolemy's observ. (oppositions)	Δt
♂	−271 Jan. 18 occultation of β Scor[14]	+139 May 27	$410^y\ 231\ 2/3^d$
♃	−240 Sept. 4 occultation of δ Can	+137 Oct. 8	$377^y\ 128^d - 1^h$
♄	−288 March 1 2 digits below γ Virg	+136 July 8	$364^y\ 219\ 3/4^d$

The approximate results for the daily mean motions in anomaly are

$$\begin{aligned} \text{for Mars: } & 0;27,42^{\circ/d} \\ \text{Jupiter: } & 0;54,9 \\ \text{Saturn: } & 0;57,8. \end{aligned} \tag{9}$$

6. Epoch Values

The mean values for the motion in longitude and in anomaly given (as tabulated in Alm. IX, 4) it is easy to go back from any known position to a situation which held at another given moment. In all cases the assumption is made that the apsidal lines are sidereally fixed, i.e. that the longitudes of the apogees and perigees increase 1° per century.

Under these assumptions Ptolemy finds for Nabonassar 1 Thoth 1 (= −746 Febr. 25) the following values (cf. also Fig. 181 to 183)

	Mars	Jupiter	Saturn	
Apogee A	♋ 16;40	♍ 2; 9	♏ 14;10	λ_A
Center C of Epicycle	3;32	184;41	296;43[15]	$\bar{\lambda}_C$
Anomaly	327;13	146; 4	34; 2	$\bar{\alpha}$

The mean longitude $\bar{\lambda}_C$ is reckoned with respect to the equant E, whereas λ_A is the same for O and E.

[14] This occultation has been utilized by Kepler; cf. Werke III, pp. 409, 422f. (trsl. Caspar, pp. 383, 396f.).

[15] Ptolemy's own computation would result in 296;44° but all MSS and also the tables in IX, 4 have 296;43. This is also the value underlying the epoch values in the "Planetary Hypotheses." Heiberg's emendation of the text in XI, 8 (p. 425, 14) is therefore misleading, in particular since accurate computation would result in 296;45° (cf. Manitius II, p. 251, note a), used in the "Handy Tables" (cf. below p. 1004, note 4).

§ 5. Planetary Tables

1. The General Method

With all the parameters of the planetary models given and with the positions of the planets at epoch (t_e) known it is possible to compute accurately the true longitude of a planet for any given moment t.

Indeed, for the time difference $\Delta t = t - t_e$ we can compute the increments $\Delta\bar{\kappa}$ and $\Delta\bar{\alpha}$ of the eccentric and of the epicyclic anomaly and therefore also $\bar{\kappa} = \bar{\kappa}_e + \Delta\bar{\kappa}$ and $\bar{\alpha} = \bar{\alpha}_e + \Delta\bar{\alpha}$ from the values $\bar{\kappa}_e$ and $\bar{\alpha}_e$ at epoch. Except for Mercury the position of the deferent is sidereally fixed; in the case of Mercury it can be found for given t from the angle $\bar{\kappa}$ (cf. Fig. 147, p. 1252). Consequently one can find, for each planet, the position of C and hence also the distance $\rho = \mathrm{OC}$ (cf. Fig. 184) since the distance $\mathrm{OE} = 2e$ is known. Hence the angle η and from it the angle $\kappa_0 = \bar{\kappa} - \eta$ at O can be found.

The position of the planet P on its epicycle of radius r is given by the mean epicyclic anomaly $\bar{\alpha}$, reckoned from the mean apogee F of the epicycle. In the triangle OCP the angle at C is $\mathrm{PCO} = 180 - (\bar{\alpha} + \eta)$; since r and ρ are known we can also find θ and therefore the true longitude

$$\lambda = \lambda_A + \kappa_0 + \theta$$

of the planet.

It is clear from the preceding considerations that we are now in a position to find for all planets by direct computation the true longitude for any given moment t. It is also evident, however, that each such computation requires a great deal of numerical work for the solving of consecutive triangles of general shape. In order to simplify the determination of planetary longitudes Ptolemy designed tables along the same lines as his tables for the lunar equations.[1] The computation of these tables must have required a great deal of labor, for each planet 270 values are tabulated, but as a result the problem is reduced to a few additions and simple interpolations.

These tables (Almagest XI, 11) are arranged for each planet in 8 columns, the first two being used for the arguments from 0° to 180° and from 180° to 360°, respectively. We assume to be found for the given moment t from the tables of mean motions (Alm. IX, 4) the values $\bar{\kappa}(t)$ and $\bar{\alpha}(t)$, as well as the longitude λ_A of the apogee, taking precession into account. Let us denote the values tabulated in columns 3 to 8 by c_3 to c_8. Then we have, if we enter the table with the argument $\bar{\kappa}$, from the third and fourth column

$$c_3(\bar{\kappa}) + c_4(\bar{\kappa}) = \eta \qquad (1)$$

where η denotes the angle shown in Fig. 184; c_3 is the angle under which OE would appear seen from D if E were the center of the deferent (cf. Fig. 185; $\mathrm{ED} = R$), while c_4 gives the correction due to the fact that the center of the epicycle is not located in D but at the correct distance $\rho = \mathrm{OC}$ from O.[2] Since the value of the argument $\bar{\kappa}$ is the same for c_3 and c_4 it would have been possible to tabulate $\eta(\bar{\kappa})$ directly and this is indeed the case in the "Handy Tables" and

[1] Cf. above p. 93 ff.

[2] For the computation of these angles cf. Fig. 173 (p. 1260).

their descendants (cf. Fig. 186). In the Almagest, however, Ptolemy still kept the two components separate in order to allow the reader to evaluate for himself the different effects of the two components[3] (cf. Fig. 187 for the case of Saturn and Mercury).

With η found from (1) one now forms the new argument

$$\alpha=\bar{\alpha}+\eta \qquad \eta \gtrless 0 \quad \text{for} \quad \begin{cases} 0\leqq\bar{\kappa}\leqq 180 \\ 180\leqq\bar{\kappa}\leqq 360 \end{cases} \tag{2}$$

which represents the true epicyclic anomaly of the planet (cf. Fig. 184). With α known the equation θ can be found, being a function of α and of the distance OC, hence of $\bar{\kappa}$. A tabulation of θ would therefore require a table of double entry; on the basis of Ptolemy's subdivision of the tabular interval in 45 steps this would mean the computation of θ in more than 2000 cases. Instead, Ptolemy computed $3\cdot 45=135$ cases, assuming that C is either at mean distance from O (i.e. $OC=R=60$)–the result being tabulated as $c_6(\alpha)$, or that C is either at maximum distance from O (i.e. C = A) or at minimum distance (i.e. C is at the perigee, $\bar{\kappa}=180$, except for Mercury with $\bar{\kappa}=\pm 120$). For the extremal distances, however, not θ itself is tabulated but only the increment (negative or positive) with respect to c_6 (cf. Figs. 188 and 189). Thus at maximum distance we have

$$\theta=c_6(\alpha)-c_5(\alpha)$$

and at perigee

$$\theta=c_6(\alpha)+c_7(\alpha).$$

For intermediate distances, however, i.e. for arbitrary values of $\bar{\kappa}$, coefficients $c_8(\bar{\kappa})$ are tabulated which have the value -1 at the apogee, $+1$ at the perigee, 0 at mean distance, and which bridge these intervals such that

$$\theta(\alpha,\bar{\kappa})=c_6(\alpha)+c_8(\bar{\kappa})\cdot\begin{cases} c_5(\alpha) & \text{if} \quad c_8\leqq 0 \\ c_7(\alpha) & \text{if} \quad c_8\geqq 0. \end{cases} \tag{3}$$

with

$$c_6(\alpha)\gtrless 0 \quad \text{for} \quad \begin{cases} 0\leqq\alpha\leqq 180 \\ 180\leqq\alpha\leqq 360 \end{cases}$$

and where the sign of c_8 is indicated (by headings) in the tables of Alm. XI, 11. The result is an approximate value of θ, computed under assumptions similar to those made for the tabulation of the lunar anomaly.[4]

With θ found one has for the true longitude of the planet

$$\lambda=\lambda_A+\bar{\kappa}+\eta+\theta \tag{4}$$

where Fig. 184 (p. 1263) allows us to control the sign that holds for η and θ, respectively.

2. Numerical Data

Ptolemy gives in Almagest XI, 10 a list of numerical data upon which he had based his computations of the tables XI, 11. We give here this list with the

[3] Alm. XI, 10 (Heib. II, p. 429).–For the Handy Tables cf. below p. 1002, (1).

[4] Cf. below p. 185.

addition of the epoch values from the tables of mean motions (IX, 4):

	Radius r of epicycle	Distance OC		Eccen-tricity e	Epoch: Nabon. 1 Thoth 1			
		max.	min.		Apogee	$\bar{\kappa}_e$	$\bar{\alpha}_e$	
♄	6;30	63;25	56;35	3;25	♏ 14;10	296;43[1]	34; 2	♄
♃	11;30	62;45	57;15	2;45	♍ 2; 9	184;41	146; 4	♃
♂	39;30	66; 0	54; 0	6; 0	♋ 16;40	3;32	327;13	♂
♀	43;10	61;15	58;45	1;15	♉ 16;10	330;45	71; 7	♀
☿	22;30	69; 0	55;34	3; 0	♎ 1;10	330;45	21;55	☿

All distances are measured in units such that the radius R of the deferent is 60. The extremal distances in the first four cases are given by $R \pm e$, the maximum distance for Mercury is $R+3e$, whereas the minimum distance is given by (1), p. 164.

For the maximum of the equations, $\eta = c_3 + c_4$ of the eccenter, θ of the epicycle (cf. Fig. 184, p. 1263) one finds from the tables[2]

	Maximum of		Maximum of θ						
			OC = max.		OC = R		OC = min.		
	$\eta = c_3 + c_4$	at $\bar{\kappa}$	$c_6 - c_5 = m$	at α	$c_6 = \mu$	at α	$c_6 + c_7 = M$	at α	
♄	6;31	90 to 93	5;53	(90 and) 96[3]	6;13	96	6;36	96 to 102	♄
♃	5;15	90 to 96	10;35	99 to 102	11; 3	102	11;35	102	♃
♂	11;25	93 to 96	36;45	126	41; 9	132	46;59	138	♂
♀	2;24	90	44;45	132 to 135	45;57	138	47;15	138	♀
☿	3; 2	93 to 99	19; 1	108	22; 2	111	23;53	114	☿

It is worth noticing that the interpolation coefficients $c_8(\bar{\kappa})$ do not represent a simple sine curve but that they are individually computed for each planet (cf. Fig. 190). Ptolemy determined first for given $\bar{\kappa}$ the maximum equation θ_0 which corresponds to a tangential position of OP with respect to the epicycle (Fig. 191). For values of $\bar{\kappa}$ which represent positions of C between the apogee and the point of mean distance from O the values of c_8 are obtained from

$$c_8(\bar{\kappa}) = \frac{\theta_0(\bar{\kappa}) - \mu}{\mu - m} \tag{1a}$$

where μ denotes the maximum of the epicyclic equation θ when C is at mean distance from O, m the same at maximum distance of C, the numerical values of μ and m being shown in the above given table.[4] For the arc between mean distance

[1] Cf. above p. 182, n. 15.

[2] Ptolemy gives the list of the maxima of θ in Alm. XI, 10 (Heib. II, p. 433, 15–19). Out of the 15 values 8 differ from the values obtainable from the tables XI, 11 by 1, 2, or 3 minutes. In the case of Mars and Venus even the value of c_6 itself is differently given in the text (41;10 and 46;0) and in the tables (41;9 and 45;57).

[3] For $\alpha = 93$ one finds only $\theta = 5;52$ which would mean that there existed one maximum at 90 and a second at 96. It follows, however, from the differences of $c_6(\alpha)$ that $c_6(90)$ should be 6;9 or 6;10 but not 6;12. On the basis of this correction one finds only one maximum at $\alpha = 96$.

[4] In the few cases where Ptolemy's values differ from the value in our table one has to use Ptolemy's values if one wishes to recompute the table in the Almagest; cf. note 2.

and perigee one uses

$$c_8(\bar{\kappa}) = \frac{\theta_0 - \mu}{M - \mu} \tag{1b}$$

a formula which also holds in the case of Mercury for the arc between the two perigees. In fact this seems to be the most accurately computed function $c_8(\bar{\kappa})$.[5] The graph in Fig. 190 shows that the c_8 for the other planets vary sometimes irregularly. It is quite obvious that analogous functions were not computed with the same accuracy for all five planets.

The ratios (1a) and (1b) obtained for the maxima of the epicyclic equations are then considered valid also for all epicyclic equations which belong to the same value of $\bar{\kappa}$, i.e., to the same position of the center C of the epicycle. This is the same method of approximation which Ptolemy used for the lunar anomaly.[6]

3. Examples

A. Ephemeris for Mars

In the following we compute the longitudes of Mars in 10-day intervals for the period from Nabon. 450 X 8 to 451 V 3, using the tables in XI, 11.

We wish to compute the longitudes to minutes of arc, i.e. to the same accuracy as Ptolemy's epoch values and similar general data. Nevertheless one must compute the values for $\bar{\kappa}$ and $\bar{\alpha}$ to seconds in order to bridge correctly an interval of 210 days. The numbers shown in the following Table 15 are rounded from a table computed to seconds, using as differences the values given for 10 days in Alm. IX, 4: $\Delta\bar{\kappa} = 5;14,26,9$ and $\Delta\bar{\alpha} = 4;36,56,43$.

In order to find the initial entry for Nabon. 450 X 8 we take from the tables of mean motion for the completed 449 Egyptian years, 9 months, 7 days:

	$\Delta\bar{\kappa}$	$\Delta\bar{\alpha}$
432^y	193;44, 7	61;14, 7
17^y	11;47,26	344; 4,35
9^m	141;29,46	124;37,31
7^d	3;40, 6	3;13,52
total	350;41,25	173;10, 5
epoch	3;32	327;13
$\bar{\lambda}=$	354;13,25	140;23, 5 $=\bar{\alpha}$
λ_A at epoch	106;40	
precession 450^y	4;30	
λ_A	111;10	
$\bar{\kappa}=\bar{\lambda}-\lambda_A=$	243; 3,25	

[5] I recomputed a sequence of values of $c_8(\bar{\kappa})$ for Mercury from $\bar{\kappa}=120$ to $\bar{\kappa}=180$ in steps of 12°. The results of these rather longish computations of θ_0 deviate only once by as much as 0;1° from Ptolemy's values. The deviations from c_8 reach in one case 0;0,17 and are otherwise 0;0,4, 0;0,2, and 0;0,1. It seems clear that Ptolemy had computed at least one more digit than the tabulated values show.

[6] Above p. 95.

By a similar computation one can check the correctness of the last entries (for Nabon. 451 V 3).

From the tables Alm. XI, 11 one forms $\eta(\bar{\kappa})=c_3(\bar{\kappa})+c_4(\bar{\kappa})$ (cf. our Table 15). Since all $\bar{\kappa}$ belong to the interval from 180° to 360° one has $\kappa_0=\bar{\kappa}+\eta$, $\alpha=\bar{\alpha}-\eta$. But the resulting α changes between No. 11 and No. 12 from a value $<180°$ to a value $>180°$; hence the corresponding epicyclic equation $c_6(\alpha)$ is positive up to No. 11 and negative thereafter. The interpolation coefficients $c_8(\bar{\kappa})$ are positive for the first 4 cases, negative thereafter. This indicates that the center of the epicycle is at first nearer to O than the mean distance, but farther away thereafter. Consequently in the first 4 cases the correction $c_7(\alpha)$ has to be used to modify $c_6(\alpha)$, but $c_5(\alpha)$ for the remaining ones. The sign of $c_7(\alpha)$ and $c_5(\alpha)$ has, of course, to agree with the sign of $c_6(\alpha)$, thus positive up to No. 11, negative from No. 12 on. From these coefficients one obtains the equation θ as in (3) p. 184 and then $\kappa=\bar{\kappa}+\theta$ for the distance of the planet from the contemporary apogee. Since we have found (p. 186) that for Nabon. 450/451 the apogee has the longitude $\lambda_A=111;10$ one finds the longitudes of Mars by adding 111;10° to κ.

A comparison with modern tables[1] shows deviations up to 2;5° in longitude. The error in Ptolemy's constant of precession tends to increase longitudes for the time around −300 by about 2° but there are many other effects which influence his results. Fig. 192 shows positions of Mars according to a scale drawing of the motion of C and P according to Ptolemy's parameters. The longitudes in combination with latitudes[2] are shown in Fig. 228a (below p. 1283), compared with the modern data.

It is clear from the longitudes listed in Table 15 that the planet should have its first station near No. 8 and the second near Nos. 15 or 16. This will be confirmed by Ptolemy's theory of retrogradations (cf. below p. 205).

B. Ephemeris for Venus

In a similar fashion we compute an ephemeris for Venus for the time from Nabonassar 442 XII 13 to 443 II 28 in 5-day intervals. The initial values for $\bar{\kappa}$ and $\bar{\alpha}$ follow from the tables in Alm. IX, 4:

	$\Delta\bar{\kappa}$	$\Delta\bar{\alpha}$
432^y	254;58,15	11; 5,50
9^y	357;48,43	225;13,52
11^m	325;15,35	203;26,52
12^d	11;49,39	7;23,53
total	229;52,12	87;10,27
epoch	330;45	81; 7
	$\bar{\lambda}=200;37,12$	$158;17,27=\bar{\alpha}$
λ_A at epoch	46;10	
precession 442^y	4;25	
λ_A	50;35	
	$\bar{\kappa}=\bar{\lambda}-\lambda_A=150;\ 2,12$	

[1] Tuckerman, Tables I, p. 184f.

[2] Cf. for their computation below p. 219.

Table 15

No.	Date: Nab.		Date: julian		from Alm. IX, 4: $\bar{\kappa}$	$\bar{\alpha}$	from Alm. XI, 11: $\eta(\bar{\kappa}) = c_3 + c_4$	κ_0	α	$c_6(\alpha)$	$c_7(\alpha)$	$c_8(\bar{\kappa})$	$c_7 \cdot c_8$	$\theta = c_6 + c_7 c_8$	κ	λ: $\lambda_A + \kappa$	mod.	$\Delta =$ Alm. − modern	No.
	450		−297																
1	X	8	Aug.	10	243; 3	140;23	10;37	253;40	129;46	+41; 8	+5;24	+0;17,36	+1;35	+42;43	296;23	47;33	46;30	+1; 3	1
2		18		20	248;18	145; 0	10;57	259;15	134; 3	41; 4	5;49	+0;12,26	+1;12	42;16	301;31	52;41	51;39	1; 2	2
3		28		30	253;32	149;37	11;12	264;44	138;25	40;41	6;17	+0; 7,25	+0;47	41;28	306;12	57;22	56;19	1; 3	3
4	XI	8	Sept.	9	258;47	154;14	11;21	270; 8	142;53	39;52	6;47	+0; 2,24	+0;16	40; 8	310;16	61;26	60;24	1; 2	4
5		18		19	264; 1	158;51	11;25	275;26	147;26	38;29	5;29	−0; 3, 3	−0;17	38;12	313;38	64;48	63;46	1; 2	5
6		28		29	269;16	163;28	11;24	280;40	152; 4	+36;21	+5;37	−0; 9,14	−0;52	+35;29	316; 9	67;19	66;14	+1; 5	6
7	XII	8	Oct.	9	274;30	168; 5	11;16	285;46	156;49	33;14	5;37	0;14,56	1;24	31;50	317;36	68;46	67;37	1; 9	7
8		18		19	279;44	172;42	11; 1	290;45	161;41	28;54	5;20	0;20,26	1;49	27; 5	317;50	69; 0	67;42	1;18	8
9		28		29	284;59	177;19	10;42	295;41	166;37	22;51	4;35	0;25,33	1;57	20;54	316;35	67;45	66;23	1;22	9
10	I	3	Nov.	8	290;13	181;56	10;19	300;32	171;37	15;22	3;18	0;30,11	1;40	13;42	314;14	65;24	63;43	1;41	10
	451																		
11		13		18	295;28	186;33	9;52	305;20	176;41	+ 6;20	+1;23	−0;34,33	−0;48	+ 5;32	310;52	62; 2	60; 5	+1;57	11
12		23		28	300;42	191; 9	9;20	310; 2	181;49	− 3;29	−0;46	0;38,40	+0;30	− 2;59	307; 2	58;13	56; 8	2; 5	12
13	II	3	Dec.	8	305;57	195;46	8;42	314;39	187; 4	−13; 8	−2;50	0;42,36	+2; 1	−11; 7	303;32	54;42	52;40	2; 2	13
14		13		18	311;11	200;23	8; 3	319;14	192;20	−21;27	−4;22	0;45,48	+3;20	−18; 7	301; 7	52;17	50;16	2; 1	14
15		23		28	316;26	205; 0	7;20	323;46	197;40	−28;12	−5;15	0;48,38	+4;15	−23;57	299;49	50;59	49;13	1;46	15
			−296																
16	III	3	Jan.	7	321;40	209;37	6;34	328;14	203; 3	−33; 8	−5;37	−0;51, 8	+4;47	−28;21	299;53	51; 3	49;26	+1;37	16
17		13		17	326;55	214;14	5;45	332;39	208;29	36;38	5;36	0;53,21	5; 0	31;38	301; 1	52;11	50;48	1;23	17
18		23		27	332; 9	218;51	4;54	337; 3	213;57	38;58	5;25	0;55,17	4;59	33;59	303; 4	54;14	53; 7	1; 7	18
19	IV	3	Febr.	6	337;23	223;28	4; 2	341;25	219;26	40;21	5; 5	0;56,54	4;49	35;32	305;53	57; 3	56;10	0;53	19
20		13		16	342;38	228; 5	3; 7	345;45	224;58	41; 2	4;45	0;57,58	4;35	36;27	309;18	60;28	59;48	0;40	20
21		23		26	347;52	232;42	2;11	350; 3	230;31	−41; 7	−4;26	−0;58,58	+4;21	−36;46	313;17	64;27	63;55	+0;32	21
22	V	3	March	7	353; 7	237;19	1;15	354;22	236; 4	40;49	4; 7	0;59,45	4; 6	36;42	317;40	68;50	68;23	0;27	22
											$c_5(\alpha)$		$c_5 \cdot c_8$	$c_6 + c_5 c_8$		$\lambda_A = 111;10$			

Table 16

No.	Date Nab.	Date julian	from Alm. IX, 4: $\bar{\kappa}$	$\bar{\alpha}$	from Alm. XI, 11: $\eta(\bar{\kappa}) = c_3 + c_4$	κ_0	α	$c_6(\alpha)$	$c_7(\alpha)$	$c_8(\bar{\kappa})$	$c_7 \cdot c_8$	$\theta = c_6 + c_7 c_8$	κ	λ: $\lambda_A + \kappa$	mod.	$\Delta =$ Alm. – modern	No.
	442	–305															
1	XII 13	Oct. 16	150; 2	158;17	1;12	148;50	159;29	+37;38	+1;51	+0;51,24	+1;35	+39;13	188; 3	238;38	239;26	–0;48	1
2	18	21	154;58	161;22	1; 1	153;57	162;23	34;39	1;52	0;53,44	1;40	36;19	190;16	240;51	241;51	1; 0	2
3	23	26	159;54	164;27	0;50	159; 4	165;17	30;58	1;49	0;55,39	1;41	32;39	191;43	242;18	243;33	1;15	3
4	28	31	164;49	167;32	0;38	164;11	168;10	26;28	1;42	0;57,24	1;38	28; 6	192;17	242;52	244;23	1;31	4
5	ep. 3	Nov. 5	169;45	170;37	0;26	169;19	171; 3	21; 9	1;27	0;58,46	1;25	22;34	191;53	242;28	244;16	1;48	5
	443																
6	I 3	10	174;41	173;42	0;14	174;27	173;56	+14;56	+1; 5	+0;59,39	+1; 5	+16; 1	190;28	241; 3	243; 5	–2; 2	6
7	8	15	179;36	176;47	0; 1	179;35	176;48	+ 8; 7	+0;37	1; 0, 0	+0;37	+ 8;44	188;19	238;54	240;59	2; 5	7
8	13	20	184;32	179;52	0;12	184;44	179;40	+ 0;51	+0; 4	0;59,47	+0; 4	+ 0;55	185;39	236;14	238;11	1;57	8
9	18	25	189;28	182;57	0;24	189;52	182;33	– 6;29	–0;31	0;58,55	–0;30	– 6;59	182;53	233;28	235;10	1;42	9
10	23	30	194;23	186; 2	0;36	194;59	185;26	–13;26	–0;59	0;57,40	–0;57	–14;23	180;36	231;11	232;25	1;14	10
11	28	Dec. 5	199;19	189; 7	0;48	200; 7	188;19	–19;47	–1;22	+0;55,56	–1;16	–21; 3	179; 4	229;39	230;23	–0;44	11
12	II 3	10	204;15	192;12	1; 0	205;15	191;12	25;18	1;39	0;54, 1	1;29	26;47	178;28	229; 3	229;19	–0;16	12
13	8	15	209;11	195;17	1;10	210;21	194; 7	30; 2	1;48	0;51,46	1;33	31;35	178;46	229;21	229;17	+0; 4	13
14	13	20	214; 6	198;22	1;20	215;26	197; 2	33;55	1;51	0;48,58	1;31	35;26	180; 0	230;35	230;14	+0;21	14
15	18	25	219; 2	201;27	1;30	220;32	199;57	37; 4	1;51	0;45,42	1;25	38;29	182; 3	232;38	232; 1	+0;37	15
16	23	–304 | 30	223;58	204;32	1;39	225;37	202;53	–39;36	–1;50	+0;41,58	–1;17	–40;53	184;44	235;19	234;22	+0;47	16
17	28	Jan. 4	228;53	207;37	1;48	230;41	205;49	41;34	1;45	0;38, 5	1; 7	42;41	188; 0	238;35	237;39	0;56	17

$\lambda_A = 50;35$

In the interval under consideration (cf. Table 16) $\bar{\kappa}$ crosses the value 180° between the positions No. 7 and No. 8, that is to say the center of the epicycle passes the perigee. Consequently

$$\kappa_0=\bar{\kappa}-\eta(\bar{\kappa}) \quad \alpha=\bar{\alpha}+\eta(\bar{\kappa}) \quad \text{until No. 7}$$
$$\kappa_0=\bar{\kappa}+\eta(\bar{\kappa}) \quad \alpha=\bar{\alpha}-\eta(\bar{\kappa}) \quad \text{from No. 8 on.}$$

The true anomaly α passes 180° between Nos. 8 and 9. Therefore $c_6(\alpha)$ and $c_7(\alpha)$ are positive until No. 8, negative after it. The coefficient $c_8(\bar{\kappa})$ is positive in the whole interval, hence the epicyclic equation is given by $c_6+c_7\cdot c_8=\theta$. Finally $\kappa=\kappa_0+\theta$ and $\lambda=\lambda_A+\kappa$, the true longitude.

The geocentric motion thus determined is represented to scale in Fig. 193. The combination with latitudes (computed below p. 225) and with modernly computed positions is shown in Fig. 233a (below p. 1285). For the stationary points cf. below p. 205.

§ 6. Theory of Retrogradation

The most puzzling phenomenon of planetary motion consists in the fact that the "wandering stars" periodically revert their forward motion and become "*retrograde*." As far as we know the first attempt at a rational explanation of this behavior consists in Eudoxus' theory of homocentric spheres[1] (4th century B.C.) which, in spite of its deficiencies, demonstrated, at least in principle, the possibility of explaining retrogradation by the superposition of circular motions which singly do not change their sense of rotation.

It was, however, not before Apollonius (end of the 3rd century B.C.) that the theory of eccentric and epicyclic motion provided a sound foundation for a mathematical description of the observed phenomena. It must be stated already here that all we know about Apollonius' theory is based on the historical remarks in Book XII of the Almagest and in particular that it remains unknown how far Apollonius tried to coordinate his cinematic theory with empirical data.[2] For our discussion of the theory of planetary retrogradation in the Almagest we simply take for granted the existence of a fundamental theorem of Apollonius which allows one to determine the positions of the stationary points on the epicycle, postponing to a later chapter[3] the proof as given by Apollonius.

The pertinent statement is the following one (cf. Fig. 194): if a planet, located at a point P of its epicycle, appears stationary to an observer O (who is located at the center of the deferent) then

$$\frac{\rho}{\mathrm{PT}}=\frac{v_{\mathrm{p}}}{v_{\mathrm{c}}} \tag{1}$$

where v_{p} is the angular velocity of the planet on its epicycle with respect to the direction OC and v_{c} the angular velocity of C as seen from O with respect to some

[1] Cf. below IV C 1, 2.
[2] Cf. p. 270f.
[3] Cf. I D 3, 1.

sidereally fixed direction, while ρ is the distance of P from O. The point P, located as shown in Fig. 194, is called the *first* stationary point; a *second* station occurs in the position symmetric to the first with respect to the radius OC. On the arc between these two stations the planet is retrograde.

The modern proof of (1) is trivial. The point P appears to be stationary when the resultant vector of the motion about O and about C points toward O. Thus (Fig. 194)

$$\sin\zeta = \frac{\rho v_c}{r v_p} = \frac{PT}{r}$$

q.e.d.

Obviously a planet will not become retrograde when its velocity v_p on the epicycle is too small in comparison with the direct motion v_c of the epicycle. Formula (1) furnishes the exact limit: no retrogradation will occur when the two stations coincide in the perigee of the epicycle, hence for

$$v_p = \frac{R-r}{r} v_c \quad (R = OC). \tag{2}$$

Actually

$$v_p > \frac{R-r}{r} v_c \tag{3}$$

is amply satisfied for all planets[4]:

	v_p	$\frac{R}{r}-1$	v_c	$\left(\frac{R}{r}-1\right)v_c$
♄	0;57°/d	≈8	0; 2°/d	0;16°/d
♃	0;54	4	0; 5	0;20
♂	0;28	0;30	0;31	0;16
♀	0;37	0;23	0;59	0;23
☿	3; 6	1;40	0;59	1;38

1. Stationary Points

So far we have disregarded eccentricities. Obviously this would imply retrograde arcs of constant length, in contradiction to the experience that these arcs differ in different parts of the ecliptic. It is fortunately not necessary for the validity of Apollonius' Theorem (1), p. 190 that O coincide with the midpoint (M) of the deferent or with the center of uniform motion (the equant E). The only assumption implicitly made in the derivation of (1) is the condition that the motion of the center C of the epicycle produces a motion of P orthogonal to OP (cf. Fig. 195, p. 1268). Such a motion also takes place in the refined Ptolemaic models when P (but not C) is a point of the apsidal line OE. These cases are properly characterized by Ptolemy as maximum (respectively minimum) distance from O

[4] The moon, seen from the sun, does not become retrograde. Indeed, we have for the moon $v_p \approx 13 v_c$ but $R/r - 1 \approx 6{,}28$. Thus (3) is not satisfied.

"of the point of opposition which bisects the arc between the stations."[1] As an intermediate case is considered a position of the center C of the epicycle at "mean distance" $R = \mathrm{OC} = 60$ from the observer at O. This, however, implies the return to the simple model (of the type shown in Fig. 194) in which O coincides with M and E. Ptolemy does not state this explicitly but it follows from his computations. The need for this simplification becomes evident if one remarks that the mean distance for OC occurs in the region of the quadratures where the motion of C about E does not induce a motion of P orthogonal to OP. Consequently one has to operate with the simplified arrangement of Fig. 194 in which not only the Apollonius Theorem holds but for which the velocities v_c and v_p can be identified with the mean velocities in longitude and in anomaly, respectively, known from the tables in Alm. IX, 4. In this way one is able to obtain an approximate mean value of the retrograde arc which then can be used in the computations for the extremal cases.[2]

A. Mean Distance

We assume for all planets, including Mercury, the following situation (Fig. 195): $\mathrm{OC} = R = 60$, P a stationary point, such that Theorem (1) is valid; we also identify O with M and with E. Since the radii of all epicycles are known,[3] we also know the product

$$\mathrm{OG} \cdot \mathrm{O}\Theta = (R+r)(R-r),$$

hence also

$$\mathrm{OQ} \cdot \mathrm{OP} = R^2 - r^2. \tag{2}$$

According to (1), p. 190 the ratio

$$\bar{c} = \frac{\rho}{\mathrm{PT}} = \bar{v}_p / \bar{v}_c \tag{1}$$

is known, e.g. as quotient of the daily mean motions of the planet in anomaly and in longitude.

Since $\mathrm{OQ} = \rho + 2\,\mathrm{PT}$ and $\mathrm{OP} = \rho$ we have with (1) and (2)

$$R^2 - r^2 = (\rho + 2\,\mathrm{PT})\rho = \mathrm{PT} \cdot \rho(\bar{c}+2) = \rho^2 \frac{\bar{c}+2}{\bar{c}}.$$

Thus

$$\rho = \sqrt{\frac{(R^2-r^2)\bar{c}}{\bar{c}+2}} \quad \text{and} \quad \mathrm{PT} = \rho/\bar{c} \tag{3}$$

can be found.[4]

It follows from Fig. 195 that

$$\zeta = \arcsin(\mathrm{PT}/r) \quad \text{and} \quad \zeta + \beta = \arcsin \frac{\rho + \mathrm{PT}}{R}. \tag{4a}$$

Hence ζ, β, and

$$\gamma = 90 - (\zeta + \beta) \tag{4b}$$

---

[1] For Saturn: Manitius II, pp. 281, 7 and 283, 14; for the remaining planets shortened to "maximum/minimum distance." Cf. also below p. 193.

[2] Below p. 195 and p. 197ff.

[3] Cf. above p. 146.

[4] Ptolemy's procedure differs from the one given here only in so far as he first finds PT and then ρ.

can be determined. The angles $180-\beta$ and $180+\beta$ give the epicyclic anomaly of the first and of the second stationary point, respectively, reckoned from the apogee G of the epicycle.

Substituting in these formulae the given values for r, $\bar{v}_c$, and $\bar{v}_p$ one finds generally good agreement with the results obtained by Ptolemy shown in Table 17, p. 197.

The difference $\Delta\lambda$ between the longitudes of the first and of the second station, as seen from O, is not given by the angle 2γ but is smaller because the epicycle moves forward during the time in which the planet on the epicycle covers the angle 2β in anomaly. But since we know 2β we also know the time Δt of retrogradation

$$\Delta t = 2\beta/\bar{v}_p. \tag{5}$$

During this time the center C of the epicycle gains in longitude the arc

$$\Delta t \cdot \bar{v}_c = 2\beta\bar{v}_c/\bar{v}_p = 2\beta/\bar{c}$$

where $\bar{c}$, as before, represents the ratio $\bar{v}_p/\bar{v}_c$. Hence the observed retrograde arc is given by

$$-\Delta\lambda = 2(\gamma - \bar{\delta}), \quad \bar{\delta} = \beta/\bar{c} \tag{6}$$

The numerical data thus obtained by Ptolemy for the retrogradations at "mean distance" are

	Saturn	Jupiter	Mars	Venus	Mercury
$\bar{\delta}=\beta/\bar{c}$	2;19°	5; 1,24°	19; 7,33°	20;35,19°	11; 4,59°
$-\Delta\lambda$	7;16,20°	9;52,16°	16;18,44°	15;17,34°	12;17,10°
Δt	138^d	121^d	73^d	$41\ 2/3^d$	$22\ 1/2^d$

B. Maximum Distance

We now consider the situation when the stationary point P on the epicycle falls in the apsidal line (cf. Fig. 196). Consequently the center C of the epicycle is not yet in the apsidal line when P is the first station; C is by the same amount beyond the apsidal line when P is at the second station. The point of symmetry between the two stations on the epicycle is the point Θ where for an outer planet the opposition to the mean sun occurs. Our assumptions are therefore equivalent to saying that the opposition occurs in the apsidal line; this is indeed Ptolemy's formulation.[1]

In order to apply the Apollonius Theorem to the new situation we need the ratio c of the angular velocities v_c and v_p, v_c being directed at a right angle to the line OP, v_p tangential to the epicycle. The velocity v_c will be smaller than the mean velocity $\bar{v}_c$ because C is farther removed from O than from E. On the other hand v_p is greater than the mean velocity $\bar{v}_p$ on the epicycle because $\bar{v}_p$ is measured from the radius CF whereas the observer at O measures the velocity of P on the epicycle with respect to the radius CG. Hence the observed velocity of P is increased by the velocity with which F moves toward G as C moves toward the apogee A.

[1] Cf. above p. 192 and note 1 there.

The negative increment of v_c and the positive increment of v_p are both measured by the rate of change of the same angle η which represents the difference between the mean and the true longitude of C and also between the mean and the true anomaly of P. Its value is tabulated as c_3+c_4 in the tables of the planetary equations[2] (Alm. XI, 11) as function of the eccentric anomaly $\bar{\kappa}$. The angular velocity $\bar{v}_c$ indicates how fast $\bar{\kappa}$ varies. Hence, if c' is the rate of change of $c_3(\bar{\kappa})+c_4(\bar{\kappa})$ per degree we have in $c'\bar{v}_c$ the rate of change of the angle η. $\left(\text{In modern terminology: } \frac{d\eta}{dt}=\frac{d(c_3+c_4)}{d\bar{\kappa}}\cdot\frac{d\bar{\kappa}}{dt}=c'\bar{v}_c.\right)$ Hence we find for the angular velocity of C with respect to O the amount $v_c=\bar{v}_c-c'\bar{v}_c$ and for the motion of P with respect to the radius CG the angular velocity $v_p=\bar{v}_p+c'\bar{v}_c$. Consequently

$$c=\frac{v_p}{v_c}=\frac{\bar{v}_p+c'v_c}{\bar{v}_c-c'\bar{v}_c} \tag{2a}$$

or, dividing numerator and denominator by $\bar{v}_c$

$$c=\frac{\bar{c}+c'}{1-c'} \qquad (\bar{c}=\bar{v}_p/\bar{v}_c). \tag{2b}$$

With these values of v_p and v_c the Apollonius Theorem applies to the present case (cf. Fig. 196[3])

$$c=\frac{v_p}{v_c}=\frac{\rho}{\mathrm{PT}} \qquad (\rho=\mathrm{OP}). \tag{3}$$

We now must determine the value of c' and the several distances from O. This is made easy in the case of Saturn and Jupiter by the assumption that OC is near enough to the apsidal line to deal with it as if C were at maximum distance from O. This is to say we assume $\bar{\kappa}\approx 0$ in which case the rate of change of $c_3(\bar{\kappa})+c_4(\bar{\kappa})$ per degree is found by means of the first entry (for $\bar{\kappa}=6°$) in the tables of the planetary equations (Alm. XI, 11) as[4]

$$c'=1/6\big(c_3(6)+c_4(6)\big). \tag{4}$$

It also follows from our assumption that

$$\mathrm{OG}\cdot\mathrm{O}\Theta\approx(R+e+r)(R+e-r).$$

Hence we have the same value also for[5]

$$\mathrm{OQ}\cdot\mathrm{OP}=\rho^2\,\frac{c+2}{c}$$

and we can find

$$\rho=\sqrt{\frac{(R+e+r)(R+e-r)c}{c+2}} \qquad \mathrm{PT}=\frac{\rho}{c}$$

and from it, in similar fashion as before, the angles ζ, β, and γ (Fig. 196).

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[2] Cf. p. 183f.

[3] In Fig. 196 v'_c denotes the product $v_c\rho$ and similarly $v'_p=v_p r$.

[4] Making use of the trivial fact that $c_3(0)+c_4(0)=0$.

[5] Cf. p. 192.

The time Δt of retrogradation is the time needed by the planet to travel on the epicycle an angle 2β with the velocity as seen from O. Hence

$$\Delta t = \frac{2\beta}{v_p} = \frac{2\beta}{\bar{v}_p + c'\bar{v}_c} = \frac{2\beta}{(\bar{c}+c')\bar{v}_c}. \tag{5}$$

The retrograde arc $\Delta\lambda$ is 2γ minus the motion in longitude of C during Δt, i.e. minus $2\delta = v_c \Delta t$. Thus with (5) and (2a)

$$\Delta\lambda = 2(\gamma - \delta) \qquad \delta = \beta v_c/v_p = \beta/c. \tag{6}$$

In this way Ptolemy finds

	Saturn	Jupiter
$\delta = \beta/c$	2;6, 6°	4;40,35°
$-\Delta\lambda$	7;4,10°	9;49,14°
Δt	140 2/3^d	123^d

i.e. values only a little different from the values obtained for mean distances.[6]

For the remaining planets we are no longer justified to act as if not only P but also C were points of the apsidal line, i.e. $\bar{\kappa}=0$ and $OC = R+e$. The distance of C from the apsidal line is in all cases measured by the angle δ which C travels during the time interval $1/2\,\Delta t$ between first station and opposition. This angle has been computed for all planets under the assumption of "mean distance," i.e. for the simple model of Fig. 195 (p. 1268). These previously found values of $\bar{\delta} = \beta/\bar{c}$ are[7]

$$\begin{array}{rlrl} \text{Saturn:} & 2;19° & \text{Mars:} & 19;\ 7,33° \\ \text{Jupiter:} & 5;1,24 & \text{Venus:} & 20;35,19 \\ & & \text{Mercury:} & 11;\ 4,59. \end{array} \tag{7}$$

These values confirm, in the case of Saturn and Jupiter, that we were entitled to find the rate of change of $c_3 + c_4$ from the first tabulated interval of 6° for which we may assume linear increase of $c_3 + c_4$ with $\bar{\kappa}$. For the remaining planets, however, c' must be found as the rate of change per degree of $c_3 + c_4$ for the specific interval in the tables of Alm. XI, 11 to which $\bar{\kappa} \approx \bar{\delta}$ belongs.

Also OC can no longer be taken as $R+e$ but only as $R+e'$ where e' is by a certain amount smaller than e because OC is inclined toward OA at an angle of approximately $\bar{\kappa} = \bar{\delta}$ (cf. Fig. 197). This distance OC can be computed in the usual fashion as function of $\bar{\kappa}$; Ptolemy had undoubtedly tabulated these distances in order to compute the tables XI, 11.[8] For the present purpose he uses

$$\begin{array}{rl} \text{Mars:} & e' = e - 0;20 = 5;40 \\ \text{Venus:} & e' = e - 0;5 = 1;10 \qquad (R=60) \\ \text{Mercury:} & 3e' = 3(e - 0;8) = 8;36. \end{array} \tag{8}$$

[6] Above p. 193. For the numerical details cf. below p. 199 (Table 18).
[7] Cf. above p. 193.
[8] Cf. above p. 183 and below p. 204.

From now on the computations follow again the previous pattern and lead to the following data for the retrogradation at maximum distance[9]

	Mars	Venus	Mercury
$\delta=\beta/c$	17;13,21°	20;19, 3°	9;48,51°
$-\Delta\lambda$	19;53,32°	16;25,26°	7;54,22°
Δt	80^d	43^d	21^d

C. Minimum Distance

The basic idea for the computation of the retrograde arcs under the assumption that the stationary point P falls in the apsidal line near the perigee Π (cf. Fig. 198) is exactly the same as in the case of maximum distance. The ratio $\bar{c}=\bar{v}_p/\bar{v}_c$ for mean distance has to be modified to

$$c=\frac{v_p}{v_c}=\frac{\bar{v}_p-c'\bar{v}_c}{\bar{v}_c+c'\bar{v}_c}=\frac{\bar{c}-c'}{1+c'}. \tag{1}$$

For Saturn and Jupiter the correction c' is computed from c_3+c_4 (tabulated in Alm. XI, 11) under the assumption that OC is the minimum distance[1] $R-e$. For the remaining planets, however, the minimum distance has to be increased by an amount which depends on the angle δ which separates OC from OΠ (approximately known as $\bar{\delta}$[2]):

$$\begin{aligned} \text{Mars: } & R-e+0;20=54;20 \\ \text{Venus: } & R-e+0;\ 5=58;50 \\ \text{Mercury: } & 55;34+0;\ 8=55;42. \end{aligned} \tag{9}$$

To deal with Mercury near its perigees (i.e. near $\bar{\kappa}=\pm 120°$) in exactly the same fashion as with the other planets is, of course, not strictly correct since one ignores the fact that the motion of the center N of the deferent about M adds one more component to the velocity of P.[3]

In this way Ptolemy finds the following retrogradations at minimum distance[4]

	Saturn	Jupiter	Mars	Venus	Mercury
$\delta=\beta/c$	2;33,28°	5;21,20°	20;33,42°	20;53,30°	11;39,30°
$-\Delta\lambda$	7;18,10°	9;54,40°	11;12,14°	14; 4,38°	15;12,46°
Δt	136^d	118^d	64 1/2^d	40 2/3^d	23^d

We have now determined the data for the retrogradations under the assumption that a planet is either at mean distance from O or in the apsidal line. Before

[9] Cf. for the details below p. 199.
[1] Cf., however, below p. 198f.
[2] Cf. p. 199.
[3] Cf. Fig. 148 (p. 1253).
[4] Cf. also Table 20 (p. 201).

turning to the discussion of the tables which give the retrogradations as function of arbitrary values of $\bar{\kappa}$ (below p. 202) we supplement the preceding outline of the general method with some details about the numerical execution.

D. Numerical Data

Through actual observation it is very difficult to say when and where a planet is accurately stationary.[1] Both before and after the theoretically exact moment at which the line of sight from us to the planet coincides with the direction of its instantaneous motion the change of position is so slow that it is of little practical importance to determine a definite moment. Nevertheless Ptolemy computes the retrogradations to seconds of arcs and gives fractions of days for the duration of the retrograde motion. There is, of course, no objection to be raised against solving a problem for an idealized geometrical model from a purely mathematical viewpoint, regardless of whether or not observations can measure up to the resulting accuracy. In many cases, however, the mathematical accuracy of Ptolemy's results is only apparent because of carelessness in the handling of the numerical material.

Table 17. Mean Distance

	Saturn	Jupiter	Mars	Venus	Mercury
r	6;30	11;30	39;30	43;10	22;30
$\bar{c}=\bar{v}_p/\bar{v}_c$	28;25,46	10;51,29	0;52,51	0;37,31	3; 9, 8
ρ	57;38,55	54; 6,44	24;58,25	20;20,11	43;30,24
PT	2; 1,40	4;59, 1	28;21, 8	32;31,29	13;48, 7
β	65;52,12°	54;21,38°	16;50,48°	12;52,24°	34;56,12°
γ	5;57,10°	9;57,32°	27;16,55°	28;14, 6°	17;13,34°

1. Mean Distance. Small deviations in square root approximations and in interpolations in trigonometric tables are not surprising. Difficult to explain, however, are discrepancies in the values of such an essential parameter as the ratio $\bar{c}$ of the mean motion of anomaly $\bar{v}_p$ to the mean motion in longitude $\bar{v}_c$. In the computation of $\bar{c}$ nothing is involved but one simple division of two values given with high accuracy in the tables of Alm. IX, 4. Nevertheless Ptolemy's values (listed under $\bar{c}$ in Table 17) deviate by the following amounts from the correct quotients:

Saturn: $-0;0,9,22$ Mars: $+0;0,0,14$
Jupiter: $+0;0,1,32$ Venus: $-0;0,0,46$
Mercury: $+0;0,0,53$.

Only in the case of Mars is one dealing with a legitimate rounding. It is therefore rather meaningless when Ptolemy gives his results with an accuracy of seconds. The initial inaccuracy in the value of $\bar{c}$ affects all subsequent numbers and this

[1] We consider here only longitudes. The real motion in longitude *and* latitude need not produce stationary points at all; cf., e.g., the orbits shown in Fig. 228 (p. 1283).

influence is still felt, at least in principle, in the computations for the extremal distances.

If one continues to the end the computations for mean distance using the corrected values of $\bar{c}$ one finds fortunately only insignificant deviations from Ptolemy's final results as listed above p. 193

Correction for	Saturn	Jupiter	Mars	Venus	Mercury
$\bar{\delta}$	+0;0, 6°	−0;0,57°	+0;0, 8°	−0;0, 9°	0°
$-\Delta\lambda$	−0;5,20°	+0;0,28°	−0;0,48°	+0;1,42°	+0;0, 2°
Δt	+0;27^d	−0;31^d	0^d	+0;6^d	−0;0,38^d

The fact remains, however, that Ptolemy could have obtained significant figures without any increase in labor.

2. Extremal Distances. As we have shown on p. 195 the time Δt of retrogradation is given by

$$\Delta t = 2\beta/v_p$$

where $v_p = \bar{v}_p \pm c'\bar{v}_p$ (the upper sign for maximum distance, the lower for minimum distance). Ptolemy uses the algebraically equivalent relation

$$\Delta t = \frac{\delta}{1 \mp c'} \cdot \frac{2}{\bar{v}_c}$$

which is, however, less satisfactory since it involves unnecessarily many rounding errors. The division by $\bar{v}_c$ is circumvented by using the tables of mean motion (Alm. IX, 4) to find the time Δt which is required to produce a mean motion in longitude of the amount $\delta/(1 \mp c')$. This procedure, however, involves again rounding errors of tabular interpolations.

A similar roundabout way is followed by Ptolemy in the computation of δ (and hence of $\Delta\lambda$).[2] Instead of simply dividing β by c (since $\delta = \beta/c$) he first corrects β from its value at O to its value at E, i.e. he forms $\beta - \eta$ where η is the equation $\eta = c_3 + c_4$ for the argument $\bar{\kappa} = \beta/(c+c')$. This angle $\beta - \eta$ divided by $\bar{c}$ is then $\bar{\delta}$ (with reference to E) and hence $(\beta-\eta)/\bar{c} - \eta = \delta$ for O. Here the inaccuracies of the tables for $c_3 + c_4$ (Alm. XI, 11), combined with the error in the value obtained for $\bar{\kappa}$, are particularly felt.

As said before these inaccuracies are of no significance in view of the unsharpness of the observable data. What is of historical interest is the fact that one has here one of the comparatively rare cases in ancient mathematical astronomy where the absence of an algebraic notation caused the use of many steps which could have been condensed into one single operation of far greater simplicity and accuracy.

Saturn and Jupiter. As we have seen[3] Saturn and Jupiter are only about $\bar{\delta} = 2\,1/3°$ and 5°, respectively distant from mean opposition when a stationary point is located in the apsidal line. The effect on the distance OC of such a small elongation is negligible and permits Ptolemy to assume for OC the extremal values $R \pm e$. It is also legitimate to use for the rate of change of $\eta(\bar{\kappa}) = c_3 + c_4$ at

[2] Described in the case of maximum distance of Mars in Alm. XII, 6 (Man. II, p. 301 f.).

[3] Above p. 193, values of $\bar{\delta}$ in the table.

maximum distance the relation

$$c' = 1/6\big(c_3(6) + c_4(6)\big) \tag{4}$$

because $\bar{\delta} < 6°$ and $\bar{\kappa} = 6°$ is the first argument tabulated in Alm. XI, 11. For $\bar{\kappa}$ near 180°, however, the entries are only 3° apart. Hence, c' for Jupiter at minimum distance should have been found from

$$c' = 1/3\big(c_3(174) + c_4(174) - (c_3(177) + c_4(177))\big) \tag{4a}$$

instead of from

$$c' = 1/3\big(c_3(177) + c_4\ 177)\big). \tag{4b}$$

Since Ptolemy uses (4b) he obtains $c' = 0;5,40$ instead of 0;6,20 from (4a).[4] Consequently[5] his $c = 9;50,5$ instead of 9;43,33. Luckily the effect on the final results is only small because the total range for $\Delta\lambda$ and Δt is very narrow.

Table 18 gives Ptolemy's results, obtained, of course, on the basis of his value of $\bar{c}$ (cf. Table 17, p. 197). The deviations from accurate computations are insignificant.

Table 18

	Maximum Distance		Minimum Distance	
	Saturn	Jupiter	Saturn	Jupiter
c'	0; 6,30	0; 5,10	0; 7,20	0; 5,40
	28;32,16	10;56,39	28;18,26	10;45,49
c	0;53,30	0;54,50	1; 7,20	1; 5,40
	(≈32; 0,18)	(≈11;58,31)	(≈25;13,27)	(≈9;50, 5)
ρ	61;11,52	57; 6,19	54; 6,22	51; 7,38
PT	1;54,44	4;46, 6	2; 8,43	5;11,55
β	67;15,17	55;55, 1	64;31,10	52;48,48
γ	5;38,11	9;35,12	6;12,33	10;18,40
δ	2; 6, 6	4;40,35	2;33,28	5;21,20

Mars, Venus, and Mercury. At mean distance (OC $= R$) we have found[6] that an angle $\bar{\delta}$ separates OC from the apsidal line:

$$\text{Mars; } 19;7,33° \quad \text{Venus: } 20;35,19° \quad \text{Mercury: } 11;4,59°. \tag{5}$$

Such comparatively great deviations from the apsidal line show an effect in two ways. First, the rate of change c' of $c_3(\bar{\kappa}) + c_4(\bar{\kappa})$ is not the same as for apogee or perigee; secondly, the distance OC is different from the extremal distance. Both modifications were taken into account by Ptolemy, unfortunately without telling us how he reached his numerical results. Hence some detailed discussion becomes necessary.

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[4] Assuming, of course, the values of c_3 and c_4 as found in Alm. XI, 11. These values themselves, however, show the effects of irregular roundings and interpolations. Cf. also p. 200, n. 7.

[5] From (2b), p. 194.

[6] Above p. 193.

Mars. Since $\bar{\delta} \approx 19°$ we have to determine c' as the rate of change of $c_3 + c_4$ in the interval $18 \leqq \bar{\kappa} \leqq 24$ and $159 \leqq \bar{\kappa} \leqq 162$, respectively. In the first case one finds $c' = 1/6(4;16 - 3;13) = 0;10,30$, in the second $c' = 1/3(4;33 - 3;55) = 0;12,40$. Ptolemy gives 0;10,20 and 0;12,40, respectively. I have no explanation for the deviation i the first case.[7]

Venus. For $\bar{\delta} \approx 20;30$ the intervals are again $18 \leqq \bar{\kappa} \leqq 24$ and $159 \leqq \bar{\kappa} \leqq 162$. The corresponding values of c' are 0;2,30 and 0;2,0, respectively. Ptolemy gives in both cases 0;2,20. One would obtain his results if one could ignore the change of c_4 from 0;1 to 0;2.

Mercury. For the whole interval $6 \leqq \bar{\kappa} \leqq 18$ one finds $c' = 0;2,30$ whereas Ptolemy gives 0;2,20. At minimum distance there is no symmetry with respect to a perigee ($\bar{\kappa} = \pm 120$). Hence one has to compute c' (with $\bar{\delta} \approx 11$) separately for the intervals $108 \leqq \bar{\kappa} \leqq 111$ and $129 \leqq \bar{\kappa} \leqq 132$. In the first case one finds $c' = 0;1$, in the second 0;2. Ptolemy uses the mean value 0;1,30.

The second problem concerns the distances OC, i.e. the explanation of the corrections (8), p. 195 and (9), p. 196. For Mars and Venus one can find the distance $OC = x$ (cf. Fig. 199) to given angle $\bar{\delta}$ and eccentricity e from

$$\begin{aligned} \delta_1 &= \arc\sin(e \sin \bar{\delta}/R) \\ \gamma' &= \bar{\delta} \mp \delta_1 \\ \gamma &= \operatorname{arc}\tan\left(R \sin \gamma'/(R \cos \gamma' \pm e)\right) \\ OC &= x = R \sin \gamma'/\sin \gamma, \end{aligned}$$

the upper sign referring to maximum, the lower to minimum, distance. Using the values (5), p. 199 for $\bar{\delta}$ one obtains in this way

		Mars	Venus
OC	max.	$1,5;45,\ 5 \approx R + e - 0;15$	$1,1;10,34 = R + e - 0;4,26$
	min.	$54;26,27 \approx R - e + 0;26$	$58;50,\ 7 = R - e + 0;5,\ 7$

Ptolemy simply takes the mean values of these corrections in both cases: $\mp 1/2(0;15 + 0;26) \approx \mp 0;20$ and $\mp 1/2\ 0;9,33 \approx \mp 0;4,47 \approx \mp 0;5$.

In the case of Mercury one has to distinguish between maximum and minimum distance. In the first case there exists symmetry with respect to the apsidal line, but not in the second case.

For the maximum distance one has (cf. Fig. 200)[8]

$$\begin{aligned} \sin \delta_1 &= \frac{2e}{R} \sin \frac{3\bar{\delta}}{2} \cos \frac{\bar{\delta}}{2} \\ EC = y &= R \sin\left(\delta_1 + \frac{3\bar{\delta}}{2}\right) \Big/ \sin \frac{3\bar{\delta}}{2} \\ \tan \gamma &= y \sin \bar{\delta}/(e + y \cos \bar{\delta}) \\ OC = x &= y \sin \bar{\delta}/\sin \gamma. \end{aligned}$$

[7] One would not only obtain Ptolemy's result but also smoother differences for $c_3 + c_4$ if one could replace $c_3(24) + c_4(24) = 4;16$ by 4;15. Unfortunately the Handy Tables confirm the value 4;16.

[8] For Ptolemy's procedure cf. p. 165 and Fig. 149.

Substituting the numerical data, $R=60$, $e=3$, $\bar{\delta}\approx 11;5$[9] one finds

$$OC = 1,8;38,46 = R + 3e - 0;21,14$$

whereas Ptolemy uses $R+3e-0;24$. I cannot explain the deviation.[10]

At minimum distance one has to distinguish two cases: C being $\bar{\delta}°$ before or $\bar{\delta}°$ after the perigee ($\bar{\kappa}=120$). By the same process as at maximum distance one can again find OC and obtains in the first case $OC=55;45=55;34+0;11$, in the second case $OC=55;41=55;34+0;7$, i.e. increments 0;11 and 0;7, respectively beyond the minimum distance 55;34.[11] Ptolemy takes 0;8 in both cases, i.e. a mean value.

It may be remarked that these computations provide us also with the possibility of checking some values of c_3+c_4 in Alm. XI, 11 since (cf. Figs. 199 and 200, p. 1269; Fig. 175, p. 1261)

$$|\bar{\delta}-\gamma| = c_3(\bar{\delta}) + c_4(\bar{\delta}) = \eta.$$

Only at the maximum distance of Mars and Venus do I find results 0;2° and 0;1° respectively greater than c_3+c_4. All other values agree.

The Tables 19 and 20 give the essential parameters of Ptolemy's computations, assuming extremal distance of C for Mars, Venus, and Mercury. The value 32;58,26 of β at maximum distance of Mercury (Table 19) is the correct result of a subtraction 76;23,58 − 43;15,32. The text, however has 32;52,26 and this erroneous value is used also later on (below p. 202).

Table 19. Maximum Distance

	Mars	Venus	Mercury
c'	0;10,20°	0; 2,20°	0; 2,20°
	1; 3,11	0;39,51	3;11,28
c	0;49,40	0;57,40	0;57,40
	(≈1;16,20)	(≈0;41,28)	(≈3;19,13)
ρ	32;42,34	21;57,38	51;11,43
PT	25;42,43	31;46,44	15;25, 9
β	22;13,19°	14; 3,47°	32;58,26°
γ	27;10, 7°	28;31,46°	13;46, 2°

Table 20. Minimum Distance

	Mars	Venus	Mercury
c'	0;12,40°	0; 2,20°	0; 1,30°
	0;40,11	0;35,11	3; 7,38
c	1;12,40	1; 2,20	1; 1,30
	(≈0;33,11)	(≈0;33,52)	(≈3; 3, 3,25)
ρ	17;21,51	18;45,16	39;36, 4
PT	31;24, 3	33;13,36	12;58,47
β	11;11, 6°	11;44,24°	35;30,15°
γ	26; 9,49°	27;55,49°	19;15,53°

9 Above p. 193.

10 It is probably only accidental that Ptolemy's correction at maximum distance (0;24) is 3 times the correction at minimum distance (0;8).

11 Cf. above p. 164, (1) and Table 14 (p. 169).

2. Tables for Retrogradations

The tables Alm. XII, 8 give for the stationary points P their positions on the epicycle as function of the mean anomaly $\bar{\kappa}$ of the center C of the epicycle (cf. Fig. 197, p. 1269). If G is the true apogee of the epicycle, Θ the corresponding perigee, then the tabulated values are the anomalies $\alpha = 180 \mp \beta$ where β is the difference of anomaly between P and Θ. The Figs. 201 and 202 give a graphic representation of the function $\alpha(\bar{\kappa})$ for the five planets.

The construction of these tables falls into two parts. First Ptolemy determines the values of α which belong to a position of C exactly at the apogee or at the perigee of the deferent. This gives the ranges for the anomaly in which stationary points are possible (cf. Fig. 203). Secondly, he finds by interpolation the values of α which correspond to intermediate values of $\bar{\kappa}$.

A. Epicycle at Extremal Distances

1. Saturn and Jupiter. The epicycles of the two outermost planets are so small that one can introduce the simplification of assuming for OC the values of the extremal distances without any correction. The values of the angle β computed under these conditions[1] correspond therefore directly to the arguments $\bar{\kappa}=0$ and $\bar{\kappa}=180$, respectively. Hence we obtain for $\alpha = 180 \mp \beta$ the following values[2]

	Saturn		Jupiter		Station
$\bar{\kappa}=0$	$180 \mp 67;15 =$	112;45	$180 \mp 55;55 =$	124; 5	1st
		247;15		235;55	2nd
$\bar{\kappa}=180$	$180 \mp 64;31 =$	115;29	$180 \mp 52;49 =$	127;11	1st
		244;31		232;49	2nd

These are the values given in the first and last line of the tables Alm. XII, 8.

2. Mars, Venus, and Mercury at Apogee. For the three remaining planets one has to take into account that the "maximum" and the "minimum" situation does not mean that the center C of the epicycle is exactly at the extremal distance. In other words, the angle β between the stationary point P and the opposition Θ is a function of $\bar{\kappa}$ of which we know only one value for a $\bar{\kappa}$ near 0 but not for exactly $\bar{\kappa}=0$. In order to find $\beta(0)$ Ptolemy argues as follows. We know the increments $\delta\beta$ from mean to maximum distance of P:[3]

	Mars	Venus	Mercury
β for P at mean dist.	16;50,48	12;52,24	34;56,12
β for P at max. dist.	22;13,19	14; 3,47	32;52,26
$\delta\beta$	+5;22,31	+1;11,23	−2; 3,46
Ptolemy: rounded	+5;22	+1;12	−2; 4

[1] Above pp. 193 and 195.
[2] From Table 18 (p. 199), slightly rounded.
[3] Cf. Tables 17 and 19 (pp. 197 and 201).

Equally known are the corresponding increments $\delta\rho$ of the distances $\rho = OC$[4]

$\delta\rho$: Mars: 5;40 Venus: 1;10 Mercury: 8;36.

When C moves into the apsidal line ($\bar{\kappa}=0$) then OC increases beyond its mean value R by the eccentricity e (or $3e$ for Mercury):

$\Delta\rho$: Mars: 6 Venus: 1;15 Mercury: 9.

Ptolemy assumes now linear increase of $\beta(\bar{\kappa})$ with ρ. Consequently one has for the increase $\Delta\beta$ of β over the value $\bar{\beta}$ at mean distance

$$\Delta\beta = \frac{\delta\beta}{\delta\rho}\,\Delta\rho \tag{1}$$

and thus

$$\beta(0) = \bar{\beta} + \Delta\rho\,\frac{\delta\beta}{\delta\rho}. \tag{2}$$

In this way one finds

	Mars	Venus	Mercury
$\delta\beta/\delta\rho$	0;56,54,53	1; 1,11,8	0;14,23,30
from (1): $\Delta\beta$	5;41	1;17	−2;10
$\bar{\beta}$[5]	16;51	12;52	34;56
$\beta(0)$	22;32	14; 9	32;46

This gives for the epicyclic anomalies $\alpha(0) = 180 \mp \beta(0)$ of the first and second station:

Mars: 157;28 Venus: 165;51 Mercury: 147;14
202;32 194;9 212;46.

These are the values of α found in the first line ($\bar{\kappa}=0$) in the tables Alm. XII, 8.

3. Mars, Venus, and Mercury at Perigee. By a similar argument one finds the values of $\beta(\bar{\kappa})$ for $\bar{\kappa}=180$ with Mars and Venus, for $\bar{\kappa}=\pm 120$ with Mercury.[6]

	Mars	Venus	Mercury
β for P at mean dist.	16;50,48	12;52,24	34;56,12
β for P at min. dist.	11;11, 6	11;44,24	35;30,15
$\delta\beta$ (Ptolemy)	−5;40	−1; 8	+0;34
$\delta\rho$[7]	−5;40	−1;10	−4;18
$\delta\beta/\delta\rho$	+1	+0;58,17	−0; 7,54,25
$\Delta\rho$	−6	−1;15	−4;26
$\Delta\beta$[8]	−6	−1;13	+0;35
$\bar{\beta}$	16;51	12;52	34;56
β at perigee	10;51	11;39	35;31

4 Cf. (8), p. 195.
5 From Table 17 (p. 197), rounded.
6 Cf. Tables 17 and 20 (pp. 197 and 201).
7 Cf. (9), p. 196.
8 From (1).

Hence the epicyclic anomalies $\alpha = 180 \mp \beta$ for the first and second stations are

Mars: 169;9	Venus: 168;21	Mercury: 144;29
190;51	191;39	215;31.

These are the values of $\alpha(\bar{\kappa})$ in Alm. XII, 8 for $\bar{\kappa}=180$, $\bar{\kappa}=180$, and $\bar{\kappa}=\pm 120$ respectively.

B. Epicycle at Arbitrary Distances; Tables

The tables in Alm. XII, 8 give in steps of 6° of the mean eccentric anomaly $\bar{\kappa}$ (reckoned from the apogee of the deferent) the epicyclic anomaly α at which for a given $\bar{\kappa}$ the first, respectively the second, station occurs (cf. the graphs Figs. 201 and 202, p. 1270).

The computation of these tables is based on the same idea which was used to find β for the center C of the epicycle at the apogee or at the perigee of Mars, Venus, or Mercury (above p. 202). Let $\bar{\kappa}$ first be a value such that the distance $OC = \rho(\bar{\kappa})$ is greater than the mean distance $R = 60$. For every $\bar{\kappa}$ one can compute $\rho(\bar{\kappa})$ and therefore also

$$\delta\rho(\bar{\kappa}) = \rho(\bar{\kappa}) - R. \qquad (1)$$

We also know the maximum distance $\rho_{\max} = \rho(0)$ for all planets ($= R + e$, excepting Mercury: $R + 3e$), hence

$$\Delta\rho = \rho_{\max} - R. \qquad (2)$$

In the preceding section we have found for each planet the value $\beta(0)$ of the angle β between stationary point and opposition at $\bar{\kappa}=0$ and the value $\bar{\beta}$ for $OC = R$. Hence we know

$$\Delta\beta = \beta(0) - \bar{\beta}. \qquad (3)$$

Assuming again that β is a linear function of ρ we can find

$$\delta\beta(\bar{\kappa}) = \frac{\Delta\beta}{\Delta\rho}\,\delta\rho(\bar{\kappa}) \qquad (4)$$

and hence

$$\beta(\bar{\kappa}) = \bar{\beta} + \delta\beta(\bar{\kappa}). \qquad (5)$$

If, however, $\bar{\kappa}$ belongs to a $\rho(\bar{\kappa}) < R$ we define:

$$\delta\rho(\bar{\kappa}) = R - \rho(\bar{\kappa}) \qquad (1\,\mathrm{a})$$

$$\Delta\rho = R - \rho_{\min} \qquad (2\,\mathrm{a})$$

$$\Delta\beta = \bar{\beta} - \beta(\bar{\kappa}_{\min}) \qquad \bar{\kappa}_{\min} = \begin{cases} 180 & \text{♄ to ♀} \\ \pm 120 & \text{Mercury} \end{cases} \qquad (3\,\mathrm{a})$$

$$\beta(\bar{\kappa}) = \bar{\beta} - \delta\beta(\bar{\kappa}) \qquad (5\,\mathrm{a})$$

and find again $\delta\beta(\bar{\kappa})$ from (4).

With $\beta(\bar{\kappa})$ known from (5) and (5a) we also know the anomaly as function of $\bar{\kappa}$

$$\alpha = 180 \mp \beta(\bar{\kappa}) \tag{6}$$

at which the stationary points occur.

The key to this whole procedure is obviously a table for the distance $\rho(\bar{\kappa})$, not given explicitly by Ptolemy,[1] but underlying also his computation of the planetary equations, (tabulated in Alm. XI, 11) and, in the case of Mercury, the computation of maximum elongations.[2]

C. Examples

The tables in Alm. XII, 8 give us the true anomaly α of a planet at a stationary point as function of the eccentric mean anomaly $\bar{\kappa}$ of the center of the epicycle. In computing planetary positions as function of the time t one needs both $\bar{\kappa}(t)$ and $\alpha(t)$. Consequently we may assume that α as function of $\bar{\kappa}$ is known.[3] In order to find the moment of a station we have only to find in a computed ephemeris such a pair $\bar{\kappa}$, $\alpha(\bar{\kappa})$ which also occurs in the tables Alm. XII, 8. Ordinarily this will require some interpolation; for the following specific examples we will do this graphically.

1. Mars. On p. 186 we computed an ephemeris of Mars in 10-day intervals for the year $-297/296$ = Nabon. 450/451. We wish to find the dates of the first and second station which occur in this interval.

We excerpt from the computed ephemeris (p. 188, Table 15) such values of α and $\bar{\kappa}$ which belong to the same moment and which are near to pairs listed in Alm. XII, 8:

Ephemeris				Alm. XII, 8	
No.	Date	α	$\bar{\kappa}$	$\bar{\kappa}$	α
7	Oct. 9	156;49	247;30	270	162;18
8	19	161;41	279;44	276	161;44
9	29	166;37	284;59	282	161;10
15	Dec. 28	197;40	316;26	312	201;42
16	Jan. 7	203; 3	321;40	318	201;26*
				324	201; 5

The graph in Fig. 204 shows that a common pair α, $\bar{\kappa}$ will be found near Oct. 18/19 (first station Φ) and near Jan. 3/4 (second station Ψ). This is in excellent agreement with the computed orbit (cf. below Fig. 228a, p. 1283), both with respect to Ptolemy's tables and to modern ones; cf. also below Fig. 192 (p. 1267).

2. Venus. The same procedure provides us with the dates of the stations of Venus in the year -305 = Nabon. 442/443. From our 5-day ephemeris (p. 189,

[1] He mentions only some special cases, e.g. for $\bar{\kappa} = 30°$ (Manitius II, p. 258).

[2] Cf. below p. 232.

[3] Since, for a short interval of time, α increases like $\bar{\alpha}$ and $\bar{\kappa}$, i.e. proportional with time, the function $\alpha(\bar{\kappa})$ will be very nearly a linear function. Cf. below the vertical graphs in Figs. 204 and 205.

Table 16) and from Alm. XII, 8 we have, respectively

Ephemeris				Alm. XII, 8	
No.	Date	α	$\bar{\kappa}$	$\bar{\kappa}$	α
3	Oct. 26	165;17	159;54	156	168;14
4	31	168;10	164;49	162	168;17
5	Nov. 5	171; 3	169;45	168	168;19
				174	168;20
12	Dec. 10	191;12	205;15	192	191;41
13	15	194; 7	209;11	198	191;43
				204	191;46
				210	191;50

The intersections of the corresponding graphs (Fig. 205) show that the first station (Ψ) occurred about Oct. 31/Nov. 1, the second (Φ) about Dec. 11/12. This is in good agreement with the computed orbit (cf. below Fig. 233a, p. 1285 and Fig. 193, p. 1268).

§ 7. Planetary Latitudes

The theory of the latitude of the moon rests on the directly observable fact that the moon's orbital plane is inclined to the ecliptic at a practically fixed angle and that the earth lies in the nodal line. Consequently the computation of lunar latitudes follows the simple pattern of the determination of solar declinations. The theory of the geocentric latitudes of the planets, however, is much more complicated because the orbital planes do not go through the earth but through the sun. It is in the theory of planetary latitudes that the heliocentric theory has a decisive advantage over the geocentric approach.

Nevertheless even a comparatively short sequence of continually recorded positions of Saturn or Jupiter will indicate the type of modification of the epicyclic model that is required to cover the latitudinal component of the planetary motion. It is quite obvious (cf. above p. 170, Figs. 155/156) that the epicycle moves along an inclined deferent. For Mars and the inner planets, however, the phenomena become much more complex and it is therefore no small achievement of Ptolemy have developed workable models for all planets for their geocentric latitudes.

It will be helpful, however, before turning to Ptolemy's theory of planetary latitudes to analyze the phenomena one has to expect to be observed from the earth, using our present knowledge of the actual situation, of course without going beyond the most general features. In particular we shall always operate with circular orbits.

1. The Basic Theory

Let in Figs. 206 and 207 the plane of the paper represent the ecliptic, O the observer, S the sun, and P the planet moving in an inclined orbit with center S. If we ignore, for the moment, all eccentricities then it is obvious that the planet moves, with respect to O, on an epicycle that is displaced parallel to itself. In the case of an inner planet (Fig. 206) the plane of the deferent coincides with the plane of the ecliptic whereas for an outer planet the plane AOBC of the deferent is inclined to the ecliptic at the same angle as the heliocentric orbit while the plane of the epicycle remains parallel to the ecliptic.

The situation becomes more complicated, however, as soon as we take eccentricities and orbital inclinations into account. Figs. 208 and 209 illustrate the situation for an inner and an outer planet, respectively: π_0 is the perihelium, e_0 the eccentricity of the earth's orbit; the nodal line of the planetary orbit goes through the sun S, the dotted part of the orbit being below the ecliptic, the rest above it.

If we transform these models to geocentric coordinates we obtain models as represented in Figs. 210 and 211, respectively. As discussed in Section I C, 1, 2[1] the eccentricities e_0 and e_1 combine to a new eccentricity e, located in a resulting apsidal line $A\Pi$. The planes of the epicycles are again displaced parallel to themselves, but the nodal line of an inner planet does not go through the center M_1 of the epicycle but through S, the true sun; and similarly for the nodal line of the deferent which passes through S′ and not through O.

We must now investigate the consequences of this situation for the computation of geocentric latitudes of a planet. Only the vector e_0 of the solar eccentricity lies in the ecliptic whereas the planetary eccentricity e_1 belongs to the inclined orbit. Therefore the points M_1 and M_1' are points which do not lie in the ecliptic unless the nodal line of the planetary orbit accidentally also contains the perihelium.

Fig. 210 shows that the parallelism of M_1M_1' to SS′ implies that the plane of the deferent of an inner planet does not coincide with the ecliptic but has a fixed distance from it. Similarly for an outer planet (Fig. 211): since the nodal line of the plane of the deferent passes through S′ the observer O will be located above or below the plane of the deferent.

2. Numerical Data

The modifications of the simple model of Figs. 206 and 207, introduced because of the eccentricities, are the cause of the complications of the Ptolemaic theory of latitudes. Ptolemy was not aware of the fact which we just derived that the observer was stationed outside of the plane of the deferent. Consequently his observations did not suggest fixed inclinations of the orbital planes to him and he had therefore to introduce secondary vibrations in order to account for the observed data.

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[1] Above p. 147.

We shall now derive, on the basis of our modern knowledge of the numerical values for the eccentricities and inclinations, the dimensions which should be given to models of the type of Figs. 208 and 210 for each of the five planets. This, then, will provide the proper background for a comparison with the numerical data of Ptolemy's theory of latitudes.

A. The Outer Planets

The modern elements for A.D. 100 are[2]:

	Mars	Jupiter	Saturn
inclination of orbit i	1;52°	1;25°	2;33°
longit. of ascend. node Ω	34;54	81;25	97; 4
longit. of perihelium π	301; 8	344; 6	55; 5
eccentricity e_1	0.091	0.045	0.062
solar orbit	$A_0 = 70;24°$		$e_0 = 0.017$

On the basis of these data we computed in I C 1, 2[3] the resulting geocentric apsidal lines and eccentricities. Taking orbital inclinations into consideration, one has to remember, however, that e_0 and e_1 belong to different planes and that Ptolemy refers the resulting eccentricity e and the apsidal line $A\Pi$ to the ecliptic. Thus one has to project the point M_1' of Figs. 210 and 211 of the orbital plane into a point M_1'' of the ecliptic (Fig. 212) in order to obtain the equivalent of Ptolemy's arrangement. The resulting parameters which are represented to scale in Fig. 212 differ only insignificantly from the previous results.

Ptolemy, for the computation of latitudes, simplifies matters by assuming for the directions of the apsidal and nodal lines convenient round numbers (cf. Fig. 213):

	♂	♃	♄
A accurate	♋ 25;30	♍ 11;0	♏ 23;0
A rounded	♌ 0	♍ 10	♏ 20
Ω	♉ 0	♋ 0	♋ 0
N	♌ 0	♎ 0	♎ 0
ω_A	0	−20	+50

The "accurate" positions mentioned first are the values which Ptolemy had found for his own time (A.D. 139 and 136[4]). The modern values for A.D. 100, would be:

	♂	♃	♄
A	♋ 23;45	♌ 23; 0	♏ 19;30
Ω	♉ 4;54	♊ 21;25	♋ 7; 3
N	♌ 4;54	♍ 21;25	♎ 7; 3
ω_A	−11; 9	−28;25	+42;27

[2] Cf. Appendix VI B 7, 2.

[3] Above p. 147.

[4] Cf. p. 179 (6).

To these deviations is to be added the fact that Ptolemy incorrectly assumes that the nodal lines go through O (cf. Fig. 212 for the actual positions).

In the computation of latitudes Ptolemy introduces two types of inclinations: i_0, the angle between ecliptic and deferent (ἔγκλισις), and i_1 (λόξωσις) between the plane of the deferent and of the epicycle when the latter is at its extremal latitude. Ptolemy's values in comparison with the modern orbital inclinations are as follows:

		♂	♃	♄
Almag.	i_0	1°	1;30°	2;30°
	i_1	2;15	2;30	4;30
modern	i	1;50	1;25	2;33

In the case of Jupiter and Saturn the inclination of the plane of the deferent is practically correct, $i_0 \approx i$, while the planes of the epicycle are inclined too much. In the case of Mars i_0 is too small, i_1 too large. Before discussing Ptolemy's methods of determining these quantities it will be useful to give a scale drawing of the dimensions of Ptolemy's model for the outer planets (Fig. 214). This figure will make it evident how difficult it must be to establish the correct inclinations from such small angles, to be measured by naked eye observations.

1. Mars. Ptolemy's parameters are based on observations of latitudes of Mars at oppositions when they occur at (or near to) the apogee A or the perigee Π of the eccenter, i.e. at longitudes near 120° or 300°. At opposition the planet is at the perigee of the epicycle, thus showing extremal geocentric latitudes. Ptolemy finds in this way (cf. Fig. 215)

$$\beta_1 = +4;20° \qquad \beta_2 = -7;0° \tag{1}$$

values which agree well with the facts (cf., e.g., the graph Fig. 228b, p. 1283). He furthermore assumes (Fig. 215):

a) the observer O is located in the plane of the deferent

b) the planes of the epicycle in A and Π are parallel

c) the apsidal line AΠ is perpendicular to the nodal line.

Looking at Fig. 215 from a purely geometrical viewpoint we can say that θ_1 and θ_2 are the equations of center which belong to an epicyclic anomaly $\alpha = 180 \pm i_1$, θ_1 under the condition of maximum distance of the epicycle from the observer, θ_2 at minimum distance. These equations are tabulated in Alm. XI, 11 in the form $c_6(\alpha) - c_5(\alpha)$ and $c_6(\alpha) + c_7(\alpha)$, respectively.[5] For small deviations from $\alpha = 180$ one may assume that the equation θ changes proportionally to this deviation. In the tables the nearest value to $\alpha = 180$ is $\alpha = 183$ (or 177). Within this range we can assume that

$$\frac{\theta_1}{\theta_2} = \frac{c_6(183) - c_5(183)}{c_6(183) + c_7(183)}$$

[5] Cf. above p. 184.

or, using the tabulated values

$$\frac{\theta_1}{\theta_2}=\frac{5;45-1;16}{5;45+2;20}=\frac{4;29}{8;5}\approx\frac{5}{9}. \tag{2}$$

On the other hand it follows from Fig. 215 that

$$\begin{aligned}\theta_1+i_0&=\beta_1\\ \theta_2+i_0&=|\beta_2|\end{aligned} \tag{3}$$

thus with (1)

$$\theta_2-\theta_1=|\beta_2|-\beta_1=\Delta\beta=2;40. \tag{4}$$

From (2) and (4) one finds

$$\theta_1=3;20 \qquad \theta_2=6$$

and thus from (3) and (1) the inclination of the plane of the deferent

$$i_0=\beta_1-\theta_1=1^\circ. \tag{5}$$

Using once more the argument that small deviations of the epicyclic anomaly from 180 will result in proportional equations of center we can say that

$$\frac{i_1}{\theta_1}=\frac{3}{c_6(183)-c_5(183)}=\frac{3}{4;29}\approx\frac{1}{1;30}. \tag{6}$$

With $\theta_1=3;20$ already found we obtain for the inclination of the plane of the epicycle toward the plane of the deferent

$$i_1=\frac{3;20}{1;30}=2;13,20\approx 2;15. \tag{7}$$

In this way Ptolemy obtained the essential parameters i_0 and i_1 for the computation of the latitudes of Mars.

We know today that one should have only one angle; namely

$$i_0=i_1=1;50^\circ. \tag{8}$$

We also know that O does not belong to the plane of the deferent and that therefore Ptolemy's model of Fig. 215 is not correct for an observer on the earth but only for an ideal observer O* located in the nodal line. We can invert Ptolemy's procedure in order to compute what an observer O* would see. From (8) and (6), (5) it follows that

$$\theta_1=1;30\,i_1=2;45 \qquad \beta_1=i_0+\theta_1=4;35$$

and similarly from (2) and (3)

$$\theta_2=9/5\,\theta_1=4;57 \qquad |\beta_2|=i_0+\theta_2=6;47.$$

This shows that Ptolemy's observations of β_1 are about 0;15° smaller than if he had really observed from the deferent plane and consequently β_2 appeared about 0;13° larger. In other words his lowering of i_0 and his increase of i_1 is caused by the fact that the earth is located nearer to the perigee than the nodal line, exactly as we have found from our investigation of the combined eccentricities of earth and Mars (cf. Fig. 212, p. 1274).

If one computes with the data of Fig. 212 one finds for the distance of O from the nodal line a value of about 0.01, i.e. 0;36 for $R=60$. If, on the other hand, one asks how far one should move O toward A such that β_1 increases by about 0;15 while β_2 decreases by the same amount one finds that a displacement of only about 0;12 is required. These values are very sensitive toward very small changes of the angles involved such that the observed data could easily be modified to result in good agreement with the expected distance of O from the nodal line. Thus one can formulate our result by saying that Ptolemy's assumption of different inclinations between ecliptic and deferent and deferent and epicycle is caused by the fact that he did not realize that the plane of the deferent does not go through the earth because of the combined eccentricities of earth and planet.

2. Jupiter and Saturn. The smallness of the epicycle radii of the two outermost planets and the smallness of their eccentricities excludes a successful repetition of the procedure followed in the case of Mars. The difference $\Delta\beta$ between the latitudes at oppositions near opposing ends of the apsidal line is no longer observable (cf. Fig. 214, p. 1276). Consequently oppositions and conjunctions either near the apogee or near the perigee have to be combined and since conjunctions themselves are not observable their latitude has to be deduced from the nearest first and last visibility of the planet.

Ptolemy reports the following results of these observations (cf. Fig. 216) of extremal latitudes:

$$\begin{array}{lll} \text{Saturn:} & \beta_1=\pm 3^\circ & \beta_2=\pm 2^\circ \\ \text{Jupiter:} & \beta_1=\pm 2^\circ & \beta_2=\pm 1^\circ. \end{array} \tag{1}$$

On the basis of these data he deduces the values of the angles of inclination i_0 and i_1, respectively by a procedure that follows the same principle as in the case of Mars with the only difference that the equations θ can now be taken simply for mean distances, i.e. only by using the coefficients $c_6(\alpha)$ for anomalies α near 0° and 180°, i.e. in the smallest tabulated interval near apogee and perigee of the epicycle. In this way we find from Alm. XI, 11

for Saturn:
$$\frac{\theta_1}{\theta_2}=\frac{c_6(186)}{c_6(6)}=\frac{0;45}{0;36}\approx\frac{23}{18}$$
$$\theta_1+\theta_2=\beta_1-\beta_2=1$$
$$\text{thus } \theta_1=23/41\approx 0;34 \qquad \theta_2=18/41\approx 0;26$$
$$\text{hence } i_0=\beta_1-\theta_1=2;26,$$

for Jupiter:
$$\frac{\theta_1}{\theta_2}=\frac{c_6(186)}{c_6(6)}=\frac{1;25}{0;58}\approx\frac{43}{29}$$
$$\theta_1+\theta_2=\beta_1-\beta_2=1$$
$$\text{thus } \theta_1=\frac{43}{1,12}\approx 0;36 \qquad \theta_2=\frac{29}{1,12}\approx 0;24$$
$$\text{hence } i_0=\beta_1-\theta_1=1;24.$$

Similarly for the fixed ratio of anomaly and equation $\frac{i_1}{\theta_1} = \frac{3}{c_6(183)}$:

for Saturn: $i_1 = \frac{3 \cdot 0;34}{0;23} \approx 4;30$

for Jupiter: $i_1 = \frac{3 \cdot 0;36}{0;43} \approx 2;30.$

These results compare as follows with the correct values

	i_0	i_1	i
Saturn	2;26	4;30	2;33
Jupiter	1;24	2;30	1;25

The main cause of errors obviously lies in the strong roundings of the observable extremal latitudes (1), leading for both planets to the difference $\beta_1 - \beta_2 = 1°$. A glance at Figs. 155/156 (below p. 1255f.) shows that observations should have given a much smaller variation, nearer 0;40 and 0;30, respectively. This is fully confirmed if one inverts Ptolemy's procedure by computing the values of β_1 and β_2 which would lead to the correct result $i_0 = i_1 = i$. One finds

for Saturn: $\beta_1 \approx 2;55 \quad \beta_2 \approx 2;15 \quad \beta_1 - \beta_2 \approx 0;40$

for Jupiter: $\beta_1 \approx 1;45 \quad \beta_2 \approx 1;10 \quad \beta_1 - \beta_2 \approx 0;35.$

Hence it is only the crudeness of the data used which caused Ptolemy to assume, in the tables of the Almagest, inclinations too great for the epicycles of Saturn and Jupiter. This error was corrected in the "Planetary Hypotheses"[6] in which i_0 and i_1 are made equal, 2;30 for Saturn, 1;30 for Jupiter, i.e. practically the correct values.

B. The Inner Planets

The following modern data are needed[7]

	Venus	Mercury
inclination of orbit i	3;23°	6;58°
longit. of ascend. node Ω	59;43	25;52
longit. of perihelium π	104;49	48; 0
eccentricity e_1	0.008	0.205
solar orbit	$A_0 = 70;24°$	$e_0 = 0.017$

[6] Cf. below p. 908.

[7] Cf. Appendix VI B 7, 2.

Taking the inclinations of the orbital planes in consideration and using as eccentricities of the circular orbits the elliptic values[8] one finds the configurations drawn to scale in Fig. 217 and the following numerical values:

	Venus	Mercury
resultant eccentricity e ($R=60$)	0;40	11;21
resultant apogee A	♉17	♏16
☊	♉29;43	♈25;52
angle from A to ☊	+12;43	+159;52

Ptolemy, for reasons to be mentioned presently, introduces nodal lines of the deferents perpendicular to the apsidal lines. The epicycles, however, intersect the deferent planes in directions parallel to the apsidal lines. For the sake of comparison with the above given data we therefore have to introduce "nodes" in a direction perpendicular to Ptolemy's nodal line. These nodes are, of course, the nodes of the heliocentric theory. In this way we obtain (cf. Fig. 218) the following data for Ptolemy:

	Venus	Mercury
eccentricity e ($R=60$)	1;15	9;0
apogee A	♉25	♎10
[☊]	♉25	♈10
angle from A to [☊]	0	180

One can also express this in the form that Ptolemy makes the directions of the actual nodal lines of Venus and Mercury parallel to their apsidal lines, whereas in fact they deviate by about 13° and 160°, respectively.

We now have to discuss the reason for the direction given by Ptolemy to the nodal line of the deferent plane. According to the simple situation depicted in Fig. 206, p. 1272 no such nodal line would exist since the plane of the ecliptic and of the deferent coincide. We have seen, however, that eccentricities in different planes, e_0 in the ecliptic, e_1 in the orbital plane, have the effect of placing the plane of the deferent into a position parallel to the ecliptic at a fixed distance $M_1'M_1''$ (Fig. 210, p. 1273 and Fig. 217, p. 1277). An observer located in the ecliptic should therefore ascribe to the center of the epicycle a latitude corresponding to the distance $M_1'M_1''$ of the deferent from the ecliptic. Ptolemy observed this fixed latitude i_0 of the deferent when the epicycle was in apogee or in perigee. Since the nodal line of the epicycle practically coincides with the apsidal line the epicycle is then seen edge on (cf. Fig. 219a and c) and the latitude of the center of the epicycle is deducible from the latitude of the points of greatest elongation. Ptolemy thus came to the correct result that in A as well as in Π the center C of the epicycle has the same latitude i_0 (+0;10° for Venus, −0;45° for Mercury). In the direction orthogonal to the apsidal line, however, he did not detect an

[8] Case (1) of p. 147 in I C 1, 2.

appreciable latitude of C. In this situation the epicycle is seen frontally (Fig. 219b); maximum elongations now produce equal latitudes which are very small whereas conjunctions are not observable directly and would lead, even if observable, to inaccurate results because of their great difference in geocentric distance (cf. the scale drawings Fig. 220 and 221). Thus Ptolemy concluded that the plane of the deferent has a maximum inclination i_0 when C is in A or Π but coincides with the ecliptic when C is 90° distant from the apsidal line. This means that he ascribed to the plane of the deferent a flapping motion about an axis perpendicular to AΠ. This axis is his "*nodal line*" for the deferent.

In principle it would be easy to check the correctness of Ptolemy's values for the "*inclination*" (ἔγκλισις) i_0 since it only depends on the value of $M_1'M_1''$ which is given by $e_1 \sin i \sin \omega$ where ω is the angle between apsidal line and nodal line in the orbital plane (thus $\omega = 44;47°$ for Venus and 22;8° for Mercury). Unfortunately it is doubtful which value for e_1 should be chosen. If one uses the simple elliptic eccentricity one finds that an observer in O would see C at a latitude of less than $-0;2°$ in the case of Venus, but for Mercury at $-0;27°$ for C at the apogee, at $-0;40$ at the perigee. These values increase, of course, with e_1. Ptolemy's values of $+0;10°$ and $-0;45°$ are probably influenced by refraction since he had to rely on observations of first and last visibilities in order to deduce the latitudes at inferior or superior conjunction. Such observations near the horizon would tend to increase the apparent latitudes.

When observational conditions are more favorable Ptolemy's results agree with modern values very well. For positions of C at A and Π (Fig. 219a and c, p. 1279) he found for the "*slant*" (λόξωσις) i_2 of the diameter em of the epicycle the values

$$\begin{aligned} &\text{for Venus:} && i_2 = 3;30° \\ &\text{for Mercury:} && i_2 = 7;0. \end{aligned} \tag{1}$$

The modern values are 3;23° and 6;58°, respectively.

At right angles to the apsidal line, where the observational conditions are more difficult, Ptolemy found for the "*inclination*" (ἔγκλισις) i_1 of the diameter cs (Fig. 219b) values about 1° too small:

$$\begin{aligned} &\text{for Venus:} && i_1 = 2;30° \\ &\text{for Mercury:} && i_1 = 6;15. \end{aligned} \tag{2}$$

He should have found the same values as for i_2. We shall see that he improved these data to almost complete correctness in the "Planetary Hypotheses".[9]

1. Determination of i_1 and i_2. Ptolemy does not decribe the individual observations from which he derived the parameters for his models of planetary latitudes. He only gives final figures, thus making it practically impossible to isolate the different sources of errors.

For the position of the epicycle of Venus and Mercury at apogee or at perigee he found a latitudinal difference $\Delta\beta$ between the two endpoints e and m (cf.

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[9] Cf. below p. 909.

Fig. 222)

$$\begin{array}{lll} \text{Venus} & \begin{cases} \text{C at apogee} \\ \text{C at perigee} \end{cases} & \Delta\beta = \text{slightly} \begin{Bmatrix} \text{less} \\ \text{more} \end{Bmatrix} \text{than } 5^\circ \\ \text{Mercury} & \begin{cases} \text{C at apogee} \\ \text{C at perigee} \end{cases} & \Delta\beta = \begin{cases} 4;30^\circ \\ 5;30^\circ. \end{cases} \end{array} \tag{1}$$

He concludes that for both planets the endpoints of the diameter (or chord) em appear at an angle of about $\pm 2;30^\circ$ from the plane of the deferent.

On the basis of this observation Ptolemy is able to determine the "slant" i_2 of the epicycle with respect to the plane of the deferent (cf. Fig. 223). Assume for OC a mean value, i.e. $\mathrm{OC} = R = 60$ for Venus and $\mathrm{OC} = 1/2(R+3e+R-e) = R+e = 63$ for Mercury. If $\beta'_0 = 1/2\,\Delta\beta = 2;30^\circ$ is the distance of the planet at maximum elongation from the plane of the deferent it follows from Fig. 223 that

$$\frac{\rho}{\mathrm{OC}} = \frac{\mathrm{GH}}{r} \quad \text{and} \quad \mathrm{GG}' = \rho \sin \beta'_0$$

and thus

$$\sin i_2 = \frac{\mathrm{GG}'}{\mathrm{GH}} = \frac{\sin \beta'_0 \cdot \mathrm{OC}}{r}.$$

Substituting on the right-hand side the above given values for β'_0 and for OC and using

$$r = 43;10 \text{ for Venus,} \qquad r = 22;30 \text{ for Mercury}$$

one finds $i_2 = 3;29$ and $7;1$, respectively, rounded to $3;30$ and $7;0$ by Ptolemy.

In order to find i_1 one observes the planet when, according to Ptolemy's model, the center of the epicycle lies in the nodal line of the deferent, i.e. at right angles to the apsidal line. From observations of first and last visibilities Ptolemy then derived for the endpoints of the diameter cs (cf. Fig. 224) the following latitudes:

$$\begin{array}{lll} \text{Venus:} & \beta_s = \pm 1^\circ & \beta_c = \pm 6;20^\circ \\ \text{Mercury:} & \beta_s = \pm 1;45 & \beta_c = \pm 4. \end{array} \tag{2}$$

Since $\mathrm{sC} = \mathrm{Cc} = r$ is the radius of the epicycle we can consider β_s and β_c as the equations of center θ caused by an epicyclic anomaly of the amount $|i_1|$ in the case of s and $|180 - i_1|$ in the case of c.

Venus in this position is practically at mean distance from O, thus the equation is given by $c_6(\alpha)$ in the tables of XI, 11. For small angles the ratio $\alpha/c_6(\alpha)$ should be constant. Thus we find from $c_6(6) = 2;31$ and $c_6(177) = 7;38$ given in XI, 11

$$i_1 = |\beta_s| \frac{6}{2;31} \approx 2;23$$

and from $i_1 = |\beta_c| \dfrac{3}{7;38}$

$$i_1 \approx 2;21.$$

Ptolemy rounds these results to

$$i_1 = 2;30. \tag{3}$$

For Mercury one cannot operate that simply because at 90° from the apsidal line OA the center of the epicycle is not at all near mean distance from O. If $\kappa=90$ one finds from Alm. XI, 11 that $\bar{\kappa}\approx 93$. Consequently the equation is given by

$$\theta=c_6(\alpha)+c_8(\bar{\kappa})c_7(\alpha)$$

where $c_8(\bar{\kappa})=c_8(93)=0;43,30$. Thus we have to find anomalies $\alpha_1(=-i_1)$ and $\alpha_2(=180-i_1)$ such that

$$c_6(\alpha_1)+0;43,30\cdot c_7(\alpha_1)=|\beta_s|=1;45$$

$$c_6(\alpha_2)+0;43,30\cdot c_7(\alpha_2)=|\beta_c|=4.$$

By interpolation in the tables of Alm. XI, 11 one finds that the first equation is approximately satisfied by $|\alpha_1|=6;11$, the second by $\alpha_2=180-6;8$. Hence $i_1\approx 6;10$. Ptolemy uses

$$i_1=6;15 \tag{4}$$

and he motivates this value by saying that the corresponding equations are $\theta_1=1;46$ and $\theta_2=4;5$, thus sufficiently close to $\beta_s=1;45$ and $\beta_c=4$. Indeed one finds for $\alpha=93$ from Alm. XI, 11 the equations $\theta_1=1;45,52$ and $\theta_2=4;4,53$ which confirms Ptolemy's statement.

3. The Tables Alm. XIII, 5

In the preceding discussion we have seen how Ptolemy developed from empirical data the features which a theory of planetary latitudes had to account for. For the outer planets this is relatively simple: the plane of the deferent maintains a fixed inclination i_0 to the ecliptic while the plane of the epicycle moves almost parallel to itself. When the center of the epicycle is located in the nodal line of the deferent then the plane of the epicycle coincides with the ecliptic.[1] Parallelism in the motion of the plane of the epicycle would mean that the angle i_1 between epicycle and deferent would be equal to i_0 when the center of the epicycle has reached the points of extremal latitude of the deferent. In fact, however, Ptolemy, when he wrote the Almagest, had found an angle $i_1>i_0$. Consequently the plane of the epicycle is made to perform a small sinusoidal motion about the diameter which is parallel to the nodal line of the deferent.

The case of the inner planets is complicated by a motion of the plane of the deferent as function of the location of the center C of the epicycle. The plane of the deferent has its full inclination $i_0(+0;10°$ for Venus, $-0;45°$ for Mercury) when C is 90° distant from the nodal line of the deferent plane. (This nodal line is perpendicular to the apsidal line of the deferent.) As C moves toward the nodes the inclination of the deferent plane decreases to zero, only to increase again to its previous value ($+0;10°$ and $-0;45°$, respectively) as C progresses to a position 90° away from the node on the other side of the nodal line.

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[1] Cf. Fig. 207, p. 1272, moving C into A or B.

The motion of the plane of the epicycle is again not quite parallel to itself. Consequently one has to define the variation of two angles, i_1 and i_2, of two perpendicular diameters of the epicycle (cf. Fig. 219, p. 1279) which vary periodically with the position of C on the deferent.

In Chap. 2 of Book XIII Ptolemy makes a feeble attempt to describe a mechanical device which would cause all planes to move according to his model, assuming that all motions vary sinusoidally between the proper extrema. He does not give any detailed construction for such a device but it seems clear that all he meant to say was that in principle sinusoidal variations of the inclination of a plane could be obtained by the rotation of a vertical disk which leads on its circumference a point of the plane up and down.

This is the only instance in the whole Almagest where something like a mechanical model for the planetary (or lunar) motion is mentioned. And even here it concerns only one component in the latitudinal motion and it is obvious that Ptolemy does not think that any such mechanism actually exists in nature. Neither here nor anywhere else in the Almagest can we find a physical hypothesis like spherical shells driven by contacting spheres, etc., constructions which later on became a favored topic of cosmological descriptions.

The chapter concludes with a delightful little dissertation about the concept of "simplicity" where Ptolemy counters the argument that his theory of planetary latitudes might appear "too complicated." He points out that human concepts cannot form a proper basis for deciding what is "simple" or not in the cosmos. Indeed the celestial bodies eternally display invariable motions while for man constancy is most difficult to achieve, if not impossible. Kepler translated this whole section in his Epitome Astronomiae Copernicanae[2] but remarked that Ptolemy himself preferred uniform circular motion "*quae nobis, hominibus de terra, videbuntur simpliciores.*"

In fact the computation of the latitudes is quite simple, the structure and the parameters of the planetary model once being given. For the outer planets are tabulated functions $c_3(\alpha)$ and $c_4(\alpha)$ which represent the latitude of a planet of known anomaly α under the assumption that the center C of the epicycle is located at the points of extremal northern or southern latitude of the deferent, respectively. For intermediate positions of C trigonometric interpolation by means of a coefficient $c_5(\omega)=|\cos\omega|$ is used, ω being the argument of latitude reckoned from the northernmost point of the deferent.

Similarly for the inner planets: $c_3(\alpha)$ is the latitude of the planet if C is in the nodal line of the deferent, $c_4(\alpha)$ assumes C to be in the direction orthogonal to the nodal line, hence either in the apogee or perigee of the deferent, i.e. when the deferent reaches its extremal inclination to the ecliptic. For general positions of C both c_3 and c_4 are modified sinusoidally such that their influence becomes zero perpendicular to the nodal line or to the apsidal line, respectively. This is achieved by means of the same coefficients $c_5(\omega)$ which are used with the outer planets. Finally a small correction, proportional to c_5^2, accounts for the variable inclination of the plane of the deferent.

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[2] Werke 7, p. 291 f.

A. Outer Planets

The parameters adopted by Ptolemy for the computation of the tables are

	Saturn	Jupiter	Mars
i_0	2;30°	1;30°	1; 0°
i_1	4;30	2;30	2;15
$\lambda_{\bar{N}}$	♎ 0	♎ 0	♌ 0
λ_A	♏ 20	♍ 10	♌ 0
ω_A	50°	−20°	0°
r	6;30	11;30	39;30
e	3;25	2;45	6; 0

A scale drawing is given in Fig. 213, p. 1275.

1. Determination of $c_3(\alpha)$ and $c_4(\alpha)$. We first describe the procedure which leads to the determination of the latitude of a planet of (equated) epicyclic anomaly α under the condition that the center C of the epicycle is either at the point N of maximum northern latitude of the deferent or at point diametrically opposite to N. The results are tabulated in Alm. XIII, 5 as $c_3(\alpha)$ and $c_4(\alpha)$ respectively.

Fig. 225 shows that the distance $\rho' = \mathrm{ON}$ is approximately given by

$$\rho' \approx R + e \cos \omega_A \qquad (R = \mathrm{MA} = 60) \tag{1}$$

and therefore (cf. Fig. 226a) the distance $\rho = \mathrm{OP}$ by

$$\rho = \sqrt{(\rho' + r \cos \alpha)^2 + (r \sin \alpha)^2}$$

assuming that the small inclination i_1 between epicycle and deferent has no appreciable influence on the distance of the planet P from O.

Since all inclinations are small we can identify sines with angles and verticals to the deferent with verticals to the ecliptic. Thus we have for the distance Q′Q″ (Fig. 226b)

$$\mathrm{Q'Q''} \approx (\rho' + r \cos \alpha) i_0.$$

If PP′ is the distance of the planet P from the plane of the deferent, PP″ the distance of P from the ecliptic, then PP″ is seen from O under the angle $\beta = c_3(\alpha)$ which we wish to determine; thus

$$\mathrm{PP''} = \rho \beta.$$

But $\mathrm{PP'} = \mathrm{QQ'} = r \cos \alpha \cdot i_1$ and $\mathrm{P'P''} = \mathrm{Q'Q''}$ found before. Thus

$$\mathrm{P'P''} = \mathrm{P'P} + \mathrm{PP''} = (\rho' + r \cos \alpha) i_0 = r \cos \alpha \cdot i_1 + \rho \beta$$

and hence

$$\beta = c_3(\alpha) = \frac{1}{\rho}\left((i_0 - i_1) r \cos \alpha + \rho' i_0\right). \tag{2}$$

Similarly for the southernmost point of the deferent. If we call

$$\begin{aligned} \rho'' &\approx R - e \cos \omega_A \\ \rho &= \sqrt{(\rho'' + r \cos \alpha)^2 + (r \sin \alpha)^2} \end{aligned} \tag{3}$$

then we find for the latitude of the planet

$$-\beta=c_4(\alpha)=\frac{1}{\rho}\left((i_0-i_1)r\cos\alpha+\rho'' i_0\right). \tag{4}$$

A numerical check of the values thus obtained from (1) and (4) shows excellent agreement with the values of $c_3(\alpha)$ and $c_4(\alpha)$ tabulated in Alm. XIII, 5, in particular for characteristic values, e.g. for $\alpha=0$, 90, and 180. But, as the graph in Fig. 227 shows, linear interpolation has been used for certain sections and in particular near the maxima in the case of Mars where the curve should end in a horizontal tangent. The resulting error is, however, negligible excepting c_4 where it can reach 0;20°.

2. Determination of $c_5(\omega)$. The tables in Alm. XIII, 5 give one more function which we denote as $c_5(\omega)$ where ω represents the "*argument of latitude*" of C, counted from the point N of maximum northern latitude of the deferent (cf. Fig. 213, p. 1275). Consequently

$$\begin{aligned} &\omega=\kappa_0 && \text{for Mars}\\ &\omega=\kappa_0-20° && \text{for Jupiter}\\ &\omega=\kappa_0+50° && \text{for Saturn}\end{aligned} \tag{5}$$

κ_0 being the normed longitude of C, i.e. counted from A. The function $c_5(\omega)$ is the same for all three outer planets and it is easy to see that the values follow the law

$$c_5(\omega)=|\cos\omega| \tag{6}$$

more or less closely although actually derived from the table of lunar latitudes in Alm. V, 8 by using a factor 0;12 in order to reduce the maximum from 5 to 1.[3]

3. Computation with the Tables Alm. XIII, 5. In order to compute the latitude β of an outer planet for a given moment t we consider to be known

the equated anomaly α of the planet
the equated normed longitude κ_0 of the center C
of the epicycle.

From κ_0 we find according to (5) the argument of latitude of ω of C. Then we obtain from the tables $c_3(\alpha)$, $c_4(\alpha)$, $c_5(\omega)$ and form

$$\beta=\pm c_5(\omega)\cdot\begin{cases} c_3(\alpha) & \text{if } 270°\leqq\omega\leqq 90°\\ c_4(\alpha) & \text{if } \ 90°\leqq\omega\leqq 270°.\end{cases} \tag{7}$$

The result in the first case is the northern latitude, in the second the southern latitude of the planet.

Examples. In I C 5, 3 A we have computed an ephemeris for longitudes of Mars for an interval for the years Nabonassar 450/451. We now complete this ephemeris by computing the latitudes.

[3] In some cases the result is 0;0,12 less than expected.

We first take from the previous computations (p. 188, Table 15) the values of κ_0. At No. 4 $\kappa_0=\omega$ crosses the value 270° which indicates that the center of the epicycle is at the ascending node. For the first three dates for which $\kappa_0=\omega<270°$ we have to use in the Tables Alm. XIII, 5 the coefficients $c_4(\alpha)$ which are always negative, thereafter $c_3(\alpha)$ which are positive. The product of these coefficients with the fraction $c_5(\omega)$ gives the latitude of the planet (cf. Table 21).

Table 21

No.	Date: Nab.	Date: julian	$\kappa_0=\omega$	$c_5(\kappa_0)$	α	$c_4(\alpha)$	β: $c_4\cdot c_5$	β: modern	$\Delta=$ Alm. − modern	No.
	450	−297								
1	X 8	Aug. 10	253;40	0;16,44	129;46	−2; 3	−0;34	−1; 8	+0;34	1
2	18	20	259;15	0;11, 9	134; 3	−2;18	−0;26	0;59	0;34	2
3	28	30	264;44	0; 5,37	138;25	−2;34	−0;14	0;49	0;35	3
4	XI 8	Sept. 9	270; 8	0; 0, 8	142;53	+2;31	0; 0	0;37	0;37	4
5	18	19	275;26	0; 5,48	147;26	+2;45	+0;16	0;22	0;38	5
6	28	29	280;40	0;11, 4	152; 4	+3; 2	+0;34	−0; 4	+0;38	6
7	XII 8	Oct. 9	285;46	0;16,10	156;49	3;19	0;54	+0;16	0;38	7
8	18	19	290;45	0;21, 9	161;41	3;37	1;17	+0;40	0;37	8
9	28	29	295;41	0;25,58	166;37	3;55	1;42	+1; 6	0;36	9
10	I 3	Nov. 8	300;32	0;30,28	171;37	4;11	2; 7	+1;34	0;33	10
	451									
11	13	18	305;20	0;34,37	176;41	+4;18	+2;29	+1;59	+0;30	11
12	23	28	310; 2	0;38,26	181;49	4;19	2;46	2;20	0;26	12
13	II 3	Dec. 8	314;39	0;41,57	187; 4	4;13	2;57	2;34	0;23	13
14	13	18	319;14	0;45,13	192;20	3;59	3; 0	2;41	0;19	14
15	23	28	323;46	0;48,15	197;40	3;39	2;56	2;42	0;14	15
		−296								
16	III 3	Jan. 7	328;14	0;50,56	203; 3	+3;20	+2;50	+2;40	+0;10	16
17	13	17	332;39	0;53, 9	208;29	3; 0	2;39	2;36	+0; 3	17
18	23	27	337; 3	0;55, 1	213;57	2;41	2;28	2;31	−0; 3	18
19	IV 3	Febr. 6	341;25	0;56,46	219;26	2;23	2;15	2;25	−0;10	19
20	13	16	345;45	0;58, 0	224;58	2; 9	2; 5	2;19	−0;14	20
21	23	26	350; 3	0;58,56	230;31	+1;54	+1;54	+2;13	−0;20	21
22	V 3	March 7	354;22	0;59,37	236; 4	1;43	1;42	2; 7	0;25	22
						$c_3(\alpha)$	$c_3\cdot c_5$			

Fig. 228a gives the position of Mars in longitude and latitude and the corresponding modern positions.[4] Two other types of loops are represented in Fig. 228b where the epicycle is located near the southernmost point of the deferent (and the perigee at $\lambda=300$) for Nabonassar 493 VII 3 to Epag. 3 and in Fig. 228c near the ascending node ($\lambda=30$) for Nabonassar 512 X 8 to 513 IV 23. Here the shift from southern to northern latitude produces an S-curve which would be exactly symmetric if it crossed the ecliptic at $\lambda=30$. Fig. 228a corresponds to longitudes not much greater than in Fig. 228c but the S-curve has already

[4] In this figure, as in all our similar graphs, the scale of the latitude is twice the scale of the longitudes. The modern positions are taken from the Tuckerman Tables.

changed to a loop at northern latitude which will increase and become more symmetric as one approaches the point N at $\lambda = 120$. For similar curves in the case of Venus cf. below p. 1285, Fig. 233.

B. Inner Planets

The basic parameters are (cf. Fig. 218, p. 1278)

	Venus	Mercury
i_0	$+0;10°$	$-0;45°$
i_1	2;30	6;15
i_2	3;30	7; 0
λ_A	♉25	♎10
nodes	♒/♌25	♋/♑10
r	43;10	22;30
e	1;15	3 to 9

1. Determination of $c_3(\alpha)$. We assume that the center C of the epicycle lies in the nodal line and hence that the plane of the deferent coincides with the ecliptic. Then the function $c_3(\alpha)$ represents the latitude of the planet at the anomaly α (cf. Fig. 229 and Fig. 219b, p. 1279).

In order to compute $c_3(\alpha)$ we need only (Fig. 229)

$$\mathrm{PP}' = r \cos \eta \sin i_1,$$

and

$$\begin{aligned}\mathrm{OP}' &= \sqrt{\mathrm{OQ}'^2 + \mathrm{Q}'\mathrm{P}'^2}\\ &= \sqrt{(\mathrm{OC} - r \cos \eta \cos i_1)^2 + (r \sin \eta)^2}.\end{aligned}$$

In the case of Venus the distance OC, if C lies in the nodal line, is given by

$$\mathrm{OC} = \sqrt{R^2 - e^2} = 59;59,13$$

as is evident from Fig. 219b, p. 1279. In the case of Mercury, however, one has to determine OC for such a position of the movable center of the eccenter that $\kappa_0 = 90°$. The accurate solution requires a rather lengthy computation[5] which leads to OC = 56;37. Ptolemy gives 56;40 without telling how he found this result.

With these quantities known the whole table for $c_3(\alpha)$ can be computed (cf. the graph Fig. 230).

2. Determination of $c_4(\alpha)$. We now assume a position of the center of the epicycle at the point of extremal latitude of the deferent (cf. Fig. 219a and c, p. 1279) while the planet has the epicyclic anomaly α. In this situation the epicycle is seen edge on and the planet appears at an angular distance β' from the plane of the deferent (cf. Fig. 231). The planet P might be seen at any point between the points F and G of maximum elongation. Their distance r_0' from C is seen from O as the maximum of the epicyclic equation and hence given in the tables

[5] One finds a cubic equation for the sine of the angle under which the eccentricity $e = 3$ is seen from C.

of anomaly Alm. X, 11[6]

$$r'_0 = \max c_6 = \begin{cases} 45;57 & \text{for Venus} \\ 22;\ 2 & \text{for Mercury.} \end{cases} \tag{1}$$

Similarly for the intermediate position, corresponding to an arbitrary value of the anomaly α, one finds the distance $PC = r'$ tabulated in Alm. XI, 11, as the epicyclic equation $c_6(\alpha)$. Thus it follows from Fig. 231 that

$$\beta'(\alpha) = \beta'_0 \frac{r'(\alpha)}{r'_0} = \beta'_0 \frac{c_6(\alpha)}{\max c_6}$$

where the values of $c_6(\alpha)$ are known from Alm. XI, 11 assuming C at mean distance from O. Also β'_0 is known; we mentioned on p. 215 Ptolemy's deductions from observations according to which

$$\beta'_0 \text{ at} \begin{Bmatrix} \text{apogee} \\ \text{perigee} \end{Bmatrix} \text{of deferent: } 2;30^\circ \text{ for Venus,} \tag{2a}$$

$$\beta'_0 \text{ at} \begin{Bmatrix} \text{apogee} \\ \text{perigee} \end{Bmatrix} \text{of deferent: } \begin{Bmatrix} 2;30 - 0;15 = \dfrac{9}{10} \cdot 2;30 \\ \\ 2;30 + 0;15 = \dfrac{11}{10} \cdot 2;30 \end{Bmatrix} \text{for Mercury.} \tag{2b}$$

For the moment we ignore the fact that β'_0 for Mercury has slightly different values at apogee and perigee, i.e. we operate at first as if C for Mercury also were at mean distance. Then we have, using (1) and $c_6(\alpha)$ from Alm. XI, 11

$$c_4(\alpha) = 2;30 \frac{c_6(\alpha)}{\max c_6} = c_6(\alpha) \cdot \begin{cases} 0;3,15 & \text{for Venus} \\ 0;6,48 & \text{for Mercury.} \end{cases} \tag{3}$$

The following table shows the excellent agreement between the results found from (3) and the values tabulated as $c_4(\alpha)$ in Alm. XIII, 5.

	Venus			Mercury		
α	XI, 11 $c_6(\alpha)$	$0;3,15 \cdot c_6$	XIII, 5 $c_4(\alpha)$	XI, 11 $c_6(\alpha)$	$0;6,50 \cdot c_6$	XIII, 5 $c_4(\alpha)$
30	12;30	0;40,38	0;41	8; 4	0;55, 7	0;55
60	24;38	1;20, 4	1;20	15;18	1;44,33	1;44
90	35;44	1;56, 8	1;57	20;33	2;20,26	2;20
120	44;12	2;23,39	2;24	21;47	2;28,51	2;29
150	43;39	2;21,52	2;22	15;31	1;46, 2	1;46

It follows from this derivation that $c_4(\alpha)$ represents for Venus the distance β' of the planet from the plane of the deferent at both ends of the apsidal line whereas this distance for Mercury will be $9/10\, c_4(\alpha)$ if C is at the apogee, and $11/10\, c_4(\alpha)$ at the perigee.

3. Computation with the Tables Alm. XIII, 5. The latitude β of an inner planet is the sum of three components, β_1, β_2, and β_3. The first two are modifications of $c_3(\alpha)$ and $c_4(\alpha)$, respectively, due to a general position of the

[6] Cf. Figs. 188 and 189 (p. 1265 f.).

center of the epicycle. Finally β_3 is the small contribution due to the inclination of the plane of the deferent. For all transitions from a special to a general position of the epicycle trigonometric interpolation is used by means of the same function $c_5(\omega)$ as with the outer planets. In Ptolemy's tables no signs are associated with the coefficients c_5. Hence we can only say that for any argument

$$c_5(x)=|\cos x|. \tag{1}$$

For a position of the planet at anomaly α the function $c_3(\alpha)$ represents the latitude of the planet under the further assumption that the normed longitude κ_0 of the center of the epicycle is either 90° or 270°. Under this condition the epicycle shows the full amount of the inclination i_1 of the diameter cs (cf. Fig. 219b). At $\kappa_0=0°$ or 180°, however, this diameter lies in the plane of the deferent (Figs. 219a and c) and its influence on the latitude is zero. Consequently Ptolemy uses for intermediate positions the formula

$$\beta_1(\alpha,\kappa_0)=\pm c_5(\kappa_0')\cdot c_3(\alpha) \tag{2a}$$

where the argument for c_5 is given by

$$\kappa_0'=\kappa_0+\begin{cases}90 & \text{for Venus}\\ 270 & \text{for Mercury}\end{cases} \tag{2b}$$

with the following rules of signs:

$$\begin{aligned} c_5(\kappa_0')&\begin{cases}>0 & \text{if}\quad 270\leqq\kappa_0'\leqq 90\\ <0 & \text{if}\quad 90\leqq\kappa_0'\leqq 270\end{cases}\\ c_3(\alpha)&\begin{cases}>0 & \text{if}\quad 90\leqq\alpha\leqq 270\\ <0 & \text{if}\quad 270\leqq\alpha\leqq 90.\end{cases}\end{aligned} \tag{2c}$$

The definition (2b) has the effect that the epicycles of Venus and Mercury tilt in opposite directions with respect to the ecliptic.

In a similar fashion β_2 takes into account the "slant" of the epicycle which reaches its full amount when the center of the epicycle lies in the apsidal line, i.e. for $\kappa_0=0$ and 180 (Figs. 219a and c). Here, however, we have to distinguish between Venus and Mercury insofar as $c_4(\alpha)$ applies directly only in the case of Venus for apogee A and perigee Π alike while one has to deal with A and Π separately in the case of Mercury.[7]

For Venus Ptolemy defines

$$\beta_2(\alpha,\kappa_0)=\pm c_5(\kappa_0'')\cdot c_4(\alpha) \tag{3a}$$

with

$$\kappa_0''=\kappa_0 \tag{3b}$$

as argument of c_5. The signs are defined by

$$\begin{aligned} c_5(\kappa_0'')&\begin{cases}>0 & \text{if}\quad 270\leqq\kappa_0''\leqq 90\\ <0 & \text{if}\quad 90\leqq\kappa_0''\leqq 270\end{cases}\\ c_4(\alpha)&\begin{cases}>0 & \text{if}\quad 0\leqq\alpha\leqq 180\\ <0 & \text{if}\quad 180\leqq\alpha\leqq 360.\end{cases}\end{aligned} \tag{3c}$$

[7] Cf. above p. 222, formula (2).

In the case of Mercury

$$\beta_2(\alpha, \kappa_0) = \pm c_5(\kappa_0'') \cdot c_4'(\alpha) \tag{4a}$$

with

$$\kappa_0'' = \kappa_0 + 180 \tag{4b}$$

and

$$c_4'(\alpha) = \begin{cases} 9/10\, c_4(\alpha) & \text{if} \quad 270 \leqq \kappa_0 \leqq 90 \\ 11/10\, c_4(\alpha) & \text{if} \quad 90 \leqq \kappa_0 \leqq 270. \end{cases} \tag{4c}$$

The rules of signs for (4a) are again given by (3c), of course with κ_0'' defined by (4b).[8]

Both components β_1 and β_2 concern the distances of the planet from the plane of the deferent. This plane itself is inclined at an angle i_0 with respect to the ecliptic, $+0;10°$ at $\kappa_0 = 0°$ and $\kappa_0 = 180°$ for Venus and $-0;45°$ at the same points for Mercury. For $\kappa_0 = \pm 90°$ deferent and ecliptic coincide, thus yielding no latitudinal contribution. Hence we can say that the inclination β_0 of the deferent with respect to the ecliptic is a function of κ_0 of the form

$$\beta_0(\kappa_0) = \begin{Bmatrix} +0;10 \\ -0;45 \end{Bmatrix} c_5(\kappa_0) \quad \begin{matrix} \text{for Venus} \\ \text{for Mercury} \end{matrix} \tag{4}$$

with c_5 as tabulated according to (1) since $\beta_0(\kappa_0)$ does not change its sign at $\kappa_0 = \pm 90°$.

If now, for an arbitrary value of κ_0, the center C of the epicycle appears at a latitude $\beta_3(\kappa_0)$ with respect to the ecliptic we have (cf. Fig. 232)

$$\mathrm{CC}' \approx \mathrm{CD} \cdot \beta_0(\kappa_0) \approx R\beta_3(\kappa_0)$$

thus

$$\beta_3(\kappa_0) \approx \frac{\mathrm{CD}}{R}\beta_0(\kappa_0) = |\cos\kappa_0| \cdot \beta_0(\kappa_0) = c_5(\kappa_0) \cdot \beta_0(\kappa_0)$$

and therefore

$$\beta_3(\kappa_0) = \begin{Bmatrix} +0;10 \\ -0;45 \end{Bmatrix} c_5(\kappa_0)^2 \quad \begin{matrix} \text{for Venus} \\ \text{for Mercury.} \end{matrix} \tag{5}$$

Ignoring the small angle between normals to the ecliptic and to the deferent we finally have for the latitude of an inner planet

$$\beta = \beta_1(\alpha, \kappa_0) + \beta_2(\alpha, \kappa_0) + \beta_3(\kappa_0) \tag{6}$$

using (2), (3), and (5) for the single components.

Examples. Table 22 shows the computations for the latitude of Venus during 80 days in Nabonassar 442/443 in 5-day intervals, continuing the computations of longitudes in I C 5, 3 B Table 16 (p. 189) from which we take the values for the equated epicyclic anomaly α and for the normed true longitude κ_0 of the center of the epicycle. Since in the given interval α is near 180° the coefficients $c_3(\alpha)$ are all positive (cf. Fig. 229, p. 1283). The values κ_0' are $<270°$ from Nos. 1 to 7 and

[8] Since $c_5(x) = |\cos x|$ one has $c_5(\kappa_0'') = c_5(\kappa_0)$ and (4b) could have been replaced by (3b).

Table 22

No.	Date: Nab.	Date: julian	α	$c_3(\alpha)$	$\kappa_0' = \kappa_0+90$	$c_5(\kappa_0')$	$\beta_1 = c_3 \cdot c_5$	κ_0	$c_4(\alpha)$	$c_5(\kappa_0)$	$\beta_2 = c_4 \cdot c_5$	$c_5(\kappa_0)^2$	$\beta_3 = 0;10 \cdot c_5^2$	β: $\beta_1+\beta_2+\beta_3$	β: modern	$\Delta =$ Alm. − modern	No.
	442	−305															
1	XII 13	Oct. 16	159;29	+4; 8	238;50	−0;31, 1	−2; 8	148;50	+2; 3	−0;51,18	−1;45	0;44	+0; 7	−3;46	−5; 0	+1;14	1
2	18	21	162;23	4;29	243;57	0;26,19	1;58	153;57	1;53	0;53,39	1;41	0;47	0; 8	3;31	4;55	1;24	2
3	23	26	165;17	4;51	249; 4	0;21,20	1;43	159; 4	1;41	0;56, 1	1;34	0;52	0; 9	3; 8	4;41	1;33	3
4	28	31	168;10	5;14	254;11	0;16,13	1;25	164;11	1;26	0;57,35	1;23	0;55	0; 9	2;39	4;16	1;37	4
5	ep. 3	Nov. 5	171; 3	5;36	259;19	0;11, 5	1; 2	169;19	1; 9	0;58,52	1; 8	0;58	0;10	2; 0	3;38	1;38	5
	443																
6	I 3	10	173;56	+5;52	264;27	−0; 5,55	−0;35	174;27	+0;48	−0;59,38	−0;48	0;59	+0;10	−1;13	−2;47	+1;34	6
7	8	15	176;48	6; 6	269;35	−0; 0,27	−0; 3	179;35	+0;27	0;59,58	−0;27	1; 0	0;10	−0;20	−1;43	1;23	7
8	13	20	179;40	6;20	274;44	+0; 5, 3	+0;32	184;44	+0; 3	0;59,41	−0; 3	0;59	0;10	+0;39	−0;28	1; 7	8
9	18	25	182;33	6; 9	279;52	+0;10,16	+1; 3	189;52	−0;21	0;59, 2	+0;21	0;58	0;10	+1;34	+0;50	0;44	9
10	23	30	185;26	5;55	284;59	+0;15,24	+1;31	194;59	−0;44	0;57,48	+0;42	0;57	0;10	+2;23	+2; 2	0;21	10
11	28	Dec. 5	188;19	+5;39	290; 7	+0;20,31	+1;56	200; 7	−1; 4	−0;56,18	+1; 0	0;52	+0; 9	+3; 5	+3; 4	+0; 1	11
12	II 3	10	191;12	5;19	295;15	0;25,34	2;16	205;15	1;22	0;54, 1	1;14	0;49	0; 8	3;38	3;52	−0;14	12
13	8	15	194; 7	4;56	300;21	0;30,18	2;29	210;21	1;38	0;51,47	1;25	0;44	0; 7	4; 1	4;26	−0;25	13
14	13	20	197; 2	4;34	305;26	0;34,43	2;39	215;26	1;52	0;48,44	1;31	0;39	0; 7	4;17	4;47	−0;30	14
15	18	25	199;57	4;13	310;32	0;38,50	2;44	220;32	2; 1	0;45,29	1;32	0;34	0; 6	4;22	4;58	−0;36	15
16	23	−304 30	202;53	+3;52	315;37	+0;42,39	+2;45	225;37	−2; 9	−0;39,45	+1;30	0;29	+0; 5	+4;20	+4;59	−0;39	16
17	28	Jan. 4	205;49	3;31	320;41	0;46,11	2;42	230;41	2;16	0;37,51	1;26	0;24	0; 4	4;12	4;54	0;42	17

$>270°$ thereafter. Hence the $c_5(\kappa'_0)$ are at first negative and therefore also $\beta_1<0$ for Nos. 1 to 7 but $\beta_1>0$ for the following dates.

Near $\kappa_0=180$ the contribution $c_4(\alpha)$ is positive for $0<\alpha<180$ and negative for $\alpha>180$ (cf. Fig. 219c, p. 1279). The coefficient $c_5(\kappa_0)$ remains negative near $\kappa_0=180$. Thus the component β_2 changes from negative to positive between Nos. 8 and 9. The last component, $\beta_3=0;10\,c_5(\kappa_0)^2$ is, of course, always positive.

Fig. 233a indicates these results in comparison with the correct positions. The motion of the planet when the epicycle is near $\kappa=270$ (hence $\lambda\approx 325$) is illustrated in Fig. 233b. For the moments of evening setting (Ω) and morning rising (Γ) cf. below p. 243f.

C. Extremal Latitudes

One condition for observing a planet at a maximum northern or southern geocentric latitude is its proximity to the observer. Thus we have to assume that the equated anomaly α of the planet is 180°.

For an outer planet the center of the epicycle must be placed at the northernmost point of the deferent or opposite to it; hence we have to assume either $\omega=0$ or $\omega=180$. For an inner planet, however, the epicycle must be located in the nodal line of the deferent.

For an outer planet we have

$$c_5(\omega)=\pm 1 \quad \text{thus} \quad \beta=\begin{cases} c_3(180) & \text{if } \omega=0 \\ -c_4(180) & \text{if } \omega=180. \end{cases}$$

Consequently, we find from the tables in Alm. XIII, 5 the following extremal latitudes:

$$\text{Saturn: } \begin{matrix} +3;2° \\ -3;5 \end{matrix} \qquad \text{Jupiter: } \begin{matrix} +2;4° \\ -2;8 \end{matrix} \qquad \text{Mars: } \begin{matrix} +4;21° \\ -7;\ 7. \end{matrix} \tag{1}$$

For the inner planets we have either $\kappa_0=90$ or 270, therefore $\kappa'_0=0$ or 180, thus $c_5(\kappa'_0)=\pm 1$ and $|\beta_1|=c_3(180)$. Since $c_4(\alpha)=0$ for $\alpha=180$ and $c_5(\kappa_0)=0$ for $\kappa_0=90$ or 270 we find $\beta_2=0$ and $\beta_3=0$ thus $\beta=\beta_1$. Hence the extremal values are $\beta=\pm c_3(180)$:

$$\text{Venus: } \pm 6;22° \qquad \text{Mercury: } \pm 4;5°. \tag{2}$$

The values given in (1) and (2) represent the extremal latitudes which can result from the tables in Alm. XIII, 5. They are slightly greater than the values recorded by Ptolemy as obtained from observations near to the ideal positions for extremal latitudes:

$$\begin{matrix} \text{Saturn: } \pm 3°\,[1] \qquad \text{Jupiter: } \pm 2°\,[1] \qquad \text{Mars: } \begin{cases} +4;20°\,[2] \\ -7 \end{cases} \\ \text{Venus: } \pm 6;20°\,[3] \qquad \text{Mercury: } 4°\,[3]. \end{matrix} \tag{3}$$

For modifications of the values (1) and (2) in the Handy Tables cf. below p. 1014.

[1] Cf. (1), p. 211.
[2] Cf. (1), p. 209.
[3] Cf. (2), p. 215.

D. Transits

In his introduction to Book IX[1] Ptolemy expresses his belief in the assumption that Venus and Mercury are located between earth and sun. The argument that consequently transits should be observable he counters by the remark that the motion of the planets in latitude could prevent the occurrence of transits,[2] exactly as the majority of syzygies are not connected with eclipses. We shall show in the following that this analogy is only too correct: just as in the case of the moon at least some conjunctions of the inferior planets result in transits. In other words Ptolemy's own theory should have convinced him of the existence of transits of Venus and Mercury. Instead this important problem remained unsolved until Kepler correctly predicted a transit of Mercury for 1631.[3]

We shall operate in the following exclusively with the parameters and cinematic models from the Almagest and demonstrate the existence of transits as a necessary consequence. Obviously transits can only be expected near the epicyclic anomalies 0° and 180°. But since we know that in the first case the planet is actually hidden behind the sun we shall discuss only the case of inferior conjunction; no new principle is involved with superior conjunctions. Ptolemy, of course, should have investigated both cases (cf. Table 23).

Table 23. Latitudes

	Venus		Mercury		
κ_0	$\alpha=0$	$\alpha=180$	$\alpha=0$	$\alpha=180$	κ_0
−36			+0;25,53		−36
−30	−0;22,50		+0;14		−30
−24	−0;16,31		+0; 2, 9		−24
−18	−0; 9,53		−0;10,15		−18
−12	−0; 3,15		−0;22, 3		−12
− 6	+0; 3,13	+0;50,41			− 6
0	+0;10	+0;10		−0;45	0
+ 6	+0;16,39	−0;30,49		−0;18,34	+ 6
+12	+0;22,47			+0; 6,41	+12
+18				+0;32,23	+18

1. Venus. We assume for the sun an apparent diameter of 0;30°. Hence we ask for latitudes $|\beta| \leq 0;15°$ near $\alpha=180°$. Such latitudes are only possible if we see the epicycle practically edge on, hence we must be very near to $\kappa_0=0°$ or 180° (cf. Figs. 219a and c, p. 1279). In this situation the center C of the epicycle is at a point of latitude $i_0=+0;10°$ and at a longitude ♉/♏25°. This is also the longitude of the mean sun; transits are, of course, only of interest with respect to the true sun. Since the solar apogee lies at ♊5;30 the true sun has near $\kappa_0=0$ a

[1] Cf. above p. 148.
[2] Proclus, Hypotyp. V, 12ff. (Manitius, p. 142ff.) repeats and elaborates (in part incorrectly) Ptolemy's arguments but he does not attempt a numerical confirmation.
[3] Kepler, Opera 7, p. 592–594; also Werke 4, p. 429–433 and for the theory Halley [1691]. Ptolemy himself eventually found in the brightness of the sun the true cause for the impossibility of a naked eye observation of transits of Mercury and Venus (Planetary Hypotheses, Goldstein [1967], p. 6). On medieval reports on alleged transits cf. Goldstein [1969].

greater longitude than the mean sun, near $\kappa_0=180$ a smaller longitude. The amount of the equation θ can be taken from Alm. III, 6; one finds

κ_0	θ	θ	κ_0
0	+0;24,30	−0;35,30	177
1	+0;22,10	−0;33,10	178
2	+0;19,50	−0;30,40	179
3	+0;17,30	−0;28	180

Next we compute the path of Venus, i.e. longitude and latitude, for $\alpha=179°$ and 180° and for 180° and 181°, respectively. This will show us that the planet crosses the disk of the sun for κ_0 from 0° to 3° and for $\kappa_0=177°$ to 180°.

For $\alpha=180°$ the longitude of the planet is the same as for C. For $\alpha=179$ it deviates near $\kappa_0=0$ by $c_6-c_5=+2;22,20°$; near $\kappa_0=180$ and for $\alpha=181$ by $c_6+c_7=-2;44,20°$ (cf. Alm. XI, 11). For the latitudes Alm. XIII, 10 gives us the following coefficients

κ_0		$c_5(\kappa_0')$	$c_5(\kappa_0)$	β_3
0	180	0	+1; 0, 0	+0;10, 0
1	179	−0;1, 4	+0;59,56	+0; 9,59
2	178	−0;2, 8	+0;59,52	+0; 9,57
3	177	−0;3,12	+0;59,48	+0; 9,56

and

α	$c_3(\alpha)$	$c_4(\alpha)$
179	+6;17	+0;8,20
180	+6;22	0
181	+6;17	−0;8,20

Hence we can compute the latitudes

$$\beta=c_3(\alpha)\cdot c_5(\kappa_0')+c_4(\alpha)\cdot c_5(\kappa_0)+\beta_3$$

and find

κ_0		β: $\alpha=179$ and 181	β: $\alpha=180$
0	180	+0;18,20	+0;10, 0
1	179	+0;11,36	+0; 3,12
2	178	+0; 4,52	−0; 3,38
3	177	+0; 1,54	−0;10,26

If we represent these data graphically (cf. Fig. 234[4]) it becomes clear that transits will occur when α is very near to 180 and κ_0 within the intervals slightly in excess of from 0 to 3 and from 177 to 180.

[4] Fig. 234 shows only the cases from $\kappa_0=0$ to 3. For $\kappa_0=180$ one has a mirrored arrangement with slightly greater equations for the sun and the planet.

We must now show that these combinations of α and κ_0 really occur. This is by no means evident. We know that 5 revolutions of the anomaly occur in 8 years.[5] If this relation were exact a transit would either occur once every 8th year or never. In fact, however, this is not the case. The tables of mean motions (Alm. IX, 4) show that α returns to the same value after 8 Egyptian years minus 8 hours whereas the mean longitude during this time does not complete 8 rotations but still lacks 2;16,25°. Hence $\alpha=180$ and a given value of κ_0 do not remain in a fixed relation forever but the 5 epicyclic positions shown in Fig. 235 will slowly rotate in the direction of smaller longitudes. Since the progress per 8-year cycle, i.e. 2;16,25°, is less than 3° the combination $\alpha=180$ and $0\leqq\kappa_0\leqq 3$ or $177\leqq\kappa_0\leqq 180$ cannot be permanently avoided. It is also clear that in general two transits will occur within 8 years, after once the critical combination α, κ_0 has been reached.[6] But each such pair of transits must be separated from the next pair by a much longer interval without transits.

To determine the length of these intervals one has only to remark that the points with $\alpha=180$ are in mean longitude 72° apart (cf. Fig. 235). The slow shift of κ_0 by 2;16,25° in 8 years, brings the perigee of the epicycle, $\alpha=180$, at intervals of 36° into contact either with $\kappa_0=0$ or $\kappa_0=180$. Hence it will take about $36/2;16,25=15;50$ 8-year periods, i.e. about 129 years, before a transit will occur near $\kappa_0=0$ after one pair near $\kappa_0=180$. Because of the asymmetric position of the sun with respect to the line from $\kappa_0=0$ to $\kappa_0=180$ we must take into consideration the fact that between $\kappa_0\approx 3°$ and $\kappa_0\approx 177°$ an interval of only $36-6=30°$ has to be bridged. Hence the next interval without transits will amount to only $30/2;16,25=13;11$ 8-year cycles or about 105 1/2 years.

These estimates are in excellent agreement with the correct values (121 1/2 or 129 1/2 and 105 1/2 or 113 1/2 years, respectively) and could easily be sharpened by reckoning the limits for κ_0 more accurately. But it is evident that Ptolemy's theory would have been amply sufficient to lead him to a correct prediction of transits of Venus. Since the brightness of the sun precludes a direct naked eye observation the above simple computations could have upset the whole ancient and mediaeval theory of planetary arrangements. On the other hand a systematic pursuit of the problem would have produced important conclusions concerning the size and structure of our planetary system, 1500 years earlier than it actually happened.

2. Mercury. A discussion, similar to that for Venus, shows that Ptolemy's theory of latitudes requires for transits an anomaly α near 183° or 184° for $3\leqq\kappa_0\leqq 11$ or near 176° or 177° for $169\leqq\kappa_0\leqq 177$. The tables of mean motions (Alm. IX, 4) show that $\bar{\alpha}$ returns to the same value in 46 years[7] 12 days 9;37^h while the mean longitude increases during this period by 1;2,18°. Hence transits are unavoidable and, in fact, should be quite frequent. Indeed, the tables of mean motions also show that $\bar{\alpha}$ returns to the same value after about 3 years 6 months

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[5] Cf. above p. 151 (4).

[6] The alternative of one single transit occurs at a nearly central position of the path across the sun. For this case one obtains as maximum duration of a transit, using Ptolemy's parameters, about 7;40^h which is only about 0;15^h too short.

[7] Cf. p. 151 (4).

minus 8^h. The mean longitude, however, increases during this time only by about $180-3;38,37°$. Since the critical interval for κ_0 is about 8° long it is clear that three consecutive transits with 3 1/2 years between them can occur, as is indeed the case.[8] Obviously Ptolemy had no basis whatsoever for denying the possibility of transits of Mercury, in contrast to the case of Venus where a superficial argument can easily lead to the belief that the latitudes exclude transits.

The actual observation of a transit of Mercury is, of course, still more difficult than in the case of Venus. It is also clear that the observation of a transit would be greatly facilitated by an accurate prediction of the time and the path across the disk of the sun. Here the inaccuracy of the ancient elements would produce serious deviations between expectation and facts. But the exclusion of transits has no basis whatever in the ancient theory of planetary latitudes.

§ 8. Heliacal Phenomena ("Phases")

Next to the phenomenon of retrogradation the periodic disappearance and reappearance of the planets could not escape attention. In the case of Venus and Mercury their being restricted to a limited elongation from the sun is another remarkable feature of their motion. It is therefore not surprising that the Babylonian astronomers concentrated their efforts in planetary theory on the prediction of these peculiar phenomena. The day by day motion could then be treated as a problem of interpolation between these characteristic points.

It must have been felt as a great success, and hence as a strongly supporting feature of the Greek planetary models, that these "*phases*" of the planets could be shown to be a direct consequence of the underlying cinematic model. At the same time the planetary phases were removed from the central position which they had held in Babylonian astronomy to a secondary place of, at best, traditional interest. Even in astrology the planetary phases hardly play any rôle.

1. Maximum Elongations

As "*elongation*"[1] of Venus and Mercury from the sun Ptolemy denotes the longitudinal difference between the true planet and the true sun, i.e. the observable distance in longitude.[2] It is clear that the maximum value of this elongation will vary with the geocentric distance of the center of the epicycle; hence it will depend on the latter's position relative to the apsidal line.

Ptolemy's goal is the tabulation of these maximum elongations as function of the longitude of the planet. He restricted himself, however, to only such

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[8] The details of the arrangement of Mercury's transits are much more intricate than for Venus. The actual intervals are 3 1/2 and 7 years or 6, 9 1/2, and 13 years. The maximum durations of about 8^h at apogee, 6^h at perigee, easily follow from Ptolemy's parameters.

[1] *Διάστασις* or *ἀπόστασις*.

[2] It should be noted that we are no longer considering "elongations" with respect to the *mean* sun as was the case in I C 2, 1 (p. 153) and I C 3, 1 (p. 160).

longitudes which are endpoints of zodiacal signs. For Venus the variations are so small that no higher accuracy is required (cf. below Fig. 238, p. 1288); but with Mercury linear interpolation can result in considerable inaccuracies. Nevertheless the problem seemed not to justify more extensive calculations, another indication of the lack of genuine interest in the planetary phases for Greek astronomy.

A. Venus

In order to construct tables which give the maximum elongation of Venus, either as morning or as evening star, Ptolemy solves the following geometrical problem. Ignoring latitudes we assume that the plane of the epicycle and of the deferent coincide with the ecliptic. As independent variable we use the longitude λ_T of the planet at maximum elongation, where T is the point of contact of the tangent OT to the epicycle (cf. Fig. 236). We restrict λ_T to the values ♈0°, ♉0°, ..., ♓0°. We wish to find for each of these longitudes λ_T the corresponding longitude $\lambda_\odot$ of the true sun. Then

$$\Delta\lambda_{max} = |\lambda_\odot - \lambda_T| \tag{1}$$

is the maximum elongation which belongs to the true longitude λ_T of the planet at T.

Fig. 236 illustrates the procedure followed by Ptolemy for the case of Venus being morning star. We know not only the given value λ_T but also the longitude λ_A of the apogee of the deferent of Venus.[3] Thus

$$\delta = \lambda_A - \lambda_T$$

is known and hence the angle ε from

$$\sin\varepsilon = (r - e\sin\delta)/R$$

and

$$\zeta = \delta - \varepsilon.$$

With ζ one obtains η from[4]

$$\tan\eta = \frac{e\sin\zeta}{R - e\cos\zeta}$$

and finally

$$\theta = \zeta + \eta.$$

When the mean sun is in the apsidal line OA the center C of the epicycle coincides with A. Since the angle θ has its vertex in the equant E it changes linearly with time. Thus the mean longitude of C is also the mean longitude of the sun and

$$\bar{\lambda}_C = \lambda_A - \theta = \bar{\lambda}_\odot.$$

Since we know the longitude $\lambda(A_\odot)$ of the solar apogee $A_\odot$[5] (cf. Fig. 237) we also have the solar anomaly

$$\bar{\kappa}_\odot = \bar{\lambda}_\odot - \lambda(A_\odot).$$

[3] $\lambda_A =$ ♉25 (cf. p. 153).

[4] Ptolemy needs two steps here, having no tables for $\tan\alpha$.

[5] ♊5;30 (cf. p. 58).

Now we can find the corresponding equation $c(\bar{\kappa}_{\odot})$ from the solar tables Alm. III, 6 and hence the longitude of the true sun

$$\lambda_{\odot}=\bar{\lambda}_{\odot}+c(\bar{\kappa}_{\odot})$$

and finally the maximum elongation (1).

The case of Venus as evening star requires only trivial modification of signs. By assigning λ_T the values 0°, 30°, ..., 330° one finds the consecutive maximum elongations tabulated in Alm. XII, 10 (graphically represented in Fig. 238).

It may be remarked that these tables could not claim validity over very long periods of time since, according to Ptolemy, the apogee A of the planet moves with the speed of precession toward the solar apogee $A_{\odot}$ of fixed longitude. Consequently the angles used in the preceding computation vary slowly with time and modify the results which were derived from data valid around A.D. 140. Ptolemy was, of course, aware of this fact but the variations are much too small to be taken into consideration in a table of only limited accuracy.

B. Mercury

In the case of Venus Ptolemy found from the given true longitude of the planet at T the longitude of C which also determines the longitude of the mean sun. Hence he could also find the longitude of the true sun and from it the desired elongation. This procedure is possible because for given direction OT (cf. Fig. 236, p. 1287) the point C is easily found since the deferent of radius $R=$MA has a fixed position and $r=$CT is known. For Mercury, however, this method is no longer applicable because the position of the deferent is variable. Consequently one has to choose consecutive positions of C and determine the corresponding longitude λ_T of T and $\lambda_{\odot}$ of the true sun. The resulting longitudinal differences $\Delta\lambda_{max}=|\lambda_{\odot}-\lambda_T|$ are therefore known only as function of $\bar{\kappa}$, i.e. of the mean eccentric anomaly which defines the position of C; similarly λ_T is obtained as a function of $\bar{\kappa}$. Ptolemy, however, wishes to tabulate $\Delta\lambda_{max}$ for $\lambda_T=0°, 30°, \ldots, 330°$. This he is only able to do by interpolation between values of λ_T which straddle sufficiently closely the integer values $k\cdot 30°$.

The geometrical problem which Ptolemy has to solve in order to carry out this program consists therefore first of all in the determination of $\lambda_T(\bar{\kappa})$. As Fig. 239 shows:

$$\lambda_T=\lambda_A+\kappa+\delta \tag{2}$$

where λ_A is the known longitude of the apogee A of the eccenter,[6] $\kappa=\bar{\kappa}-\eta$ and $\delta=\arcsin(r/\rho)$. The angle η is a known function of $\bar{\kappa}$, obtainable from Alm. XI, 11 in the form

$$\eta=c_3(\bar{\kappa})+c_4(\bar{\kappa}).$$

The distance $\rho=$OC can be computed by the method described above p. 165; hence λ_T will be known as function of $\bar{\kappa}$.

The longitude $\lambda_{\odot}$ of the true sun can be found as before from the known mean longitude of C:

$$\bar{\lambda}_C=\lambda_A+\bar{\kappa}=\bar{\lambda}_{\odot}.$$

[6] $\lambda_A=$ ♎ 10 (cf. above p. 159).

Hence $\bar{\kappa}_{\odot}=\bar{\lambda}_C-\lambda(A_{\odot})$ and from Alm. III, 6 the equation $c(\bar{\kappa}_{\odot})$; finally $\lambda_{\odot}=\bar{\lambda}_{\odot}+c(\bar{\kappa}_{\odot})$ and thus

$$\Delta\lambda_{max}(\bar{\kappa})=|\lambda_T(\bar{\kappa})-\lambda_{\odot}(\bar{\kappa})|. \tag{3}$$

C. The Tables (Alm. XII, 10)

For Venus Ptolemy computes only one numerical example, namely the case $\lambda(T)=♈0°$, both for the situation as morning star and as evening star. The results, $\Delta\lambda_{max}=45;15°$ and $46;22°$, respectively, are also found as the first entries in the table Alm. XII, 10. The remaining entries can easily be computed and present no problem (cf. Fig. 238, p. 1288).

The situation for Mercury is quite different. Ptolemy gives the numerical details for two examples: $\lambda(T)=♏0°$ for Mercury as evening star and $\lambda(T)=♉0°$ as morning star, examples chosen in view of later application to a visibility problem.[7]

From the basic observations concerning the apogee of Mercury it is known[8] that the maximum elongation of Mercury from the mean sun varies between 19;3° when $C=A=♎10$ and 23;15° at $C=\Pi=♈10$. If we wish T to be located at about ♏0 as evening star we must expect $C=\bar{\odot}\approx ♏0-19=♎11$. Consequently Ptolemy computes the maximum elongations for $\bar{\kappa}=0=♎10$ and $\bar{\kappa}=3=♎13$ which can be expected to straddle the position of C belonging to $T=♏0$. Similarly for $T=♉0$ one can assume $C=\bar{\odot}\approx ♉0+23=♈10+43$. Ptolemy, again operating with multiples of 3, chooses $|\bar{\kappa}|=180+42=222$ and $|\bar{\kappa}|=180+39=219$ for the interval which should contain the C that belongs to $T=♉0$.

For each of these four values of $\bar{\kappa}$ the position of C has to be found (cf. Fig. 239). Consequently Ptolemy needs the angle $\delta=\arcsin(r/\rho)$ where $r=22;30$ and $\rho(\bar{\kappa})$ is the geocentric distance of C. Ptolemy quotes for ρ the following values which we can check by using his known procedure[9]:

$\bar{\kappa}=0°$	Ptolemy: $\rho=69$	check: 69
3	68;58	68;58,20
219 = −141	55;59	55;59,10
222 = −138	55;50	55;52,50.

From the tables in Alm. XI, 11 one can find for each $\bar{\kappa}$ the equation $\eta=c_3+c_4$ hence $\kappa=\bar{\kappa}+\eta$. Then $\lambda_T(\bar{\kappa})=♎10+\kappa+\delta$. From $\bar{\lambda}_C=\bar{\lambda}_{\odot}=♎10+\bar{\kappa}$ one knows $\bar{\kappa}_{\odot}=\bar{\lambda}_{\odot}-♊5;30$ and from Alm. III, 6 the solar equation $c(\bar{\kappa}_{\odot})$ and hence $\lambda_{\odot}=\bar{\lambda}_{\odot}+c$ for the true sun. In this way Ptolemy obtains

$\bar{\kappa}=0$	$\lambda_T=♎29;\ 2$	$\lambda_{\odot}=♎\ 8$	$\Delta\lambda_{max}=21;\ 2°$
3	♏ 1;55	♎ 11;4	20;51

hence for $\lambda_T=♏0$ by linear interpolation $\Delta\lambda_{max}\approx 20;58$.

[7] Below p. 241.

[8] Above p. 159 and Fig. 144, p. 1252.

[9] Cf. above p. 165f. and Table 14 (p. 169). In principle one could reconstruct the table for $\rho(\bar{\kappa})$ from the relation $\rho=e\sin\bar{\kappa}/\sin\eta$ with η known from Alm. XI, 11 as $c_3(\bar{\kappa})+c_4(\bar{\kappa})$. Unfortunately the roundings in the tabulated values of c_3 and c_4 have a great influence on the small values of $\sin\eta$ and hence produce large errors for ρ, much too large for the accuracy required in our present problem.

Similarly for

$$\begin{array}{llll} \bar{\kappa}=219 & \lambda_T = \text{♈} 27;15 & \lambda_\odot = \text{♉} 19;38 & \Delta\lambda_{max} = 22;23^\circ \\ 222 & \text{♉}\ 0;19 & \text{♉} 22;31 & 22;12 \end{array}$$

and therefore $\Delta\lambda_{max} \approx 22;13$ for $\lambda_T = \text{♉} 0$.

In this fashion Ptolemy determined the 24 entries in the table of maximum elongations (Alm. XII, 10).[10]

2. The "Normal Arcus Visionis"

The last chapters of the Almagest (XIII, 7 to 10) are devoted to the heliacal risings and settings of the planets. Since it is obvious that the nearness to the sun causes the invisibility of the planet it is natural to consider the elongation $\Delta\lambda$ between planet and sun as the basic parameter for the prediction of the appearances and disappearances of the stars. Ptolemy introduced instead an element of much greater significance for the theory of all heliacal phenomena, namely the depression of the sun below the horizon required to make the planet visible in the horizon, i.e. the "*arcus visionis*", to use modern terminology.[1]

A. Ptolemy's Procedure

Since the heliacal phenomena are related to the horizon one is dealing with a problem which depends on the geographical latitude. Ptolemy eliminates this variable from his discussion in the Almagest[2] by referring all his data to one intermediate latitude of the mediterranean area, i.e. to the latitude of Phoenicia where the longest day is $M = 14\ 1/4^h$ which is the mean value between $M = 14^h$ at Alexandria and $M = 14\ 1/2^h$ at Rhodes.[3] This choice has the added advantage that $\frac{M}{m} = \frac{14;15}{9;45} \approx 1;28$ is very near to the ratio $\frac{M}{m} = 3/2$ which is characteristic for Babylon[4] from where the majority of reliable observations originated.

Another simplification consists in the replacement of spherical triangles by plane ones. Obviously, high mathematical accuracy makes no sense for the description of phenomena which are not too well defined. Consequently Ptolemy considers the following configuration (Fig. 240): the ecliptic intersects at an

[10] The entry 19;14 for Mercury as evening star in Capricorn is incorrect and should be 18;54. Halma H.T. III, p. 32 gives 18;14. Fig. 238 shows the correct value.

[1] I do not know where this term originated. Nallino, Battani II, p. 256 considers it to be of Arabic origin. Petavius says "arcus ille, qui *fulsionis*, vel *visionis* vulgo nuncupatur" (De doctrina temporum III, Var. Diss. Lib. I, cap. III, p. 5 [Verona 1736]). Neither Brahe nor Kepler seem to use the term, though Regiomontanus knows it (Epitoma in Almag., XIII, propos. 23 [1496]).

[2] As we shall see later Ptolemy in the Handy Tables determined planetary phases for each of the seven climata (cf. below p. 257 and V C 4, 5 C) and phases for the fixed stars for the five climata II to VI (cf. below V B 8, 1).

[3] Cf., e.g., Strabo, Geogr. II, 5, 39 (Loeb I, p. 511).

[4] Cf. below p. 367; cf., however, below p. 249, note 12.

angle v the horizon at H, P is the planet in the horizon at its first or last visibility, β its latitude. The sun is at S, $\Delta\lambda$ is the elongation of the planet.

Under these conditions he considers the situation when the point H is the summer solstitial point (♋0°) because then the same angle v between ecliptic and horizon occurs at rising and at setting. From observations made under the favorable atmospheric conditions near the summer solstice Ptolemy assembled the following elongations $\Delta\lambda$ which correspond to first or last visibility

Saturn		$\Delta\lambda = 14°$	
Jupiter		12 3/4	
Mars		14 1/2	(1)
Venus	at evening	5 2/3	
Mercury	rising (Ξ)	11 1/2.	

Taking these elongations as the empirical basis he computes the corresponding arc $h = SS'$ as follows:

Because

$$\sin v = h/(\Delta\lambda + P'H) \qquad \tan v = \beta/P'H$$

one can find h from

$$h = \sin v(\Delta\lambda + \beta \cot v) = \Delta\lambda \sin v + \beta \cos v. \tag{2}$$

For v at the assumed geographical latitude and for H = ♋0° he uses[5] $v = 51;30$ hence

$$\sin v = 0;47 \qquad \cos v = 0;37\,^{6}. \tag{3}$$

The latitude β can be ignored in the case of Saturn and Jupiter because the northernmost points of the deferent planes of these planets lie near ♎0°[7]; thus ♋0° is near the nodal line and hence $\beta \approx 0$. Substituting (1) and (3) into (2) we find

$$\begin{aligned} &\text{Saturn:} && h = 14 \cdot 0;47 \approx 11° \\ &\text{Jupiter:} && 12;45 \cdot 0;47 \approx 10°. \end{aligned} \tag{4a}$$

For the remaining planets the determination of their latitudes near ♋0° requires a detailed discussion which we postpone to the next section.[8] The latitudes used are

$$\text{Mars: } \beta = 0;12° \qquad \text{Venus: } \beta = 1° \qquad \text{Mercury: } \beta = 1;40°. \tag{5}$$

Consequently

$$\begin{aligned} &\text{Mars:} && h = 14;30 \cdot 0;47 + 0;12 \cdot 0;37 = 11;28,55 \approx 11;30° \\ &\text{Venus:} && 5;40 \cdot 0;47 + 0;37 = 5;3,20 \approx 5° \\ &\text{Mercury:} && 11;30 \cdot 0;47 + 1;40 \cdot 0;37 = 10;2,10 \approx 10°. \end{aligned} \tag{4b}$$

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[5] Cf. for this value below p. 236.
[6] Accurately 0;46,57,23 and 0;37,21,3, respectively.
[7] Cf. above p. 218.
[8] Cf. below p. 237f.

The values (4a) and (4b) of h are henceforth considered as "*normal arcus visionis*" in the sense that, independent of the longitude λ of the planet, a depression of the sun below the horizon in the amount h is considered characteristic for the moments of heliacal setting or rising. In other words h is a parameter specific for each planet under the assumed geographical latitude. In the Canobic Inscription[9] Ptolemy gives the same values (4a) and (4b) for h, except for $h=10;30°$ for Mercury. This change could be explained by an increase of $\Delta\lambda$ from 11;30° to 12°. Much more drastic changes were introduced in the Handy Tables and in the Planetary Hypotheses.[10]

It could be remarked that Ptolemy's procedure in determining h by means of a configuration like Fig. 240 is reminiscent of similar methods used in Babylonian astronomy in connection with the visibility of the moon.[11] There is no need, however, to assume any direct influence for such simple geometrical arguments.

B. *Numerical Details*

The Angles between Ecliptic and Horizon

From the tables in Alm. II, 13 one obtains for the eastern angle v between ecliptic and horizon at ♋0° for clima III (longest daylight $M=14^h$) and clima IV ($M=14;30^h$) the values $v=56;28$ and $v=50;1$, respectively.[1] Linear interpolation for $M=14;15^h$ would therefore result in $v=53;15$ whereas Ptolemy uses $v=51;30$. Similar discrepancies appear in subsequent discussions where Ptolemy gives his values of v for ♉0° and ♍0°.[2] The same values appear again in the computations of the tables Alm. XIII, 10 for the first and last visibility of the planets and this will give us the possibility of detecting the data which Ptolemy used for these computations[3]; namely, $M=14;15^h$ in combination with $\varepsilon=24°$ for the obliquity of the ecliptic. It is surprising to find such a crude parameter used in the Almagest in such sophisticated trigonometric computations as are required for the angles v.

At the moment it will suffice to show in the following table that at least two of Ptolemy's values were not derived by interpolation from the Almagest (II, 13).

0° of	interpol.	Ptolemy
♈	32;58	
♉ ♓	35; 1	34;30
♊ ♒	41;43	
♋ ♑	53;15	51;30
♌ ♐	66;43	
♍ ♏	77; 1	77; 0
♎	80;40	

[9] Ptol. Opera II, p. 153, 15.
[10] Cf. below p. 1017 (2).
[11] Cf. below p. 535.
[1] Cf. Table 3, p. 47 and Fig. 41, p. 1218.
[2] Cf. below pp. 239 and 241.
[3] Cf. below p. 245ff.

Planetary Latitudes

Mars. As we have seen (p. 235) the latitudes of Saturn and Jupiter must be so near to zero as to be negligible. To Mars Ptolemy ascribes the latitude $\beta = +0;12°$, assuming for the planet P (Fig. 241) the longitude ♋0° at an elongation $\Delta\lambda = 14\ 1/2°$ from the sun ((5) and (1), p. 235). In order to compute β from [5]

$$\beta = c_5(\omega) \cdot c_3(\alpha) \tag{5}$$

we need the values of ω and α which correspond to the configuration shown in Fig. 241.[6] Now[7]

$$\omega = \theta + 30 = \bar{\kappa} - \eta(\bar{\kappa}) \tag{6}$$

where $\eta(\bar{\kappa}) = c_3(\bar{\kappa}) + c_4(\bar{\kappa})$ is tabulated in Alm. XI, 11.

From $\Delta\lambda = 14;30$ it follows that the true sun is in ♋14;30 = ♊5;30 + 39°. At that point the equation of center of the sun is about $-1;27°$ (Alm. III, 6). Since CP must be parallel to $M_\odot\odot$ we know that the radius r of the epicycle makes with PO an angle $\gamma = 14;30 + 1;27 = 15;57°$. Hence, with $r = 39;30$,[8] one has OC sin $\theta = r \sin\gamma = 10;51,16$. At the mean distance OC $= R = 60$ this would mean $\theta \approx 10;25°$, thus $\omega = 40;25°$ and $\alpha = \gamma + \theta = 26;22$. In fact OC $> R$ and therefore θ should be smaller, hence also ω and α. If we nevertheless substitute these preliminary values for ω and α into (5) we obtain $\beta = 0;45,27 \cdot 0;13,24 \approx 0;10,9°$ instead of Ptolemy's 0;12°.

Even an accurate computation of β does not change this result.[9] A discrepancy of about 0;2° for β is, of course, not surprising since these computations require a great deal of interpolation in tables of limited accuracy. For the final purpose, the determination of the normal arcus visionis h in (4b), p. 235, it is irrelevant whether one uses 0;12 or 0;10 for β.[10]

Venus. Venus at first visibility as evening star (P in Fig. 242) in ♋0° supposedly has an elongation $\Delta\lambda = 5;40°$ from the sun (cf. (1), p. 235). The eccentricities

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[5] Cf. p. 219, (7).

[6] Fig. 241 is drawn to scale, representing the situation at first visibility of the planet. At last visibility the point C and the direction O$\odot$ would have a mirrored position with respect to the line OP. Cf. also below note 10.

[7] Cf. above p. 208 and Fig. 213, p. 1275.

[8] Above p. 180.

[9] From OC $= \rho = 2e \sin\bar{\kappa}/\sin\eta$ it follows (cf. Fig. 241) that one should find $\bar{\kappa}$ such that

$$10;51,16 = \rho \sin\theta = 2e \cdot \sin\bar{\kappa} \cdot \sin\theta/\sin\eta(\bar{\kappa}) \tag{7}$$

where, because of (6), the right-hand side is a function of $\bar{\kappa}$ alone. By interpolation one finds that $\bar{\kappa} = 47;38$ is a solution of (7), with $\eta(\bar{\kappa}) = 7;54$, hence $\theta = 9;44$. This result can be checked by means of the tables Alm. XI, 11. Since $\alpha = \gamma + \theta = 15;57 + 9;44 \approx 25;40$ we can compute θ from

$$\theta = c_6(\alpha) + c_8(\bar{\kappa}) \cdot c_5(\alpha) \tag{8}$$

and find $\theta \approx 10;9 - 0;35,30 \cdot 0;46,29 \approx 9;42$ instead of the previous 9;44. For β this small discrepancy remains without effect and one finds from the tables Alm. XIII, 5

$$\beta = 0;45,54 \cdot 0;13,17 \approx 0;10,10°$$

as before.

[10] The situation at last visibility is not exactly symmetric to Fig. 241 because the apsidal line remains fixed. Graphically I find $\alpha \approx 23$ and $\omega \approx 31;30$ and finally $\beta \approx 0;11,45°$.

are so small that we may locate the center C of the epicycle on the line O⊙. Hence[11]

$$\kappa_0 = ♋0° - \Delta\lambda - ♉25 = 29;20°.$$

In this situation $\Delta\lambda$ represents the equation which belongs to the anomaly α. Now we can find in Alm. XI, 11, from $c_6(\alpha) = \Delta\lambda = 5;40°$ the value of α, again ignoring the small influence of the eccentricity. This gives $\alpha = 13°$. With κ_0 and α known we can compute the components of the latitude[12]:

$$c_3(\alpha) = -1;1 \qquad c_5(\kappa_0') = -0;29,33 \quad \text{hence} \quad c_3 c_5 = 0;29,52 = \beta_1$$
$$c_4(\alpha) = +0;17,30 \qquad c_5(\kappa_0) = +0;52,17 \quad \text{hence} \quad c_4 c_5 = 0;15,15 = \beta_2$$
$$c_5(\kappa_0)^2 = 0;45,24 \quad \text{hence} \quad 0;10 \cdot c_5^2 = 0;7,34 = \beta_3$$

and finally

$$\beta = \beta_1 + \beta_2 + \beta_3 = 0;52,41°.$$

Ptolemy gives $\beta = 1°$.

It may be remarked that practically the same result ($\beta = 0;59,53°$) would be obtained for the morning setting of Venus. Near inferior conjunction, however, (Q in Fig. 242) the latitude is about $-4°$.

Mercury. No new principles are involved. We are given $\Delta\lambda = 11;30$ and we know that the planet in ♋0° is 260° ahead of the apogee in ♎10°.[13] As with Venus we may identify the longitude of the center C of the epicycle with the longitude of the sun, i.e. with ♋0° $- \Delta\lambda =$ ♊18;30. Hence

$$\kappa_0 = ♊18;30 - ♎10 = +248;30 = -111;30.$$

This shows that C is near the minimum distance from O (at $\kappa_0 = 120$). The tables in Alm. XI, 11 show that $\alpha = 41$ corresponds to the equation $c_6 + c_7 = 11;30$ as required by our $\Delta\lambda$.[14] Entering with these values of κ_0 and α the tables of latitudes one finds with $\kappa_0' = \kappa_0 + 270 = 158;30$ and $\kappa_0'' = \kappa_0 + 180 = 68;30$:

$$c_3(\alpha) = -1;24,10 \qquad c_5(\kappa_0') = -0;55,46 \quad \text{hence} \quad c_3 c_5 = 1;18,14 = \beta_1$$
$$c_4' = \frac{11}{10} c_4(\alpha) = 1;22 \qquad c_5(\kappa_0'') = 0;21,54 \quad \text{hence} \quad c_4' c_5 = 0;29,51 = \beta_2$$
$$c_5(\kappa_0) = 0;21,54 \quad \text{hence} \quad -0;45 \cdot c_5^2 = -0;6 = \beta_3$$

and thus

$$\beta = \beta_1 + \beta_2 + \beta_3 = 1;42°$$

as compared with Ptolemy's 1;40°.

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[11] According to p. 153, A = ♉25.

[12] Following the rules given p. 222ff.

[13] Above p. 159.

[14] By constructing a figure to scale, similar to Figs. 241 and 242, one finds that actually $\alpha \approx 39°$. The influence of this correction on the value for β is negligible.

3. Extremal Cases for Venus and Mercury

Proclus in his "Hypotyposis" says that whole books had been written about the "paradoxical phases of Venus,"[1] referring to the fact that the time of invisibility of Venus at inferior conjunction may vary from only 2 days to as much as 16 days. Unfortunately nothing of this older literature has survived.

Still more irregular are the phases of Mercury. Normally one can expect the same phase about three times each year; but occasionally a whole evening appearance, or a whole morning appearance is omitted, i.e. Mercury does not reach enough elongation from the sun to become visible as evening or as morning star,

In Alm. XIII, 8 Ptolemy sets out to show that these strange phenomena are direct consequences of data derived from the before established values of the "normal arcus visionis" for Venus and Mercury.

A. Venus

If one considers the motion of an inferior planet on its epicycle (cf. Fig. 243) it is evident that the planet will be invisible twice during each synodic period, once in direct motion near superior conjunction between Σ and Ξ, a second time at inferior conjunction in retrograde motion between Ω and Γ. It is the great variability of the length of the arc $\Omega\Gamma$ which constitutes the "paradoxical" behavior of Venus.

Since OC is the direction from the observer to the sun the planet cannot be seen when it enters a cone of axis OC and with a certain angle ω at the vertex O. Great variations in the length of $\Omega\Gamma$ are only surprising as long as one assumes that the planes of the epicycle and of the deferent coincide. If, however, these planes are inclined toward one another, as is known to be the case,[2] then it is clear that the length of $\Omega\Gamma$ will depend on the specific way in which the plane of the epicycle intersects the cone of invisibility. It is for this reason that Ptolemy considers the influence of the latitude of Venus in combination with the variation of the inclination ν between ecliptic and horizon.

The specific phenomenon which Ptolemy wishes to explain consists in the observation that Venus at the beginning of Pisces is invisible only 2 days at inferior conjunction, but 16 days at the beginning of Virgo. From the theory of latitudes it is known that the nodal line of the deferent goes through ♒25 and ♌25.[3] Hence we may say that the observations in question concern a situation which is practically identical with the one represented in Fig. 219b, p. 1279, the planet being near point c (inferior conjunction). It has also been established that at these points the latitude of the planet has the value $\approx \pm 6;20°$ (positive in ♓, negative in ♍).[4]

Ptolemy now considers the configurations Figs. 244a and b which represent the planet at Ω in last visibility in ♓ and ♍, respectively, and c and d for reappear-

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[1] I, 17; I, 22; VII, 18. Translation Manitius, pp. 11, 13, 221, respectively.

[2] Cf. I C 7, 2 B.

[3] Cf. Fig. 218 (p. 1278).

[4] Above p. 226 (2), and p. 215 (2).

ance in Γ. For the angles ν he gives 77;0° and 34;30°, respectively, values which we have discussed above p. 236.

In order to operate with consistent formulae we define for any two points A and B on the ecliptic

$$AB = \lambda(A) - \lambda(B) = -BA,$$

in particular,

$$\Delta\lambda = SP' = \lambda(\odot) - \lambda(P').$$

Furthermore: angles between horizon and ecliptic will always be reckoned in counterclockwise direction (seen from the center of the celestial sphere) such that increasing azimuths (from east over north to west) turn into increasing longitudes. Such angles will be denoted by $\nu(\lambda)$ if the point of longitude λ is rising, by $\bar{\nu}(\lambda)$ if it is setting. It follows from this definition that

$$\bar{\nu}(\lambda + 180) = -\nu(\lambda) \qquad \bar{\nu}(\lambda) = -\nu(\lambda + 180)$$

(as is evident if one considers, e.g., a stereographic projection of the celestial sphere onto the plane of the horizon as in Fig. 245). In this connection it is also useful to remark that always

$$\nu(\lambda) = \nu(-\lambda)$$

because (cf. Fig. 246) to equal ecliptic arcs V_1H_1 and V_2H_2 belong equal oblique ascensions $V_1E = EV_2$ (cf. p. 35).

The arcus visionis h is reckoned always positive, the latitude β of the planet with its proper sign. Then it holds for all four cases in Fig. 244 that

$$\Delta\lambda = \frac{h}{\sin\nu} - \beta\cot\nu. \tag{1}$$

We now have to substitute in (1) the following values

$$|\nu| = \begin{cases} 77;\ 0 \\ 34;30 \end{cases} \quad \sin|\nu| = \begin{cases} 0;58,28 \\ 0;33,59 \end{cases} \quad \cot|\nu| = \begin{cases} 0;13,51 \\ 1;27,18. \end{cases} \tag{2}$$

With $h = 5$ (above p. 235, (4b)) and $\beta = \pm 6;20$ one finds (cf. Fig. 244):

$$\left.\begin{matrix}\text{(a)} \\ \text{(b)}\end{matrix}\right\} \text{West } (\Omega) \qquad \left.\begin{matrix}\text{(c)} \\ \text{(d)}\end{matrix}\right\} \text{East } (\Gamma) \qquad \Delta\lambda = \begin{cases} -\ 3;40,\ 9 & \text{Ptolemy: } 3;38 \\ -18;\ 2,34 & 18;\ 2 \\ -\ 0;23,14 & 0;24 \\ +\ 6;35,35 & 6;38. \end{cases} \tag{3}$$

Combining (a) and (c) we obtain the longitudinal difference between Ω and Γ in Pisces from $\Delta\lambda(\Omega) - \Delta\lambda(\Gamma)$ and similarly with (b) and (d) in Virgo:

$$\begin{aligned} \Omega\Gamma &= 3;38 - 0;24 = 3;14° \quad \text{in } ♓ \\ \Omega\Gamma &= 18;\ 2 + 6;38 = 24;40° \quad \text{in } ♍. \end{aligned} \tag{4}$$

These two arcs represent retrograde motions near the perigee of the epicycle, measured relative to the sun. Hence we may say (cf. Fig. 247) that an observed angle $\omega = \Omega\Gamma$ corresponds to an angle α of epicyclic anomaly approximately given by

$$\alpha = \frac{R - r}{r}\,\omega. \tag{5}$$

Hence, with $r=43;10$,[5] $\alpha\approx 0;23,30\omega$ and then from (4)

$$\alpha\approx 1;16^\circ \quad \text{in } ♓ \qquad \alpha\approx 9;40^\circ \quad \text{in } ♍.$$

The tables of mean motions (Alm. IX, 4) show that the first angle corresponds to about 2 days (1;14°), the second to about 16 days (9;52°); q.e.d.

The fact that $\Delta\lambda$ in (3a) and (3c) has the same sign indicates that both Ω and Γ lie to the east of the radius OC whereas normally OC would be between $O\Omega$ and $O\Gamma$ (as in Fig. 243, p. 1290). This paradoxical effect of the latitudinal deviation of the planet from the ecliptic is in some manuscripts[6] emphasized by the word ἑπόμεναι "following" i.e. the phases follow the sun in the sense of the daily rotation.[7]

B. Mercury

The "abnormal" behavior of Mercury consists in the omission of the whole visible sector $\Xi\Omega$ (Fig. 243, p. 1290) at the beginning of Scorpio and similarly in the omission of $\Gamma\Sigma$ when at the beginning of Taurus. We shall see presently that these empirical facts were also known to the Babylonian astronomers.[8] Ptolemy's explanation is again based on the consideration of the planet's motion in latitude.

The diameter of steepest ascent of Mercury's deferent goes through the points ♈10° and ♎10°,[9] hence deviates by only 20° from the line ♉0°/♏0° under consideration. We now assume that the planet is at maximum elongation from the sun, i.e. at a distance of 20;58° to the east of the sun in Scorpio and 22;13° to the west of it in Taurus.[10] Ptolemy then demonstrates that even under these most favorable conditions the planet does not reach the elongations necessary for visibility.

First Ptolemy determines the latitudes β and finds about -3° and $-3;10^\circ$ respectively.[11] The angle ν between ecliptic and horizon is the same for ♍ and ♏ and for ♉ and ♓ (cf. above p. 240). The angle in the east at one sign equals the negative angle in the west at the diametrically opposite sign. Hence we can say as in the case of Venus[12]

$$\begin{cases} \nu(♉)=-\bar{\nu}(♏)=|\nu|=34;30 \\ \sin|\nu|=0;33,59 \qquad \cot|\nu|=1;27,18. \end{cases} \tag{1}$$

We know that the planet can only be visible when the sun is

$$h=10^\circ \tag{2}$$

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[5] Cf. above p. 154 (8).

[6] E.g. Vat. gr. 1291 fol. 89ᵛ.

[7] This was recognized by A. Aaboe [1960], p. 20.

[8] Below p. 403.

[9] Cf. Fig. 218 (p. 1278).

[10] According to the tables for maximum elongation, Alm. XII, 10. Cf. also Fig. 238 (p. 1288).

[11] The computation of these latitudes causes no difficulties since the basic parameters, κ_0 and α (cf. p. 223) are readily available as Fig. 248 shows. In the first case, P_1 in ♏0°, one has $-\kappa_{01}=\Delta\lambda_1-20=20;58-20\approx 1$ and $\alpha_1=90+\Delta\lambda_1\approx 111$. In the second case, P_2 in ♉0°, $\kappa_{02}=180+20+\Delta\lambda_2\approx 222$ and $\alpha_2=180+90-\Delta\lambda_2\approx 248$. Computing with these elements one finds from the tables Alm. XIII, 5 $\beta_1\approx -3;1$ and $\beta_2\approx -3;7$ in agreement with Ptolemy's rounded values.

[12] Above pp. 236 and 240.

below the horizon.[13] To given h, v, and β then belongs a definite elongation (cf. Fig. 249):

$$\left.\begin{matrix}SP'\\ SQ'\end{matrix}\right\}=\frac{h}{\sin v}-\beta\cot v=\mp 17;39,19\mp\left\{\begin{matrix}4;21,54\approx -22\\ 4;36,27\approx +22;16.\end{matrix}\right. \tag{3}$$

These are the elongations required for visibility. They are numerically greater than the above-mentioned maximum elongations $-20;58$ (in ♏) and $+22;13$ (in ♉); hence the planet will remain invisible.

4. The Tables (Alm. XIII, 10)

The last chapter of the Almagest is a table for the elongations $\Delta\lambda = \mathrm{SP}'$ (cf. Fig. 244, p. 1291) which correspond to one of the following phases (henceforth to be denoted by Greek letters):

$$\text{outer planets:}\left\{\begin{matrix}\Gamma & \text{first visibility in the morning}\\ \Omega & \text{last visibility in the evening}\end{matrix}\right.$$

$$\text{inner planets:}\quad\begin{matrix}\Xi & \text{first visibility}\\ \Omega & \text{last visibility}\end{matrix}\Big\}\text{ as evening star}\qquad \begin{matrix}\Gamma & \text{first visibility}\\ \Sigma & \text{last visibility}\end{matrix}\Big\}\text{ as morning star.}$$

The tabulated values are considered to be valid for a position of the planet at the beginning of a zodiacal sign. Consequently "λ" usually means in the following a longitude $k \cdot 30°$ ($k = 0, 1, \ldots, 11$). The geographical latitude is the latitude of "Phoenicia" (cf. above p. 234 and below p. 249). A graphical representation of these $\Delta\lambda$'s as listed in Table 24 is given in Fig. 250.[1]

Ptolemy gives only few details concerning the method of computation of these tables. One gets the impression that he got tired at the end of his work or that he did not have sufficient interest in a problem which was merely a relic from Babylonian astronomy, now obsolete in view of the Greek cinematical models. Yet the problem must have seemed to him important enough to compute new tables for all seven climates and with new (presumably improved) parameters, eventually incorporated in the Handy Tables.[2] He furthermore extended his investigations to the phases of the fixed stars and wrote a special treatise about them.[3] Nevertheless it is quite clear that the planetary phases must have soon lost all practical interest for ancient astronomy.[4]

[13] Above p. 235 (4b). In the Handy Tables (cf. below p. 257) the value of h for Mercury is increased to 12°, raising the lower limits of visibility given in (3) to $-25;33$ and $+25;48$, respectively.

[1] For some corrections that must be made in the text as accepted by Heiberg (II, p. 606f. ≈ Manitius II, p. 394) cf. p. 248, note 9 and 11, p. 252, note 2, p. 256, note 2. According to our norm (p. 240) the elongations $\Delta\lambda$ for the evening phenomena (Ξ and Ω) are reckoned negative.

[2] Cf. below p. 256f.

[3] Cf. below p. 261.

[4] Cf. also below p. 260f.

Table 24

	Saturn		Jupiter		Mars		
	Γ	Ω	Γ	Ω	Γ	Ω	
♈	23;30	11;28	20;10	10;19	21;12	11;40	♈
♉	21;57	11;41	19; 6	10;29	20;19	11;48	♉
♊	17;52	12;26	15;51	11;10	17;21	12;30	♊
♋	14; 2	14; 2	12;46	12;46	14;33	14;33	♋
♌	11;34	15;34	10;40	14;31	12;28	17;19	♌
♍	10;53	16;53	10; 1	16;12	11;46	20; 5	♍
♎	10;48	17; 6	9;57	16;34	11;38	21; 1	♎
♏	10;53	16;53	10; 1	16;12	11;48	20;19	♏
♐	11;34	15;34	10;40	14;31	12;34	17;32	♐
♑	14; 2	14; 2	12;46	12;46	14;45	14;45	♑
♒	17;52	12;26	15;51	11;10	17;35	12;36	♒
♓	21;57	11;41	19; 6	10;29	20;26	11;49	♓

	Venus				Mercury				
	Ξ	Ω	Γ	Σ	Ξ	Ω	Γ	Σ	
♈	5;10	4; 9	3; 0	10;28	9;58	9;43	23;58	23;38	♈
♉	5; 8	4;16	6;16	9;40	10; 4	10;15	22;15	22;15	♉
♊	5;12	5; 7	9;15	7;36	10;18	11;47	18; 0	16;44	♊
♋	5;36	8;23	9;50	5;59	11;22	15;34	14; 4	12;30	♋
♌	6;16	13; 3	8; 2	5; 5	13;43	19;59	11;25	10;21	♌
♍	7;22	18; 2	6;38	4;54	18;31	23;13	10;21	9;59	♍
♎	7;53	17;43	5;41	4;54	22;49	23;16	9;57	10; 0	♎
♏	8;20	13;47	5;28	4;55	22; 1	22; 1	9;44	10;15	♏
♐	7;49	8; 1	4;39	5;16	18;11	17;25	9;25	11;19	♐
♑	6;52	4; 8	2;43	6;35	13;54	12;10	9;36	14; 5	♑
♒	5;51	3;16	0;30	8;33	11;10	9;50	12;27	17;50	♒
♓	5;22	3;38	−0;24	10;16	10;11	9;20	19;15	21;46	♓

The problem of reconstructing Ptolemy's methods by means of which he computed the tables in Alm. XIII, 10 requires rather complicated discussions. The practical computation of phases, however, presents no difficulties and it will suffice to give one example.

A. Example

We wish to determine the dates of evening setting (Ω) and morning rising (Γ) of Venus near the inferior conjunction which occurs at the beginning of the year Nabonassar 443 (November −305), making use of the longitudes λ of Venus computed in I C 5, 3 B (p. 189 and Fig. 233a, p. 1285).

What we need are, of course, the elongations $\Delta\lambda = \lambda_{\odot} - \lambda$. Hence we compute the longitudes of the sun for the dates under consideration from the tables in Alm. III, 2 and III, 6 and find (cf. p. 189, Table 16):

No.	Nabon. 443	−305	$\lambda_{\odot}$	Venus λ	$\Delta\lambda$
6	I 3	Nov. 10	224;24	241; 3	−16;39
7	8	15	229;32	238;54	− 9;22
8	13	20	234;39	236;14	− 1;35
9	18	25	239;48	233;28	+ 6;20
10	23	30	244;57	231;11	+13;46

According to the tables in Alm. XIII, 10 the elongations which are characteristic for Ω and Γ near $\lambda=240=\text{♐}0°$ are

$$\begin{array}{llll} \lambda=210 & \Omega\colon \Delta\lambda=-13;47 & \Gamma\colon \Delta\lambda=+5;28 \\ 240 & \qquad\quad -\ 8;\ 1 & \qquad\quad +4;39 \\ 270 & \qquad\quad -\ 4;\ 8. & \end{array}$$

By linear interpolation we can determine the values of $\Delta\lambda$ which are required for Ω and Γ for the above given values of λ. One finds

$$\begin{array}{rll} \text{No. } 6 & \Omega\colon \Delta\lambda=-7;53 & \\ 7 & \qquad\quad -8;13 & \\ 8 & \qquad\quad -8;46 & \Gamma\colon \Delta\lambda=+4;45 \\ 9 & & \qquad\quad +4;50 \\ 10 & & \qquad\quad +4;52. \end{array}$$

Again by linear interpolation (or graphically as in Fig. 251) one finds that the planet becomes invisible at No. 7 (Nov. 15) and that it should reappear as morning star at No. 9 (Nov. 25) or perhaps one day earlier (cf. also p. 1285, Fig. 233a). This agrees with the interval given by modern tables (Schoch).

B. Method of Computing the Tables

For the computation of the elongations $\Delta\lambda$ which are characteristic for the planetary phases Ptolemy operates with the basic formula (cf. p. 240)

$$\Delta\lambda=\frac{h}{\sin v}-\beta\cot v \tag{1a}$$

or

$$\Delta\lambda=A-B \qquad A=h/\sin v \qquad B=\beta\cot v \tag{1b}$$

where h is the known arcus visionis (above p. 235), v the angle between the horizon at the given geographical latitude and the ecliptic at $\lambda=k\cdot 30$ ($k=0, 1, \ldots, 11$), β the corresponding latitude of the planet.

In order to follow Ptolemy's method of computation we must also find those angles v which he did not mention explicitly (cf. p. 236); furthermore one must find out how he computed the latitudes β. Both problems were first clearly formulated by A. Aaboe[1] who succeeded in explaining the main features of the tables. In the following we shall somewhat sharpen and extend these results but only by making a detailed analysis of all aspects of the problem.

[1] Aaboe [1960].

1. The Angles between Ecliptic and Horizon

Ptolemy states explicitly[2] that the angles ν between ecliptic and horizon refer to the beginnings of the zodiacal signs. On the other hand the tables (Alm. XIII, 10) are supposed to give the elongations $\Delta\lambda$ also under the assumption that the longitude of the planet is of the form $k \cdot 30$ ($k=0, 1, \ldots, 11$). In other words Ptolemy ignores for the determination of the angles ν the distance $\mathrm{HP}'=\beta \cot \nu = B$ (cf. Fig. 244, p. 1291). This represents a very necessary simplification of the problem because otherwise ν would depend in a complicated fashion on the position of each specific planet[3] and hence on $\Delta\lambda$. But with this simplification Ptolemy is allowed to use the same ν for all planets at the same zodiacal sign.[4] For us it means that we may use the data from whatever planet we wish in order to determine the underlying angle ν.

We first recall two laws of symmetry (cf. above p. 240) which must hold for the angles ν and which, combined with symmetries observed in the tables for $\Delta\lambda$, lead to conclusions about the latitudes β as well as to a derivation of all values of ν underlying the computation of the $\Delta\lambda$'s.

First:

$$\nu(\lambda)=\nu(-\lambda). \tag{2}$$

Hence, if we observe in the tables that $\Delta\lambda(\lambda)=\Delta\lambda(-\lambda)$, as is indeed the case for Saturn and Jupiter, then we can conclude from (1) that also $\beta(\lambda)=\beta(-\lambda)$. We shall make use presently of this consequence (p. 246).

Secondly: If ν and $\bar{\nu}$ denote angles at opposite sides of the horizon respectively, then

$$\nu(\lambda)=-\bar{\nu}(\lambda+180). \tag{3}$$

We apply this relation to the case when the planet at the longitude λ[5] is for the first time visible in the morning (Γ) and again, at the diametrically opposite point,[6] for the last time in the evening (Ω). The angle ν in (1) is $\nu(\lambda)$ for Γ but $-\nu(\lambda)$ for Ω if P′ is at the longitude $\lambda+180$ (cf. Fig. 252). Hence

$$\Delta\lambda_\Gamma(\lambda)=\frac{h}{\sin\nu(\lambda)}-\beta(\lambda)\cot\nu(\lambda), \tag{4a}$$

$$\Delta\lambda_\Omega(\lambda+180)=-\frac{h}{\sin\nu(\lambda)}+\beta(\lambda+180)\cot\nu(\lambda). \tag{4b}$$

Subtracting (4b) from (4a) we obtain

$$\Delta\lambda_\Gamma(\lambda)-\Delta\lambda_\Omega(\lambda+180)=\frac{2h}{\sin\nu(\lambda)}-\cot\nu(\lambda)(\beta(\lambda)+\beta(\lambda+180)). \tag{5}$$

This shows: if the latitudes β are symmetric such that $\beta(\lambda)=-\beta(\lambda+180)$ then $\Delta\lambda_\Gamma(\lambda)-\Delta\lambda_\Omega(\lambda+180)=2A$ is independent of β.

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[2] Alm. XIII, 9 (Manitius II, p. 393).

[3] $|B|$ can reach about 5;30° for Mercury and almost 10° for Venus.

[4] Cf., e.g., p. 235 for all planets, p. 240 (2) for Venus, p. 241 (1) for Mercury.

[5] In Fig. 245, p. 1291 λ is associated with the rising point of the ecliptic, not with the planet.

[6] These are, of course, not coordinated phases Ω and Γ but two independent cases.

We make use now of the previously stated conclusion (p. 245) that for Saturn and Jupiter

$$\beta(\lambda) = \beta(-\lambda) \tag{6}$$

because of the symmetry $\Delta\lambda(\lambda) = \Delta\lambda(-\lambda)$ observed in the tables Alm. XIII, 10. We say that (6) is only possible if the epicyclic anomaly α for which β was computed is zero. Indeed, let us assume $\alpha \neq 0$ and consider, e.g., $\Gamma(\lambda)$ and $\Gamma(-\lambda)$ (cf. Fig. 253). We know that the northernmost point for Saturn and Jupiter has the longitude ♎0° (cf. p. 208). Two points with longitude λ and $-\lambda$ are therefore symmetric with respect to the diameter of maximum inclination of the deferent. This does not hold, however, for the center C of the epicycles. Consequently the arguments of latitude for the two positions of C are symmetric to ♈0°/♎0°, hence (6) is excluded unless $\alpha = 0$.

Knowing now that the latitudes of Saturn and Jupiter were computed under the assumption that $\alpha = 0$ we can also make use of the relation (5) because for $\alpha = 0$ we may expect that

$$\beta(\lambda) \approx -\beta(\lambda + 180). \tag{7}$$

This relation would be exact if the deferent had no eccentricity. But even if we accurately compute β for $\alpha = 0$ we find only small deviations from (7) as in seen in the following table[7]:

λ	$c_5(\omega)$	β_0 Saturn	β_0 Jupiter
♈	−1	−2; 2	−1; 5
♉ ♓	−0;52	−1;45,44	−0;56,20
♊ ♒	−0;30	−1; 1	−0;32,30
♋ ♑	0	0	0
♌ ♐	+0;30	+1; 2	+0;33,30
♍ ♏	+0;52	+1;48,20	+0;58, 4
♎	1	+2; 4	+1; 7

Considering (7) sufficiently well satisfied we see from (5) that for Saturn and Jupiter

$$\Delta\lambda_\Gamma(\lambda) - \Delta\lambda_\Omega(\lambda + 180) = \frac{2h}{\sin \nu(\lambda)} = 2A(\lambda). \tag{8}$$

Since h is given and the $\Delta\lambda$'s are known from the tables we can compute from (8) $\sin\nu$ and hence ν.

[7] Cf. the scheme given p. 219. The coefficients $c_3(0)$ and $c_4(0)$ which one needs for the computation of β_0 are unfortunately not listed in the tables Alm. XIII, 5 which begin, strangely enough, only with $\alpha = 6$. Hence we must compute $c_3(0)$ and $c_4(0)$ by means of the general formulae (above p. 218). In this way one finds that $c(0) \approx c(6)$ or

Saturn: $c_3(0) = 2;4$ $c_4(0) = 2;2$
Jupiter: $c_3(0) = 1;7$ $c_4(0) = 1;5$.

(Only for Saturn does one find $c_4(0) = 2;2,43$, i.e. $> c(6)$ instead of $\leq c(6)$.)

The observational values used by Ptolemy in determining the inclinations i_0 and i_1 (above p. 211), ± 2 and ± 1, respectively, are too inaccurate for the present purpose.

Before doing so, however, we can check the validity of (8) for the cases where $\nu(\lambda)$ is given to us by Ptolemy, i.e. for $\lambda =$ ♈, ♓; ♋, ♑; ♍, ♏. For these angles we can compute $2A = 2h/\sin\nu$ and find

λ	ν	$\sin\nu$	$1/\sin\nu$	A	
				Saturn $h=11$	Jupiter $h=10$
♈ ♓	34;30	0;33,59	1;45,56	19;25,16	17;39,20
♋ ♑	51;30	0;46,57	1;16,41	14; 3,31	12;46,50
♍ ♏	77; 0	0;58,28	1; 1,34	11;17,14	10;15,40

Hence

λ	Saturn				
	Γ		$2A$ computed	Ω	
	ν	$\Delta\lambda$		$\Delta\lambda$	ν
♈ ♓	34;30	21;57	+38;51 −	−11;41	−77; 0
♋ ♑	51;30	14; 2	+28; 7 −	−14; 2	−51;30
♍ ♏	77; 0	10;53	+22;34 −	−16;53	−34;30

λ	Jupiter				
	Γ		$2A$ computed	Ω	
	ν	$\Delta\lambda$		$\Delta\lambda$	ν
♈ ♓	34;30	19; 6	+35;19 −	−10;29	−77; 0
♋ ♑	51;30	12;46	+25;34 −	−12;46	−51;30
♍ ♏	77; 0	10; 1	+20;31 −	−16;12	−34;30

If we take, e.g., for Saturn $\nu = 77;0$ we have

$$\Delta\lambda_{\Gamma}(\text{♍}) - \Delta\lambda_{\Omega}(\text{♓}) = 10;53 + 11;41 = 22;34 = 2A(\text{♍})$$

in agreement with (8). In general A deviates in no case by more than 0;3 from the required value. Such deviations are to be expected as a consequence of roundings in the $\Delta\lambda$'s and of the inaccuracy of trigonometric tables.[8] Hence we are justified in using (8) for the determination of the missing angles ν.

[8] One should not forget that the ancient equivalent of our formula (1) is

$$\Delta\lambda = \frac{2{,}0 \cdot h}{\mathrm{Crd}\, 2\nu} - \beta \frac{\sqrt{4{,}0{,}0 - \mathrm{Crd}^2\, 2\nu}}{\mathrm{Crd}\, 2\nu}.$$

As an example of deviations which actually occurred, we have Ptolemy's discussion of the abnormal behavior of Venus (IC8, 3A) where (3) p. 240 is showing the following differences between his and modern computations: +0;2, +0;1, −0;1, +0;2°.

As an additional control we can utilize $\Delta\lambda$ in ♉ and ♏ for Mars[9] because these signs define the nodal line of the deferent (cf. above p. 218); hence $\beta=0$ and $\sin v = h/\Delta\lambda$.

The following table shows the values of $\sin v$ and v obtained in this way in comparison with the angles v given by Ptolemy or restored by Aaboe[10]

λ	$\sin v$			v				
	♄	♃	♂	♄	♃	♂	Ptol.	Aaboe
♈	0;32,30,44[11]	0;32,40, 4		32;49	32;59			33; 0
♉ ♓	0;33,59,29	0;33,59,39	0;33,57,42	34;31	34;31	34;28	34;30	
♊ ♒	0;39,28,53′	0;39,31, 0		41; 9	41;12			41;10
♋ ♑	0;47, 1,51	0;46,59,50		51;37	51;34		51;30	
♌ ♐	0;55, 0, 0	0;54,57,42		66;26	66;21			66;30
♍ ♏	0;58,29,36	0;58,32,11	0;58,28,28	77; 8	77;19	77; 3	77; 0	
♎	0;59,16,53	0;59,12,38		81; 7	80;42			80;50

It is possible to improve the restored values a little and at the same time to obtain some information about the geographical latitude φ of "Phoenicia" for which the inclinations v were computed.

We denote the equinoctial angles by v_1 and v_2, respectively. Then we know (cf. Fig. 254) that

$$90-\varphi = v_1+\varepsilon = v_2-\varepsilon \tag{9}$$

ε being the obliquity of the ecliptic. The numbers in this relation are known to us only within certain limits. We therefore write

$$\begin{aligned} \varphi &= 33;\ 0+x \qquad & v_1 &= 33;\ 0+x_1 \\ \varepsilon &= 23;50+y \qquad & v_2 &= 80;50+x_2 \end{aligned} \tag{10}$$

and try to determine the corrections x, y, x_1, x_2. For ε we can assume that y is either 0, or 0;1,20 or 0;10. The second value is so near to the first one that we may restrict y to 0 and 0;10.

It follows from (9) that

$$\begin{aligned} v_1+v_2 &= 2(90-\varphi) \\ v_2-v_1 &= 2\varepsilon \end{aligned}$$

[9] The MSS give for $\Delta\lambda_r(♉)$ either 20;8 or 20;16 but $\Delta\lambda_\Omega(♏)=20;19$. The first value, though accepted by Heiberg and Manitius, cannot be correct since it would mean that $v=34;50$ instead of Ptolemy's 34;30. For 20;16 one finds $v=34;34$ but $v=34;28$ for 20;19 which I therefore use for Table 24, p. 243.

[10] Aaboe [1960], p. 7.

[11] The first entry of Saturn $\Delta\lambda_r=23;1$ is definitely wrong as the computation shows (cf. below Table 25, p. 251). Obviously one has to accept the variant 23;30 given by Halma H.T. III, p. 30; cf. also the MSS D and K. An emendation 23;[2]1 would agree better with computation and also with Jupiter ($\sin v = 0;32,37,58$, hence $v=32;57$).

hence from (10)

$$v_1+v_2=113;50+x_1+x_2=114-2x$$
$$v_2-v_1=\ 47;50+x_2-x_1=47;40+2y$$

or

$$\begin{aligned} x_1+x_2&=+0;10-2x \\ x_2-x_1&=-0;10+2y. \end{aligned} \tag{11}$$

If we assume $y=0$ (i.e. the accurate value for ε) we find

$$\begin{aligned} x_1&=0;10-x \quad \text{hence} \quad v_1=33;10-x \\ x_2&=-x \qquad\quad \text{hence} \quad v_2=80;50-x. \end{aligned} \tag{12a}$$

For $y=0;10$, however, i.e. if $\varepsilon=24$, we get

$$\begin{aligned} x_1&=-x \qquad\quad \text{hence} \quad v_1=33;0-x \\ x_2&=0;10-x \quad \text{hence} \quad v_2=81;0-x. \end{aligned} \tag{12b}$$

We can decide between the two possibilities (12a) and (12b) by looking at the values v deducible from the $\Delta\lambda$'s (p. 248). Obviously

$$v_1=32;55 \qquad v_2=80;55 \tag{13}$$

would appear a proper set of values. This excludes (12a) but fits excellently (12b) with $x=0;5$. Hence we know now

$$\varepsilon=24 \qquad \varphi\approx 33;5.^{12} \tag{14}$$

Finally we can completely free ourselves from considering the visibility tables by making use of Ptolemy's explicit statement that the computations concern "Phoenicia where the longest day is 14 1/4^h." Substituting in the relation (cf. above p. 37 (4))

$$-\tan\varphi=\cos\frac{M}{2}\cot\varepsilon$$

$M=14;15^h=213;45°$ and $\varepsilon=24°$ one finds $\varphi=33;6,15$. Hence we see that

$$M=14;15^h \quad \text{i.e.} \quad \varphi=33;6° \qquad \varepsilon=24°$$

underlies the computation of the angles v.

It is important to realize that this result made use only of the relation (9) which hold for the equinoctial angles v_1 and v_2, no matter how the angles for other longitudes were computed. This leaves us with the problem of detecting the method of computation for the remaining angles. We have already seen that linear interpolation in the tables of the Almagest does not explain Ptolemy's values.[13] The only alternative we can check is the use of the trigonometric methods of the Almagest for the determination of the angles between the ecliptic

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[12] This fits also very well Ptolemy's geographical data for Phoenicia; cf. above p. 44, Table 2, No. 10 ($\varphi=33;18$) and Geogr. V 15, 5 (ed. Nobbe, p. 58). Babylon, however, is given a latitude of 35° (Geogr. V 20, 6 ed. Nobbe, p. 78).

[13] Above p. 236.

and the horizon ($\varphi = 33;6$), accepting $\varepsilon = 24°$. These computations are rather involved, if carried out accurately,[14] and yield results which deviate only little from the values obtainable by linear interpolation and therefore do not explain the numbers used in the visibility tables:

λ	comp.	Alm.
0	32;54	[32;54]
30	35; 1	34;30
60	41;46	[41;10]
90	53;17	51;30
120	66;46	[66;25]
150	77;11	77; 0
180	80;54	[80;54]

Hence the conclusion seems to me inevitable that the angles v for the planetary phases were not only based on the round value $\varepsilon = 24$ but also computed with methods more primitive than the tables Alm. II, 8 and II, 13. I see no way to a reconstruction of these (earlier?) methods.

2. The Outer Planets

We are now in a position to recompute the tables of the phases for Saturn and Jupiter since we have all angles v and since we know that $\beta = \beta_0$, i.e. the latitude for the epicyclic anomaly $\alpha = 0$ (cf. p. 246). The use of β_0 is fully justifiable by the smallness of the epicycles of the two outermost planets. Although α at Ω and Γ is actually different from zero it is clear that the latitude does not change appreciably on either side of the conjunction ($\alpha = 0$). Table 25 shows on the example of $\Delta\lambda_\Gamma$ of Saturn the good agreement between modern computation using β_0 and the tables in Alm. XIII, 10.

For Mars the situation is more involved. The tables no longer possess exactly the symmetry $\Delta\lambda(\lambda) = \Delta\lambda(-\lambda)$ and hence we cannot directly conclude anything about the β's. In fact such symmetry cannot be expected for Mars since its latitudes could only be symmetric about the diameter ♌0°/♒0° (cf. above p. 218)

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[14] In order to apply

$$\sin v = \sin r_1 / \sin s_2$$

with $r_1 = 90 - \varphi - \delta(\overline{\mathrm{M}})$ and $s_2 = \lambda(\overline{\mathrm{M}}) - \lambda(\mathrm{H})$ ($\mathrm{M} = \overline{\mathrm{M}} + 180$ the culminating point of the ecliptic, H the rising point; cf. above p. 46) one has to find δ from p. 31 (2)

$$\sin \delta = \sin \varepsilon \cdot \sin \lambda$$

and M from p. 42 (6)

$$\alpha(\mathrm{M}) = \rho(\mathrm{H}) - 90$$

hence $\lambda(\mathrm{M})$ from p. 32 (5)

$$\tan \lambda = \tan \alpha / \cos \varepsilon.$$

The oblique ascension for our φ is obtainable from p. 36 (1) and (2)

$$\rho(\lambda) = \alpha(\lambda) - n_2(\lambda)$$

where $\sin n_2(\lambda) = \tan \varphi \cdot \tan \delta(\lambda)$.

The results come very close to the values obtainable by means of modern tables (e.g. van der Waerden [1943], p. 52–54).

Table 25. Saturn Γ

λ	v	A	cot v	β_0	B	$\Delta\lambda_\Gamma$		Alm.-$(A-B)$
						$A-B$	Alm.	
♈	[32;54]	20;15, 5	1;32,46	−2; 2	−3; 8,38	23;24	23;[30]	+0;6
♉ ♓	34;30	19;25,14	1;27,18	−1;45,44	−2;33,51·	21;59	21;57	−0;2
♊ ♒	[41;10]	16;42,40	1; 8,37	−1; 1	−1; 9,46	17;52	17;52	0
♋ ♑	51;30	14; 3,33	0;47,44	0	0	14; 4	14; 2	−0;2
♌ ♐	[66;25]	12; 0, 9	0;26,12	+1; 2	+0;27, 4	11;33	11;34	+0;1
♍ ♏	77; 0	11;17,22	0;13,51	+1;47,28	+0;24,48	10;53	10;53	0
♎	[80;54]	11; 8,25	0; 9,37	+2; 4	+0;19,52	10;49	10;48	−0;1

Table 26. Mars Γ

λ	v	A	cot v	β_0	B	$\Delta\lambda_\Gamma$		Alm.-$(A-B)$
						$A-B$	Alm.	
♈	[32;54]	21;10,18	1;32,46	−0;1,30	−0;2,29	21;13	21;12	−0;1
♉	34;30	20;18,12	1;27,18	0	0	20;18	20;16	−0;2
♊	[41;10]	17;28,14	1; 8,37	+0;4	+0;4,34	17;24	17;21	−0;3
♋	51;30	14;41,40	0;47,44	+0;6,56	+0;5,31	14;36	14;33	−0;3
♌	[66;25]	12;32,53	0;26,12	+0;8	+0;3,30	12;29	12;28	−0;1
♍	77; 0	11;48, 9	0;13,51	+0;6,56	+0;1,36	11;47	11;46	+0;1
♎	[80;54]	11;38,48	0; 9,37	+0;4	+0;0,38	11;38	11;38	0
♏	77; 0	11;48, 9	0;13,51	0	0	11;48	11;48	0
♐	[66;25]	12;32,53	0;26,12	−0;1,30	−0;0,26	12;33	12;34	+0;1
♑	51;30	14;41,40	0;47,44	−0;2,36	−0;2, 4	14;44	14;45	+0;1
♒	[41;10]	17;28,14	1; 8,37	−0;3	−0;3,26	17;32	17;35	+0;3
♓	34;30	20;18,12	1;27,18	−0;2,36	−0;3,47	20;22	20;26	+0;4

Table 27. Mars Ω

λ	v	A	cot v	β_0	B	$\Delta\lambda_\Omega$		Alm.-$(A-B)$
						$A-B$	Alm.	
♈	−[80;54]	−11;38,48	−0; 9,37	−0;1,30	+0; 0,14	−11;39	−11;40	−0;1
♉	− 77; 0	−11;48, 9	−0;13,51	0	0	−11;48	−11;48	0
♊	−[66;25]	−12;32,53	−0;26,12	+0;4	−0; 1,45	−12;31	−12;30	· +0;1
♋	− 51;30	−14;41,40	−0;47,44	+0;6,56	−0; 5,31	−14;36	−14;33	+0;3
♌	−[41;10]	−17;28,14	−1; 8,37	+0;8	−0;12, 9	−17;19	−17;19	0
♍	− 34;30	−20;18,12	−1;27,18	+0;6,56	−0;10, 5	−20; 8	−20; 5	+0;3
♎	−[32;54]	−21;10,18	−1;32,46	+0;4	−0; 6,11	−21; 4	−21; 1	+0;3
♏	− 34;30	−20;18,12	−1;27,18	0	0	−20;18	−20;19	−0;1
♐	−[41;10]	−17;28,14	−1; 8,37	−0;1,30	+0; 1,43	−17;30	−17;32	−0;2
♑	− 51;30	−14;41,40	−0;47,44	−0;2,36	+0; 2,24	−14;44	−14;45	−0;1
♒	−[66;25]	−12;32,53	−0;26,12	−0;3	+0; 1,19	−12;34	−12;36	−0;2
♓	− 77; 0	−11;48, 9	−0;13,51	−0;2,36	+0; 0,36	−11;49	−11;49	0

and not about ♈0°/♎0°, the axis of symmetry for the angles ν (cf. (2), p. 245). It is only a lucky accident that these two axes coincide for Saturn and Jupiter.

But it is easy to determine the values of β used by Ptolemy in computing the tables in Alm. XIII, 10. With h, ν, and $\Delta\lambda$ known one can obtain β from

$$\beta = (A - \Delta\lambda) \tan \nu$$

and finds that all β's are very small. This is exactly what we can expect for $\alpha = 0$ (cf. Fig. 213, p. 1275) and it is also clear that the latitudes remain practically constant for a considerable range near $\alpha = 0$. Hence we compute the $\Delta\lambda$'s by again using $\beta = \beta_0$[1]. The values of this are symmetric with respect to ♌ whereas the values A are symmetric with respect to ♎. The resulting slightly asymmetric values for $\Delta\lambda$ agree excellently with the tables in the Almagest[2] (cf. Table 26 and 27).

3. The Inner Planets

We have seen that the elongations for the phases of the outer planets were computed with the latitudes β_0, i.e. with β for $\alpha = 0$. From a theoretical viewpoint, however, the use of β_0 implies a logical contradiction: $\alpha = 0$ means that the planet has the same longitude as the mean sun. Hence the elongation $\Delta\lambda$ of the planet from the true sun would simply be the solar equation, a quantity which has nothing to do with planetary visibility. Obviously the assumption $\alpha = 0$ can be taken only as a convenient approximation for the computation of β, justifiable by the fact that β varies very little near $\alpha = 0$.

The computation of latitudes from given epicyclic anomalies α raises another problem of principle: is it really true that the elongations $\Delta\lambda$ for the phases can be found exclusively from the normal arcus visionis (above p. 234) and the angles ν (which are independent of planetary theory)? Or are there new empirical data involved in the determination of the α's?

The formulation of the problem as a sharp alternative is, however, not quite adequate. To arbitrary selected α's (and given planetary longitude λ) one can determine the corresponding κ_0's and elongations $\Delta\lambda$ from the true sun as well as the latitudes β. Hence, by varying α one must be able to find a pair $\Delta\lambda$ and β such that the relation (1), p. 253 is satisfied. This shows that theoretically no new data are necessary to compute the elongations for the phases. In practice, however, this process would be by far too complicated and some definite values of α must be selected which directly give the latitude β for the planet in the horizon. For the outer planets the whole problem is circumvented by the assumption that β does not vary with α near $\alpha = 0$. For Venus we shall show that a constant value of α was chosen (of course different near superior and inferior conjunction) and it seems plausible that the same is true for Mercury. We do not know, however, how these values of α were determined and at least the possibility that new empirical data were introduced cannot be excluded.

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[1] For the computation of β_0 we need $c_3(0)$ and $c_4(0)$ not given in the tables (cf. above p. 246, note 7). I found $c_3(0) = 0;8$ $(= c_3(6))$ and $c_4(0) = 0;3$.

[2] The best of the attested values for Γ in ♉ is 20;16. It should be the same as for Ω in ♏ which is 20;19 and which shows better agreement; cf. also above p. 248, note 9.

Venus. We know that Ptolemy operated with the formula

$$\Delta\lambda = A - \beta \cot v. \tag{1}$$

From it we can deduce the values

$$\beta = (A - \Delta\lambda) \tan v \tag{2}$$

which we wish to explain. Since the superior and inferior conjunctions lie between the morning and evening phases it seems reasonable to compare the latitudes of the phases as derived from (2) with the latitudes at the conjunctions, i.e. at $\alpha = 0$ and $\alpha = 180$. The result is shown in Fig. 255,[3] computed, as always, for the longitudes $\lambda = k \cdot 30$. The general symmetry of the curves for the latitudes of the phases with respect to the latitudes at conjunction suggests fixed values of α for the phases, symmetric to $\alpha = 0$ and $\alpha = 180$, respectively. A systematic investigation of the dependence of β on α confirms this conjecture. As will be shown presently the proper values of α are about $\pm 13°$ near superior conjunction and about $180 \pm 2°$ near inferior conjunction.

It is of interest to notice that these anomalies localize the phases in such a fashion on the epicycle that Ω and Ξ on the one side, Γ and Σ on the other side of OC appear from O in the same direction (cf. Fig. 243[4], p. 1290). Indeed, one finds in Alm. XI, 11 the equation of center (at mean distance) 5;25° for $\alpha = 13$ and 5;5° for $\alpha = 178$, i.e. essentially the same equation as it should be for points seen from O on a straight line. It seems to me likely that such a geometrical consideration played a role in the choice of one of the α's.

Before discussing the computation of tabulated values $\Delta\lambda$ a few general remarks may be added. In computing latitudes one needs not only the epicyclic anomaly α but also the distance κ_0 of the center of the epicycle from the apsidal line[5] (cf. Fig. 256). One might think that it would suffice to assign to κ_0 the same differences of 30° as to the longitudes of the planet, i.e. to assume a fixed distance of C from OP or in other words to ignore the eccentricity of the deferent. It turns out that this is correct for Σ and Ξ (i.e. near superior conjunction) but that serious differences appear for Ω and Γ (near inferior conjunction). The values of β and hence of $\Delta\lambda$ are far more sensitive against variations in κ_0 than in α. Hence it is clear that after α had been chosen Ptolemy computed for each λ accurately the corresponding κ_0 and from it with the full accuracy of his theory of latitudes the values of β which determine the $\Delta\lambda$.

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[3] The fact that the three curves intersect near ♍ and ♓ is not surprising. At ♌25 and ♒25 the center C of the epicycle lies in the ecliptic and the nodal line of the epicycle is perpendicular to OC. Hence at $\alpha = 0$ or $\alpha = 180$ the rim of the epicycle is practically parallel to the ecliptic and the latitude will vary only little (cf. p.   , Fig. 219b). Thus

$$\beta_\Xi \approx \beta_\Sigma \approx \beta(0) \approx \pm 1° \quad \text{in} \begin{cases} ♍ \\ ♓ \end{cases} \quad \text{and} \quad \beta_\Gamma \approx \beta_\Omega \approx \beta(180) \approx \pm 6;20° \quad \text{in} \begin{cases} ♓ \\ ♍ \end{cases}$$

(cf. p. 215 (2); p. 226 (2); p. 239). Cf., however, also the subsequent discussion on p. 254.

[4] This does not imply that the longitudes of the phases are symmetric to the longitudes of the mean conjunctions since Fig. 243 does not show the influence of the latitudes on the longitudes. Cf. for this, e.g., the discussion of the "paradoxical" phases of Venus, above p. 239.

[5] Cf. above p. 223.

Venus at Σ and Ξ. Above, p. 238 we have seen that Ptolemy found $\beta \approx +1°$ for Venus at Ξ in ♋0° on the basis of an epicyclic anomaly of $\alpha \approx 13°$. We also shall discuss presently (p. 259) textual evidence, independent from the Almagest, that assigns Venus at Ξ and Σ the anomaly $\alpha = \pm 12;24$. Of course the tables for latitudes (Alm. XIII, 5) and for $\Delta\lambda$ (Alm. XIII, 10) are not accurate enough to distinguish between $\alpha = \pm 13$ and $\alpha = \pm 12;24$ on the basis of final results. Hence it will suffice to check the hypothesis of a constant anomaly α with the round values $+13°$ for Ξ and $-13°$ for Σ.

In Alm. XI, 11 we find for $\alpha = 13$ at the apogee (♉25) the equation $\theta(\alpha) = 5;26 - 0;3 = 5;23$. Thus the center C of the epicycle will be in ♉24;37 ≈ ♉25 when Venus in ♊0° at Ξ is ahead of the sun (cf. Fig. 257), but in ♊5;23 when the sun is ahead of Venus at Σ. In the first case we have $\kappa_0 \approx 0$ because ♉25 is the apogee of the deferent; in the second case $\kappa_0 \approx 10$. With $\alpha = 13$ and $\kappa_0 = n \cdot 30$ and with $\alpha = -13$ and $\kappa_0 = 10 + n \cdot 30$ ($n = 0, 1, \ldots, 11$) we can compute all latitudes β and hence the elongations $\Delta\lambda$. This leads to excellent agreements with Ptolemy's tables.[6]

Venus at Ω and Γ. We have no direct information about the values of α which were used by Ptolemy in the computation of the tables Alm. XIII, 10,[7] but as before one can determine relatively accurately the underlying latitudes.[8] If one then takes arbitrary values of α near 180, e.g. $\alpha = 181$, 182, etc., one can find the corresponding positions of the center of the epicycle (i.e. the angles κ_0) and finally the β's from α and κ_0. This procedure leads to the result that only an α near to 180 ± 2 can produce the proper latitudes which in turn lead to elongations sufficiently close to the tabulated values.

For ♍ and ♓, however, $\alpha = 180$ itself must have been used. This is particularly evident for Γ in ♓ where the tables give $\Delta\lambda = -0;24$. Only $\alpha = 180$ leads to the required $\beta \approx 6;18,30$ whereas $\alpha = 182$ gives $\beta = 6;3,5$ and hence $\Delta\lambda \approx +0;1$ instead of the exceptional $-0;24$.[9] Also for Γ in ♍ and Ω in ♓ and ♍ the use of $\alpha = 180$ produces exactly the tabulated $\Delta\lambda$ through the deviations for $\alpha = 182$ or 178 are not so outspoken.[10]

For all remaining cases we compute with $\alpha = 178$ (at Ω) or $\alpha = 182$ (at Γ). In the tables for the planetary equations (Alm. XI, 11) one finds for the mean distance $c_6(\alpha) = 5;5,20$. From this one can estimate (as before, p. 254 and Fig. 257) that $\kappa_0 \approx 10°$ for Γ in ♊0° and $\kappa_0 \approx 0°$ for Ω in ♊0°. But for the computation of the latitudes one needs more accurate values of κ_0. At the apogee of the epicycle the equation is $\theta(\alpha) = c_6(\alpha) - c_5(\alpha) = 5;5,20 - 0;20,40 = 4;44,40$. The equation increases to $c_6(\alpha) - c_8(30) c_5(\alpha) = 5;5,20 - 0;52,55 \cdot 0;20,40 = 4;47,6$ at ♊25, etc., until $c_6(\alpha) + c_7(\alpha) = 5;28,40$ at ♏25. This gives us the equations for Ω. For Γ we know

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6 The maximum deviations are: for Σ once $+0;9°$, for Ξ twice $+0;7°$. The arithmetical mean of the deviations is $+0;1°$ for Ξ, zero for Σ.

7 The rounded values ($\pm 6;20$) for the latitudes at Ω and Γ in ♓ and ♍ used by Ptolemy in explaining the greatly variable duration of invisibility of Venus (above p. 239) are not accurate enough for the present purpose. Indeed the tables require the latitudes $\beta_\Gamma \approx 6;18$ $\beta_\Omega \approx 6;29$ in ♓ and $\beta_\Gamma \approx -6;30$ $\beta_\Omega \approx -6;20$ in ♍.

8 Cf. p. 253 (2).

9 For the significance of a negative elongation at Γ cf. above p. 241.

10 For Γ in ♍ $+0;6$, for Ω $+0;4$ and $-0;11$, respectively.

that C is about 10° farther ahead than at Ω, hence we interpolate accordingly between the equations obtained for Ω. This gives the following table

λ		θ at Ω	θ at Γ	
♊		$-4;44,40$	4;45,9	
♋	♉	$-4;47,\ 6$	4;49,33	4;45,55
♌	♈	$-4;54,26$	4;57,53	4;53, 0
♍	♓	$-5;\ 4,46$	5; 8,37	5; 1,19
♎	♒	$-5;16,19$	5;19,19	5;12,28
♏	♑	$-5;25,19$	5;26,26	5;24,19
♐		$-5;28,40$	5;27,33	

From it we can immediately obtain the angles κ_0 and hence compute β and finally $\Delta\lambda$.

The results are as good as can be expected in computations which require many interpolations without any possibility of evaluating the effects of ancient rounding errors. If we ignore the two cases ♍ and ♓ already discussed (p. 254) we find for Ω in 6 of the remaining 10 cases errors which are within the range of $\pm 0;10°$, 3 within $\pm 0;15°$ and once (in ♌) $+0;29°$. The arithmetical mean of all errors is $+0;1°$. For Γ four cases remain within $\pm 0;10°$, four within $\pm 0;20°$, once (in ♉) the deviation is 0;23° and in ♈ even 0;58°. This last discrepancy would disappear if one could emend the $\Delta\lambda = 3°$ of the text to $\Delta\lambda = 2°$; the corresponding $\beta \approx +4;40$ would also fit the curve for Γ in Fig. 255, p. 1295 much better. Without any such correction the mean value of the deviations for Γ is $+0;8°$, with it again only $+0;1°$.

Mercury. If one had only the tables in Alm. XIII, 10 one would not realize that Ptolemy was aware of the fact that under certain conditions a pair of phases would be omitted. Indeed, the tables give for every zodiacal sign and for every phase a $\Delta\lambda$ which is characteristic for the phase and only if one also looks at the table of maximum elongations of Mercury (Alm. XII, 10) does one see that the $\Delta\lambda = -22;1$, required for Ξ and Ω in ♏, exceeds the maximum elongation ($-20;58$) in this sign and similarly for Σ and Γ (22;15) in ♉ (22;13). It is exactly this situation which Ptolemy described in his discussion of the "paradoxical" behavior of Mercury (above p. 241).

A modern compiler of tables would have given no entries at all for Ξ and Ω in ♏ and for Σ and Γ in ♉. Hence the question arises how the numerical values ($-22;1$ and 22;15, respectively) originated which Ptolemy listed as required elongations $\Delta\lambda$. Now if one computes the latitudes β which belong to these $\Delta\lambda$ according to the relation (2), p. 253, one finds $\beta = -2;59,51 \approx -3;0$ in ♏ and $\beta = -3;9,28 \approx -3;10$ in ♉ and these are exactly the latitudes obtained by Ptolemy for Mercury at maximum elongation in ♏ and ♉.[1] Hence, starting from the latitudes at maximum elongation Ptolemy computed a $\Delta\lambda$ as always with (1), p. 253 and put it in the tables although the planet can never reach such a distance from the sun and hence remains invisible (cf. Fig. 258).

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[1] Cf. above p. 241 and note 11 there.

Having disposed of the exceptional cases in which no position of the planet exists on the epicycle to reach the elongation necessary for visibility we now ask how Ptolemy computed the latitudes β in the remaining cases.[2] As with Venus we compare them with the latitudes found for $\alpha = 0$ and $\alpha = 180$ (cf. Fig. 259[3]). Unfortunately there is no longer a clear symmetry recognizable between the latitudes at the phases in comparison with the latitudes at superior and inferior conjunction.

From all the preceding experience it seems very unlikely that Ptolemy operated alone for Mercury with variable α. We have furthermore an explicit reference to a fixed anomaly for Ξ and Σ ($\alpha = \pm 38°$) in the same texts which gave for Venus $\alpha = \pm 12;24°$ and which we could confirm by computation as essentially correct for the tables Alm. XIII, 10.[4] Unfortunately the same cannot be said for Mercury. Fig. 260 illustrates the situation in the example of the latitudes for Σ. The full drawn curve represents the required latitudes as derived from Alm. XIII, 10 (the same as in Fig. 259). Computing the latitudes for $\alpha = -38$ one gets good agreement from ♌ to ♒ but very bad results for the remaining signs. On the other hand if one tries to obtain the correct latitudes for the first half of the zodiac one needs an anomaly near -102, which, however, yields very bad values for the other signs. The deviations in $\Delta\lambda$ are somewhat modified by the influence of cot v such that the worst error for $\alpha = -38$ is located in ♈ (about 2°) and for $\alpha = -102$ in ♐ (about 1;10°). Obviously a constant α near -70 would reduce these extrema but it is also clear that the agreement with the tables will be nowhere as good as for the other planets.

Hence it seems as if the assumption of constant α does not suffice to explain the $\Delta\lambda$'s for Mercury at Σ and Ξ and the chances for Γ and Ω are even smaller since the latitudes near inferior conjunction are even more sensitive to variations of α and κ_0 than for Σ and Ξ. We have to admit that the last chapter of the Almagest is the least understood in the planetary theory.

5. The Planetary Phases in the Handy Tables and Other Sources

Ptolemy's discussion at the end of Book XIII of the Almagest leaves little room for doubt that it was here that for the first time the concept of the "arcus visionis" was made the basis for the computation of the planetary phases. Nevertheless, the tables in Alm. XIII, 10 do not represent Ptolemy's last preoccupation

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[2] There are some corrections to be made in these tables Alm. XIII, 10. For Ξ at ♋0° Ptolemy used $\Delta\lambda = 11;30$ as empirically given (cf. above p. 235 (1)) and he derived from it the normal arcus visionis h of the planet. Hence the $\Delta\lambda = 12;22$ of the tables (in all MSS) must be emended to 11;22. For Ξ at ♏0° one must use the reading 22;1 of the MSS H and K because the version 20;1 (accepted by Heiberg) would be smaller than the maximum elongation 20;58 given for ♏0° in Alm. XII, 10. Hence the planet would become visible, contrary to Ptolemy's discussion in the text (cf. above p. 241). For Γ in ♎0° Heiberg accepted the reading 9;51. The trend of the latitudes, however, shows that the reading 9;54 of D is preferable; still better is 9;57 of H and K. In ♏0° for Σ the reading 10;15 of D is better than 10;19 accepted by Heiberg. For Ω in ♓0° the value 9;43 leads to a grossly wrong latitude in contrast to 9;20 given by H.

[3] The omitted cases are marked by a ×.

[4] Above p. 254; below p. 259.

with this problem. In the "Handy Tables" he removed the restriction to the geographical latitude of Phoenicia and tables for each of the seven climates are given. Since their way of computation contrasts significantly with the methods used in the Almagest we shall summarize here the new procedures, again following the lead of results obtained by Aaboe [1960]. The discussion of the numerical details, however, we postpone to a later chapter.[1]

That new principles of computation are involved is evident if one takes the data for clima III and IV and compares them with the tables in Alm. XIII, 10. Fig. 261 shows as an example the case of Venus; obviously there exists no possibility of obtaining the tables in the Almagest, which concern a latitude halfway between clima III and IV, by interpolation from the Handy Tables.

There are several causes for this divergence. As we have seen[2] the angles v between ecliptic and horizon for the tables Alm. XIII, 10 cannot be derived from the angles tabulated in Alm. II, 13. The Handy Tables, on the contrary, are based on the very tables of Alm. II, 13.[3] This furthermore implies the use of the accurate value $\varepsilon = 23;51$,[4] not $\varepsilon = 24°$.[5]

A second fundamental change consists in the adoption of new values for the arcus visionis h, values which were also adopted in Ptolemy's "Planetary Hypotheses"[6]

	Handy Tables	Almagest
Saturn:	13°	11°
Jupiter:	9	10
Mars:	14;30	11;30
Venus:	Γ, Ω: 5 Σ, Ξ: 7	Ξ: 5
Mercury:	12	Ξ: 10

One can only conjecture that Ptolemy derived these values from more recent observations whereas he had utilized older data in the Almagest.

Thirdly, the computation of the planetary latitudes is greatly simplified. We have shown (above p. 250ff.) that the tables for Mars used variable latitudes which, however, never reach as much as 0;10° (cf. p. 251, Tables 26 and 27). The Handy Tables simply assume $\beta = 0°$, hence compute with $\Delta\lambda = h/\sin v$.[7] For the remaining planets the latitudes are computed (of course with the parameters of the Handy Tables[8]) under greatly simplifying assumptions: (a) the center of the epicycle is at mean distance, (b) the epicyclic anomaly is either zero or 180 (for an inner planet at Γ and Ω), and hence (c) the longitude of the planet is the same as the longitude of the center of the epicycle, i.e. ♈0°, ♉0°, ... In other words the inner planets are now treated in the same fashion as Saturn and Jupiter in the Almagest. It seems quite possible that these simplifications do more than

[1] Below p. 1017ff.
[2] Above p. 236; p. 245ff.
[3] Aaboe [1960], p. 12.
[4] For all climates $v(♎) - v(♈) = 47;42 = 2\varepsilon$ holds (cf. Table 3, p. 47).
[5] Above p. 249 (14).
[6] Cf. Goldstein [1967], p. 9 and below p. 1017.
[7] Aaboe [1960], p. 12.
[8] Cf. below p. 1006ff.

outweigh the improvements which are presumably the purpose of introducing new values for the arcus visionis.

The tables for the phases at the seven climates do not exhaust the material concerning the planetary phases in the extant copies of the Handy Tables. One obviously later addition is computed (with the same methods as the main tables[9]) for the geographical latitude of Byzantium.

Added to this uniform group of tables, however, is in all MSS one more table which is closely related to the tables in the Almagest. It is known through Halma's publication[10] and through the variants given by Heiberg in his apparatus to Alm. XIII, 10[11] under the sigla H (for Marc. gr. 303) and K (for Vat. gr. 1291). For the outer planets and for Mercury we are undoubtedly dealing with the tables from the Almagest and are thus entitled to use variants in the ordinary fashion. For Venus, however, this is not the case. Although based on the methods of the Almagest the phases of Venus are computed with somewhat different parameters (cf. the dotted curve in Fig. 262) and therefore should have been excluded from the variants in the tables of Venus in Alm. XIII, 10.

A cursory investigation of these tables for Venus makes it quite clear that the geographical latitude is the latitude of Alexandria. It seems likely that the angles v agree now with Alm. II, 13 but that the latitudes β are not quite identical with the β's in Alm. XIII, 10. Such a table is of more than accidental interest since it demonstrates the existence of tables which belong neither to the final form of the Almagest nor to the Handy Tables proper in the Theonic version known to us.

Ptolemy's approach to the visibility problem of the planetary phases, characterized by the use of the arcus visionis as the decisive element, is by no means the only one known to us from Greek sources. The most primitive version of visibility conditions consists in the assumption of fixed round values of elongations from the sun. Porphyry (who wrote in the second half of the third century A.D.[12]) gives[13] the following limits:

Saturn and Jupiter:	9°
Mars:	8
Venus and Mercury:	7

whereas Rhetorius (around A.D. 500[14]) assumes 10° as necessary elongation[15] for all three outer planets. For the inner planets he (or rather his source) distinguishes between the morning and evening phases but the numbers in the extant text are so badly garbled that no reasonable data can be established. Again Ptolemy's great methodological superiority is evident when one remembers that

[9] Cf. below p. 1024.

[10] Halma, H.T. III, p. 30 to 32.

[11] Ptolemy, Opera I, 2, Heiberg, p. 606/607.

[12] Cf., e.g., Cumont [1934].

[13] CCAG 5, 4, p. 228, 15 to 19; cf. also below V A 3, 2.

[14] In CCAG 8, 4, p. 180, 19 and 29 the longitudes of Aldebaran and of Antares are given, respectively as Taurus and Scorpio 16;20°, i.e. 3;40° greater than in the Almagest (VII, 5/VIII, 1; cf. also p. 980). Hence the epoch is A.D. 138 + 366 = 504 (Cumont [1918], p. 43). We also have horoscopes in the works of Rhetorius which confirm this date; cf. Neugebauer-Van Hoesen, Gr. Hor., p. 187f.

[15] CCAG 7, p. 214 to 224.

he gives limiting elongations only under accurately specified observational conditions.[16]

Much nearer to Ptolemy's methods seem to be the data which are known from Byzantine manuscripts, in part dated by examples to the 5th and 6th centuries.[16a] In these texts the planetary phases are related to the epicyclic anomaly α. The numbers are marred by copyist errors but nevertheless can be restored securely on the basis of symmetries and parallels. Thus one finds for the outer planets at Ω and Γ the following values for α:

$$\text{Saturn: } \pm 17^\circ \quad \text{Jupiter: } \pm 16^\circ \quad \text{Mars: } \pm 42^\circ. \tag{1}$$

The phases of the outer planets were computed in the Almagest for the latitudes β_0 which correspond to $\alpha = 0^\circ$ (cf. above p. 250). Yet, the actual phases must correspond to anomalies different from zero and one could therefore expect that the above given values (1) agree at least with a mean value of the elongations found in Ptolemy's tables (Alm. XIII, 10). Ignoring eccentricities one should have approximately (cf. Fig. 263)

$$\Delta\lambda_\Gamma = \alpha - c_6(\alpha) \tag{2}$$

where $c_6(\alpha)$ can be found in Alm. XI, 11. Denoting the mean values of the elongations given in the tables Alm. XIII, 10 by $\overline{\Delta\lambda}$ we obtain from (1) and (2) the following comparison

	α	$c_6(\alpha)$	$\Delta\lambda_\Gamma$	$\overline{\Delta\lambda}$
Saturn:	17°	1;40°	15;20°	15;35°
Jupiter:	16	2;33	13;27	13;55
Mars:	42	16;29	25;31	15;32.

For the two outermost planets the agreement is as good as one may expect for mean values. For Mars, however, $\alpha = 42^\circ$ is much too great; about $\alpha = 26^\circ$ would be required.

For the inner planets no fixed anomalies are assigned near inferior conjunction (Ω and Γ) but for the superior conjunction (Σ and Ξ) the following values are assumed

$$\text{Venus: } \pm 12;24^\circ \quad \text{Mercury: } \pm 38^\circ. \tag{3}$$

As we have seen (above, p. 254) the values for Venus agree very well with Ptolemy's tables whereas $\alpha = \pm 38^\circ$ for Mercury cannot have been used in the Almagest (cf. above p. 256). Hence for only three of the five planets do the data in (1) and (3) seem to be related to Ptolemy's parameters.

The problem does not end with Greek sources. Islamic astronomers could copy tables for the planetary phases either from the Handy Tables or from the Almagest (in the latter case, as we have seen on p. 258, not necessarily from the Almagest directly but also from Handy Tables which had incorporated the Almagest tables). That such has been the case was shown by Kennedy-Agha

[16] Cf. above p. 235 (1).

[16a] One version, Monac. 287 and Vat gr. 208, is published CCAG 7, p. 119ff. and Neugebauer [1958, 2]. A slightly different version comes from a group of notes to the Handy Tables, published by Tihon [1973], No. XIV. Cf. below pp. 1053f.

[1960].[17] One group of tables, however, in the Sanjarī Zīj of al-Khāzinī (written about 1120) has tables for all seven climates, not from the Handy Tables, as we know them, but belonging to the Almagest type. This has been shown by comparing the Almagest tables with the Sanjarī tables for the climates III and IV.[18] In contrast to the situation illustrated in our Fig. 261 (below p. 1298 f.) the Almagest tables now fall neatly between the tables for the neighbouring climates. Hence it is clear that the computational methods for the Sanjarī tables must be the methods of the Almagest and not of the Handy Tables. As is evident from our previous discussion[19] the basic parameters for the tables in the Almagest (angles ν, obliquity ε, anomalies α) are by no means easily deducible from the extant tables and it is also difficult to see how these methods could be extended to tables for all seven climates. The conclusion seems to me inevitable that al-Khāzinī did not construct his tables from an analysis of the Almagest tables but that he is depending on a source which eventually goes back to Greek tables for the seven climates, constructed at a time when the methods and parameters used in the Almagest were still fully known, perhaps to Ptolemy himself. In favor of such an extension of the Almagest material to all climates speaks the fact that Ptolemy also computed tables for the climates II to VI for the fixed star phases as he himself tells us.[20] It is certainly not too far fetched to assume a similar extension for the planetary phases. Again we are led to the assumption of tables which belong to the time between Ptolemy and Theon and which are not represented by the only tables which we accidentally possess from the two extrema of this interval.

In Fig. 262, below p. 1300 f., we have given a graphical comparison of tables for the phases of Venus belonging to the Almagest type but not taken from the Almagest, apparently computed for Alexandria, i.e. for climate III. It would be nice if the corresponding tables in the Sanjarī Zīj agreed with this table. This is, however, not the case. The Islamic tables for climate III deviate (in an irregular fashion) by about the same amount, but in the opposite direction, from the Greek tables as the latter from the Almagest tables. The variety of basically different tables for the planetary phases is apparently greater than one would expect in a topic no longer of real significance, either for theoretical astronomy or for astrology.

The practical uselessness of these tables can be demonstrated even in a direct fashion. Tables which show no obvious trends (in contrast, e.g., to mean motion tables or equations) are naturally liable to accumulate scribal errors and indeed such errors abound in all our manuscripts, Greek as well as Arabic. Many of these errors, however, are so gross that any intelligent user would have, if not corrected, at least queried them; but we have no evidence of any such action in our manuscripts. Worst of all is the fact that the earliest extant manuscript of the Handy Tables, the Vat. gr. 1291 of the 9th century, has in the tables for Venus and Mercury the headings interchanged for Σ and Ξ. Halma's manuscripts add Γ of Venus to the confusion and even Battānī's headings interchange Γ and Σ. A glance, e.g., at our Fig. 262 (p. 1300 f.) will show what predictions

[17] Cf. also Nallino, Batt. II, p. 255 to 268.
[18] Cf. Kennedy-Agha [1960], p. 138, Fig. 2.
[19] Above p. 244 ff.
[20] Ptolemy, Opera II, p. 4 Heiberg.

must result from an interchange of Σ and Ξ or Σ and Γ. The handing down of such errors through centuries would have been impossible for tables ever exposed to a confrontation with observable facts. The same can be said about the disregard for the effects of precession during the centuries between the Roman imperial period and the Islamic Middle Ages.

The problems of heliacal risings and settings of fixed stars near the ecliptic are not essentially different from the determination of the phases for the outer planets, in particular since stars of the first magnitude were assumed by Ptolemy[21] (and, perhaps, Hipparchus) to be of the same apparent diameter as Mars (i.e. 1/20 of the solar diameter). Hence the arcus visionis of such stars was estimated to be 15° (as compared with 14;30° for Mars[22]). For stars not near the ecliptic, however, it is no longer admissible to operate with a plane configuration near the rising or setting point of the ecliptic and spherical trigonometry must come into play. Ptolemy discussed these problems not only in a special chapter of the Almagest (VIII, 6) but he also wrote a separate treatise on the "Phases of the Fixed Stars." We shall return to these problems in a later section (V B 8, I B).

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[21] Plan. Hyp. I; cf. Goldstein [1967], p. 8.
[22] Cf. above p. 257.

D. Apollonius

§ 1. Biographical Data

Apollonius of Perga has always been rated with Euclid and Archimedes as one of the greatest mathematicians of antiquity. A good indication of the high esteem his work enjoyed is that Edmund Halley untertook the edition of his "Conic Sections", the first four books of which survived in Greek and three of the remaining four in Arabic.[1] For Kepler it was a matter of course to consult Apollonius in his search for approximations to the oval curve he had found for the orbit of Mars.[2]

It is mainly from the introductions to the individual books of the Conic Sections that the time of life of Apollonius can be determined within narrow limits to about 240 to 170 B.C. We also know that he lived for some time in Alexandria.[3]

It is evident that Apollonius had a decisive influence on the development of Greek mathematical astronomy from Ptolemy's discussion in the Almagest of his theory of eccentric and epicyclic motion. It seems clear that these models for planetary motions were actually invented by Apollonius, thus opening the way for a rational astronomy in contrast to the speculative cosmogony of his predecessors.[4] As we shall discuss presently, however, it is not known to what extent Apollonius himself tried to determine empirically the parameters of his models; nor do we know whether or not he had any knowledge of contemporary Babylonian astronomy.

No reference to Apollonius is made in the Almagest in connection with lunar theory. Surprisingly it is always the moon which is related to Apollonius in the few accidental references to his astronomical work found in other ancient sources. Hippolytus in his treatise "Against Heresies" (about A.D. 230) ascribes to him the estimate of 5000000 stades for the distance from the earth to the moon.[5] A quotation in the "Library" of Photius ($\approx$A.D. 835) from Ptolemaios Chennos ($\approx$A.D. 190) tells us that Apollonius was nicknamed "ε" because the shape of his letter is reminiscent of the moon about which he knew so much.[6]

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[1] No printed edition of the Arabic text exists; Halley's edition (Oxford 1710) gives only a Latin translation. An epigram on the Conic Sections from the Byzantine period is found in the Greek Anthology (Loeb III, p. 323, No. 578).

[2] Kepler, Astronomia Nova (Werke III), Chaps. 59 and 60.

[3] A careful discussion of these biographical data has been given by G.J. Toomer in the Dictionary of Scientific Biography I (1970), p. 179f.

[4] Cf. for this earlier phase below IV B 3, 4.

[5] Apollonius, Opera II (ed. Heiberg), p. 139 frgm. 60. Cf. also below pp. 650 and 655.

[6] Opera II, p. 139 frgm. 61 or Photius, ed. Henry, vol. III, p. 66 (Collection Budé). The connection of the letter ε with the moon probably originated in the coordination of the seven vowels of the Greek alphabet with the seven planets; cf. the restoration of P. Ryl. 63 in Neugebauer-Van Hoesen [1964], p. 64, No. 131 and Dornseiff, Alph., p. 43.

The most specific reference, however, is a much discussed passage of Vettius Valens ($\approx$ A.D. 160) where he says[7] that he used for the computation of eclipses Hipparchus for the sun, Sudines, Kidenas, and Apollonius for the moon and for both types of eclipses, placing equinoxes and solstices at 8° of their signs. We have here factually correct references to Babylonian astronomy[8] and related to it would be not only Hipparchus but also Apollonius.

Concerning "Apollonius", however, modern scholars usually follow Cumont, suggesting that Vettius Valens had Apollonius of Myndos in mind. He is perhaps contemporary with the mathematician, or a little earlier, calling himself a pupil of the Chaldeans.[9] This relationship of the Myndian has been supported by a reference in a treatise of the "Anonymus of 379"[10] specifically to Apollonius Myndius. This, however, does not exclude Babylonian connections for anyone else around 200 B.C. nor does it explain how Vettius Valens, writing more than three centuries later, could expect his readers to think of the Myndian when he only spoke about Hipparchus and Apollonius.

§ 2. Equivalence of Eccenters and Epicycles

One can well imagine that the discovery of the inequality of the seasons suggested an eccentric position of the observer with respect to the uniformly revolving sun. Similarly the quite noticeable variations in the velocity of the moon can be readily explained by the assumption of an eccentric orbit. The planets, however, behave in a different manner: an oscillatory motion is superimposed on the mean progress along a circular orbit. Again one can consider it as a quite natural step to describe such a phenomenon as produced by a planet rotating in a small circular orbit which in turn is carried around us on another circular track. It is, however, a brillant geometric discovery to see that the solar and lunar motion can also be described as an epicyclic motion while the planetary phenomena can equally well be generated by an eccenter model.

The concept of eccentric motion is much older than Apollonius and is explicitly attested to by the remark that Polemarchus purposely ignored eccentricities in order not to spoil the beauty of the Eudoxan arrangement of homocentric spheres.[1] We do not know at what time epicycles were introduced to explain planetary phenomena but the brilliant mathematical treatment of both cinematic models as one common structure is undoubtedly the work of Apollonius. We can reconstruct the basic ideas of Apollonius' theory because of references by Ptolemy in the Almagest (mainly in XII,1 and IV,6) to the alternative use of eccenter and epicycle models, culminating in a theory of planetary stations expressly ascribed to Apollonius.

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[7] Vettius Valens, Anthol., ed. Kroll, p. 354, 4–7; Cumont [1910], p. 161. Cf. also below p. 602.

[8] On Sudines and Kidenas cf. below p. 611; on the norm with 8° below IV A 4, 2 A.

[9] Cf. Cumont [1910], p. 163, n. 2; also Kroll, RE Suppl. V, col. 45 (No. 114) and Honigmann in Mich. Pap. III, p. 310. The date of the Myndian is extremely insecure, based on a huge web of very tenuous arguments.

[10] CCAG 5, 1, p. 204, 16; 5, 2, p. 128, 16 and note 1; CCAG 1, p. 80, 8 and p. 113, note 1.

[1] Cf. below p. 658, n. 15.

There are two levels discernible in the discussion of the equivalence of eccenters and epicycles. One consists in a simple proof based on the parallelogram with epicycle radius r and eccentricity e as sides, shown, e.g., in Alm. III, 3[2] and innumerable times thereafter.[3] The other approach does not keep the position of the observer unchanged but operates with two positions with respect to one fixed circle which carries the planet: for an exterior observer O the circle serves as epicycle, for an interior observer $\bar{\mathrm{O}}$ as an eccenter, while O and $\bar{\mathrm{O}}$ are related to one another by an inversion with reciprocal radii on the given circle. This relationship is then used to determine the position of two points on the circle which separate direct from retrograde motion of the planet, i.e. the "stations" (cf. below I D 3, 1). Ptolemy names Apollonius as the author of this procedure[4]; it also fits in perfectly with the fact that the above-mentioned transformation $\mathrm{O}-\bar{\mathrm{O}}$ is a special case of the pole-polar relation discussed in full generality by Apollonius in Book III of his Conic Sections.[5] Evidently Apollonius was also in possession of the preliminary considerations which lead up to this treatment of the stationary points, even if his name is not explicitly associated with each individual step.

In order to make clear the main ideas of Apollonius' theory I shall use modern notation and terminology, sometimes rearranging a little the material which is embedded in different sections of the Almagest or Theon's commentary. At no point, however, is it necessary to supply essential considerations which are not in one form or another explicitly attested.

Needless to say the cinematic models under consideration are always the "simple" models, i.e. epicycles observed from the center of the deferent or eccenters where the center of uniform rotation coincides with the center of the eccenter.

1. Transformation by Inversion

We begin with a seemingly useless generalization of the standard equivalence proof for epicycles and eccenters. Instead of making CSMO a parallelogram (cf. Fig. 264), i.e. instead of assuming $r=e$ and $R=R'$, we observe that all phenomena remain unchanged for an observer in O as long as the star S is located anywhere on OS, be it at S′ farther away, or at S″, nearer to O.[1] Hence the equivalence requires only equal ratios:

$$\frac{r}{R}=\frac{e}{R'}. \tag{1}$$

[2] Cf. above Fig. 51, p. 1220.

[3] E.g. in Copernicus, De revol. III, 15. Theon of Smyrna (2nd cent. A.D.) says that Hipparchus considered it worth the attention of mathematicians to investigate the cause of so greatly different explanations of the phenomena. Theon gives the impression that Adrastus (around A.D. 100) first proved the mathematical equivalence (ed. Hiller, p. 166, 6–12; Dupuis, p. 268/269). This only goes to show that even an ancient author may have an incorrect view of the chronological sequence of events.

[4] Alm. XII, 1 (Manitius II, pp. 268, 1 and 272, 18).

[5] Apollonius, Opera I, p. 402–413; ed. Heiberg; trsl. Ver Eecke, p. 249–255.

[1] Alm. III, 3 (Manitius I, p. 162).

If we in particular assume

$$R' = r = M''S''$$

then we have

$$r^2 = eR \tag{2a}$$

or its algebraic equivalent

$$\frac{R+r}{R-r} = \frac{r+e}{r-e} \tag{2b}$$

which we find in Alm. XII, 1.[2] The observer can be said to be at a distance R outside an epicycle of radius r, or, with the same right, to be inside an eccenter of radius r with eccentricity $e = OM''$. Hence we can replace Fig. 264 by a new figure which operates with only one circle of radius r but with two positions, O and $\bar{O}$, for the observer (cf. Fig. 265). It follows from (2a) that $\bar{O}$ is located at the intersection of OM with the chord that connects the points T and U at which the tangents from O contact the circle. This is, of course, a special case of the pole-polar relation with respect to a conic section.

2. Lunar Theory

In connection with Book IV of the Almagest we have discussed the method by means of which Ptolemy determined from three lunar eclipses the radius of the lunar epicycle,[1] assuming the "simple" model which, as he had shown, remains valid in the case of eclipses.[2] We furthermore know from an appendix (Chap. 11) to Book IV that Hipparchus also had used triples of eclipses for the determination of the eccentricity of the lunar orbit,[3] using both eccenter and epicycle models. As we shall see in the next section Apollonius developed the theory of planetary stations again for both models, making use of his principle of inversion on a fixed circle by means of reciprocal radii. It seems therefore plausible to assume that Hipparchus' method for finding the lunar eccentricity from three eclipses is simply the eccenter-equivalent of the epicycle procedure which we know in all details from the Almagest. Indeed Ptolemy refers explicitly to such an equivalent method[4] and Theon in his commentary to Book IV[5] completes the details which belong to a figure in the Almagest. Ptolemy's remark about another use of these transformations is obviously directed at Apollonius' theory of the planetary stations and it is therefore likely that the whole mathematical background for the determination of the basic parameters of the simple lunar theory belongs to Apollonius, representing a close parallel to his theory of planetary stations. Hipparchus' contribution would then be the selection of convenient empirical data from recorded lunar eclipses and the numerical execution of the quite involved trigonometric computations as we know them from the Almagest.

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[2] Manitius II, p. 270. The relation (2a) motivates the term "reciprocal radii" since $e = 1/R$ for $r = 1$.

[1] Above I B 3, 4 A.

[2] Cf. above I B 4, 1.

[3] Cf. for his results below p. 315.

[4] Alm. IV, 6, Manitius I, p. 223.

[5] Rome CA III, p. 1053–1056.

We now repeat, step by step, the procedure which furnished us in I B 3, 4 A the radius of the lunar epicycle, but using the terminology for an eccenter model (cf. Fig. 266 [6]). We assume that two lunar eclipses give us the difference $\Delta\lambda$ of the true longitudes for two lunar positions P_1 and P_2, a time interval Δt apart, such that also the corresponding increments $\Delta\bar{\lambda}$ and $\Delta\alpha$ of mean longitude and anomaly are known.

The center of the eccenter is moving during Δt from M_1 to M_2 by an angle $\Theta = \Delta\bar{\lambda} - \Delta\alpha$. If we turn M_1 into M_2 the point P_1 obtains a position P_1' on the circle of center M_2 and radius R. From $\bar{O}$ the arc $P_1' P_2$ appears under the angle $\Delta\lambda - \Theta = \Delta\lambda - \Delta\bar{\lambda} + \Delta\alpha$; from M_2 it is seen under the angle $\alpha_2 - \alpha_1 = \Delta\alpha$, that is under the same angle as in the epicycle model from its center C. If we deal with a third eclipse at P_3 in the same fashion we can formulate the following problem: three points are seen on a circle of radius R from its center M under the angles $\bar{\delta}_1$ and $\bar{\delta}_2$; one should find within this circle the position of a point $\bar{O}$ from which these three points appear under given angles δ_1 and δ_2 (cf. Fig. 268); then $M\bar{O} = e$ is the sought for eccentricity of the model.

As in the case of the epicycle model the place of the observer cannot be chosen arbitrarily. The locus of all points which see an arc I II under the same angle as it is seen from M is a circle through the three points I, II, and M (cf. Fig. 267). If $\delta_1 < \bar{\delta}_1$ then $\bar{O}$ must lie in the exterior of this circle. Using for an example the same numerical data as before [7] we have for the three eclipses

	$\Delta\lambda$	$\Delta\bar{\lambda}$	$\Delta\alpha$	Θ	$\Delta\lambda - \Theta$
I → II	349;15	345;51	306;25	39;26	309;49
II → III	169;30	170;7	150;26	19;41	149;49

hence

	seen from M	seen from $\bar{O}$
II → I	$\bar{\delta}_1 = 360 - 306;25 = 53;35$	$\delta_1 = 360 - 309;49 = 50;11 < \bar{\delta}_1$
I → III	$\bar{\delta}_2 = 150;26 - 53;35 = 96;51$	$\delta_2 = 149;49 - 50;11 = 99;38 > \bar{\delta}_2$
III → II	$\bar{\delta}_3 = 209;34$	$\delta_3 = 210;11 > \bar{\delta}_3$.

The comparison of these two sets of angles shows that $\bar{O}$ must be located in the zone numbered "4" in Fig. 267, exactly as O was found to belong to the exterior zone "4" of the epicycle.[8]

After the position of $\bar{O}$ relative to the three points on the circle has been in principle determined it is only a problem of plane trigonometry to find the eccentricity $M\bar{O}$, a problem which can be solved in the same fashion as before for an exterior position of O. The details of the computation are presented in Theon's commentary to Alm. IV, 6.[9] We reproduce in the following the method of solution without the numerical details which lead Theon (with some fudging), as expected, to almost exactly Ptolemy's result ($e = 5;13$ as against 5;14 chosen by Ptolemy).

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[6] Fig. 266 is the exact equivalent of Fig. 63, p. 1225.
[7] Cf. above p. 74.
[8] Cf. Fig. 67, p. 1227.
[9] Rome, CA III, p. 1053–1056.

Given three points P_1, P_2, P_3 on a circle of radius R and midpoint M (cf. Fig. 268 [10]). The arcs $P_1 P_2$ and $P_2 P_3$ appear from M under the given angles $\bar{\delta}_1, \bar{\delta}_2$, from an eccentric point $\bar{O}$ under δ_1, δ_2. Find $e = M\bar{O}$.

Let Q be a point on the circle and on the straight line $\bar{O}P_1$. In the right triangle QZP_2 one has

$$P_2Q = Q\bar{O} \sin \delta_1 / \sin \gamma_1$$

with γ_1 known from $\delta_1 = \gamma_1 + \beta_1 = \gamma_1 + 1/2\,\bar{\delta}_1$. In the right triangle QHP_3 one has

$$P_3Q = Q\bar{O} \sin(180 - \delta_1 - \delta_2)/\sin \gamma_2$$

with γ_2 known from $\delta_1 + \delta_2 = \gamma_2 + \beta_2 = \gamma_2 + 1/2(\bar{\delta}_1 + \bar{\delta}_2)$. In the right triangle $Q\Theta P_3$ one has

$$P_3\Theta = P_3Q \sin \beta_3 \qquad Q\Theta = P_3Q \cos \beta_3 \qquad \text{with } \beta_3 = 1/2\,\bar{\delta}_2.$$

Thus $P_2\Theta = P_2Q - Q\Theta$ and therefore

$$P_2P_3 = \sqrt{P_2\Theta^2 + P_3\Theta^2}$$

is known in terms of $Q\bar{O}$. But P_2P_3 is also known in terms of $R = 60$ because

$$P_2P_3 = \mathrm{Crd}\,\bar{\delta}_2.$$

Consequently $Q\bar{O}$ and P_3Q are known in terms of $R = 60$. Therefore we also know

$$\alpha_4 = \mathrm{arc\,Crd}\, P_3Q.$$

Hence

$$P_1Q = \mathrm{Crd}\,(360 - \bar{\delta}_1 - \bar{\delta}_2 - \alpha_4)$$

and

$$P_1\bar{O} = P_1Q - Q\bar{O}$$

are known and e can be found from

$$(R + e)(R - e) = Q\bar{O} \cdot P_1\bar{O}.$$

§ 3. Planetary Motion; Stationary Points

1. Apollonius' Theorem for the Stations

We know from the discussion about planetary retrogradations and stationary points in Book VII of the Almagest[1] that Apollonius had established for the simple epicyclic model a relation which characterizes a stationary point by the ratio

$$\frac{OP}{PT} = \frac{v_p}{v_c} \tag{1}$$

[10] For the sake of greater clarity the points on the circle in Fig. 268 have been spaced more conveniently than in Fig. 267.

[1] Cf. above I C 6.

where v_p and v_c are the angular velocities of P (the planet) and C (the center of the epicycle), respectively and T the midpoint of the chord PQ on the line OPQ (cf. Fig. 269). We have also remarked that (1) is simply the expression for the direction of the velocity vector at P toward the observer O, being the resultant of the rotation of P about C and of C about O.[2]

Similarly for the eccenter model (cf. Fig. 269): a point P is a station, seen from $\bar{\mathrm{O}}$, when

$$\frac{\bar{\mathrm{O}}\mathrm{P}}{\mathrm{PS}}=\frac{v_p}{v_m} \tag{2}$$

where

$$v_m=v_c+v_p$$

is the angular velocity of the center M of the eccenter with respect to some fixed direction from $\bar{\mathrm{O}}$.[3]

Apollonius had no vectorial definition of a stationary point at his disposal. He conceived of a station as the boundary between the set of points on the epicycle which appear to have direct motion and the set of points which are seen to move retrograde. In other words Apollonius applied to cinematics essentially the same mathematical concepts upon which Eudoxus had built the theory of irrational quantities, known to us from the Books V and XII of Euclid's Elements.[4]

It is clear that the simple relations (1) or (2) could not have been found by the set-theoretical considerations which Apollonius introduced for a rigid proof. Perhaps it is the special case in which the retrograde arc collapses that suggested a relation of the form (1). Indeed it is explicitly remarked in the Almagest[5] that stations occur for an epicyclic model only when

$$\frac{\mathrm{PC}}{\mathrm{OP}}=\frac{r}{R-r}>\frac{v_c}{v_p}. \tag{3}$$

Obviously $r v_p=(R-r)\,v_c$ means that the rotation of P on the epicycle and the motion of C on the deferent compensate each other at the perigee of the epicycle. It therefore seems plausible to look for a direction OPTQ such that (3) appears to be the limiting case.

We now turn to Apollonius' proof that the points P which satisfy the relations (1) or (2) respectively are stationary points, i.e. that they separate points of direct motion from points of retrograde motion.

Thanks to the equivalence theorem in the form given it by Apollonius the same circle can be used as carrier of the planet,[6] representing the epicycle for an outside observer O, and eccenter for an inside observer $\bar{\mathrm{O}}$, assuming that O and $\bar{\mathrm{O}}$ are related to one another by a transformation by reciprocal radii. Only for the sake of simplicity of the figures do we represent the different cases in different drawings.

[2] Cf. Fig. 195, p. 1268.
[3] Cf., e.g., Fig. 134, p. 1248.
[4] Cf. for these problems and the role of Eudoxus: Hasse-Scholz, Die Grundlagenkrisis der Griechischen Mathematik, Charlottenburg 1928 (Pan Bücherei, Philosophie No. 3).
[5] Alm. XII, 1 (Manitius II, p. 277). Cf. also above p. 191.
[6] Cf. above p. 264f.

The first step consists in locating a point P on the circle such that the relations (1) and (2) are satisfied. Make PB = BP′ (cf. Fig. 269, p. 1304) and define $\bar{\mathrm{O}}$ on the diameter AB through the intersection with DP. Then we have

$$\frac{\mathrm{AO}}{\mathrm{BO}}=\frac{\mathrm{A\bar{O}}}{\mathrm{\bar{O}B}} \tag{4}$$

i.e. O and $\bar{\mathrm{O}}$ are mapped on each other by inversion on the given circle. Furthermore we have

$$\frac{\mathrm{DO}}{\mathrm{DP'}}=\frac{\mathrm{D\bar{O}}}{\mathrm{\bar{O}P}} \tag{5}$$

and hence

$$\frac{\mathrm{TP}}{\mathrm{PO}}=\frac{\mathrm{S\bar{O}}}{\mathrm{\bar{O}P}} \tag{6}$$

We now assume that P (and P′) had been chosen such that (1) is satisfied, hence with (6)

$$\frac{\mathrm{TP}}{\mathrm{PO}}=\frac{v_{\mathrm{c}}}{v_{\mathrm{p}}}=\frac{\mathrm{S\bar{O}}}{\mathrm{\bar{O}P}} \tag{7}$$

and therefore

$$\frac{v_{\mathrm{c}}+v_{\mathrm{p}}}{v_{\mathrm{p}}}=\frac{\mathrm{SP}}{\mathrm{\bar{O}P}}.$$

But

$$v_{\mathrm{c}}+v_{\mathrm{p}}=v_{m} \tag{8}$$

and thus

$$\frac{v_{\mathrm{m}}}{v_{\mathrm{p}}}=\frac{\mathrm{SP}}{\mathrm{\bar{O}P}}$$

which is the criterium (2). Hence we have shown: if O and $\bar{\mathrm{O}}$ are related by inversion and P is chosen such that it satisfies (1) with respect to O it automatically also satisfies (2) with respect to $\bar{\mathrm{O}}$, and, obviously, vice versa.

The next step makes use of the following lemma: if in a triangle

$$c \leqq d < a \tag{9a}$$

then

$$\frac{d}{a-d}>\frac{\gamma}{\beta}. \tag{9b}$$

The proof compares the circular sectors of vertex A (cf. Fig. 270) with the corresponding triangles.[7]

We now look at a point K located somewhere between P and Q (case of epicycle model). Hence (Fig. 271)

$$\mathrm{QK}<\mathrm{QP}<\mathrm{QO}$$

and from (9)

$$\frac{\mathrm{QP}}{\mathrm{PO}}>\frac{\gamma}{\beta}.$$

[7] Alm. XII, 1 (Manitius II, p. 272 f.).

Therefore from (1)

$$\frac{v_c}{v_p}=\frac{\mathrm{TP}}{\mathrm{PO}}=\frac{1/2\,\mathrm{QP}}{\mathrm{PO}}>\frac{\gamma}{2\beta}. \tag{10}$$

If N is a point on the deferent such that its angular distance from the line OPQ is $\gamma'>\gamma$, satisfying exactly

$$\frac{\gamma'}{2\beta}=\frac{1/2\,\mathrm{QP}}{\mathrm{PO}}=\frac{c_c}{v_p} \tag{11}$$

then we seen that the planet moves from K to P in the same time $2\beta v_p$ in which the epicycle progresses (during γ'/v_c) from T′ to N. Seen from O the arc $\widehat{\mathrm{KP}}$ appears under the angle γ, the arc T′N under the angle $\gamma'>\gamma$. Hence the planet is seen to move forward by the amount $\gamma'-\gamma>0$.

In the case of the eccenter model (cf. Fig. 272) we have as before $\mathrm{QP}/\mathrm{PO}>\gamma/\beta$, hence

$$\frac{\mathrm{QO}}{\mathrm{PO}}>\frac{\gamma+\beta}{\beta}=\frac{\delta}{\beta}. \tag{12}$$

Since QF = DE both QF and DE are seen from K under the same angle δ. Similarly KP is seen both from Q and from D under the angle β. Since $\mathrm{QO}/\mathrm{PO}=\mathrm{D}\bar{\mathrm{O}}/\bar{\mathrm{O}}\mathrm{P}$ we have from (12)

$$\frac{\mathrm{DP}}{\bar{\mathrm{O}}\mathrm{P}}>\frac{\delta+\beta}{\beta}=\frac{\varepsilon}{\beta}. \tag{13}$$

But according to (2) we have

$$\frac{v_m}{v_p}=\frac{\mathrm{PS}}{\bar{\mathrm{O}}\mathrm{P}}=\frac{1/2\,\mathrm{DP}}{\bar{\mathrm{O}}\mathrm{P}}$$

thus with (13)

$$\frac{v_m}{v_p}>\frac{\varepsilon}{2\beta}. \tag{14}$$

If ε' is an angle such that exactly

$$\frac{v_m}{v_p}=\frac{\varepsilon'}{2\beta} \tag{15}$$

we see that the corresponding point K′ is at a greater distance from P than K. From $\varepsilon'>\varepsilon$ it follows that seen from O the eccenter moves forward through a greater angle than K. Hence K appears to have direct motion.

Similarly one can show for both models that a point on the arc PP′ appears to be retrograde. Hence P and P′ are stationary points satisfying (1) or (2).

2. Empirical Data

Ptolemy never connects numerical parameters with the theory of planetary motion of Apollonius, in marked contrast to what he tells us about Hipparchus. Nevertheless we have no proof that this omission is significant and one can argue

that any theory of epicyclic or eccentric motion will naturally have to face the question of the relative size of the radii involved, in particularly since the maximum elongations of Venus and Mercury immediately lead to a numerical estimate for the size of the epicycle. Hence one seems bound to ask for methods to answer the same question for the outer planets. On the other hand there is much in the astronomy of Eudoxus, Aristarchus, and Archimedes (i.e. in the period just preceding Apollonius) that shows a lack of interest in empirical numerical data in contrast to the emphasis on the purely mathematical structure.[1] It would therefore be a perfectly defensible position, in view of Ptolemy's silence, to assume that Apollonius also was primarily interested only in the mathematical aspects of the theory of planetary motion and not in the numerical agreement with observational facts. One could even imagine a very valid argument in favor of such an attitude. Apollonius' theory in all its mathematical elegance is nevertheless based on a model which is obviously insufficient to explain the phenomena more than qualitatively. A model based on a simple epicyclic motion, seen from its center, can only result in a strictly periodic repetition of all phenomena, in flagrant contradiction, e.g., to the greatly variable shape and amplitude of the retrograde arcs of the planets.

I see no way to decide between these alternative attitudes in the interpretation of our sources. Nevertheless it may be useful to show that a geometric analysis of the simple epicyclic motion makes it possible to determine the order of magnitude of the radius of the epicycle for an outer planet. Then we can at least say that it was fully within the grasp of Apollonius' planetary theory to establish numerical data for its models on the basis of some simple observations. It seems to me obvious that either Apollonius, or at the latest Hipparchus, must have investigated such numerical consequences. Any improvement of the theory had to start from empirically established deviations from earlier estimates based on the simple cinematic models.

The problem of determining the radius of the epicycle can be considered as mathematically similar to the determination of the radius r of the lunar epicycle from three eclipses.[2] There the value of r can be found (for $R=60$) from the relation[3]

$$(R+r)(R-r)=\mathrm{P_1O}\cdot\mathrm{QO} \tag{1}$$

where the segments $\mathrm{P_1O}$ and QO are trigonometrically determined by the given angles at the center C of the epicycle and at the observer O.

In the theory of the planetary stations we have a very similar configuration (cf. above p. 269, Fig. 269) with

$$(R+r)(R-r)=\mathrm{QO}\cdot\mathrm{PO}. \tag{2}$$

In this case, however, we do not know the two distances QO and PO but only their ratio: using (1), p. 267 we have

$$\frac{\mathrm{PT}}{\mathrm{OP}}=\frac{1/2\,\mathrm{PQ}}{\mathrm{OP}}=\frac{v_c}{v_p} \tag{3}$$

[1] Cf. below p. 643.
[2] Cf. above I B 3, 4 A and p. 267.
[3] Cf. Fig. 68, p. 1227 (and similarly Fig. 268, p. 1303).

hence

$$\frac{\mathrm{OQ}}{\mathrm{OP}}=\frac{v_p+2v_c}{v_p}. \tag{4}$$

The values of v_p and v_c can be consider known since they are directly obtainable from the basic parameters for the synodic and the sidereal periods. If one furthermore assumes the length 2γ of the retrograde arc PP′ as seen from O one can also determine OP and OQ individually. From (3) we have

$$\frac{\mathrm{TO}}{\mathrm{OP}}=\frac{v_c+v_p}{v_p}$$

and, assuming γ known (cf. Fig. 273)

$$\mathrm{TO}=R\cos\gamma$$

hence

$$\mathrm{OP}=\frac{v_p}{v_c+v_p}R\cos\gamma$$

and thus from (4)

$$\mathrm{OQ}=\frac{v_p+2v_c}{v_p+v_c}R\cos\gamma.$$

Finally from (2)

$$R^2-r^2=R^2k\cos^2\gamma \qquad k=v_p(v_p+2v_c)/(v_p+v_c)^2 \tag{5}$$

which gives us r for $R=60$.

Unfortunately the angle γ is not directly observable because the center of the epicycle moves by the amount $v_c\Delta t$ during the time Δt of retrogradation. Hence

$$\gamma=1/2(v_c\Delta t-\Delta\lambda). \tag{6}$$

In this equation the retrograde arc $-\Delta\lambda$ can be easily observed but the time between two stations is certainly not a sharply defined quantity. Still, it is possible to get at least a rough idea what results for r are obtainable by using, e.g., Ptolemy's data for retrogradations which surely agree with estimates obtainable from observations.[4] In I C 6, 1 p. 195, 193, 196, respectively we have values for Δt which lead to some plausible estimates as follows:

	max.	mean	min.	Δt estimate	v_c	$1/2\,v_c\Delta t$
Saturn	140 2/3^d	138^d	136^d	140^d	0;2$^{\circ/d}$	2;20°
Jupiter	123	121	118	120	0;5	5
Mars	80	73	64 1/2	70	0;30	17;30

Similarly for $\Delta\lambda$: for Saturn and Jupiter the retrograde arcs vary little and 7° and 10° respectively may be accepted as fair estimates. For Mars we have variations between about 10° and 20°; hence about 15° seems acceptable. This leads to the

[4] Also Babylonian data for Δt and $\Delta\lambda$ lead to essentially the same values (cf. ACT II, p. 303–315).

following results

	$-1/2\,\Delta\lambda$	$1/2\,v_c\Delta t$	γ	$\cos\gamma$	k	$kR^2\cos^2\gamma$	r^2	r
Saturn	3;30°	2;20	5;50°	0;59,40	1	59,20	40	6;30
Jupiter	5	5	10	0;59	0;59,30	57,30	2,30	12
Mars	7;30	17;30	25	0;54,20	0;45	36,50	23,10	37

as compared with Ptolemy's results $r = 6;30$, $11;30$, and $39;30$, respectively. In other words: relatively crude estimates for the time and the amount of retrogradation give quite correct values for the sizes of the epicycles.

Another possibility for determining the radius r of the epicycle of an outer planet has been suggested by A. Aaboe.[5] The method is algebraically simple but rather sensitive to changes in the empirical data which are neither easy to obtain nor attested in Greek sources. In Babylonian procedure texts, however, one finds estimates for the daily velocities of the outer planets[6] such that we may assume, at least in principle, as known for the sections of invisibility (Ω to Γ) and for the retrograde arcs (at opposition Θ) the extremal values v_{max} and v_{min} for the angular velocity of the planet as seen from O. These extrema are related to v_c and v_p by

$$v_{max}(R+r) = v_c(R+r) + r\,v_p \quad \text{at the apogee}$$

and

$$v_{min}(R-r) = v_c(R-r) - r\,v_p \quad \text{at the perigee}.$$

Hence one can find r from

$$r = R\frac{v_{max} - v_c}{v_c + v_p - v_{max}} \quad \text{or from} \quad r = R\frac{v_c - v_{min}}{v_c + v_p - v_{min}}. \tag{7}$$

Unfortunately the Babylonian parameters are adapted to arithmetical, not cinematic models and do not result in identical values for r when substituted in (7).

Both methods for finding r, described in the preceding pages, require a relatively high degree of mathematical sophistication in arranging the steps which we have presented here in algebraic notation. Furthermore the required empirical data are not easily obtainable with the necessary accuracy, neither duration and amount of the retrogradation nor the extremal direct or retrograde velocity of the planet. Hence one can hardly assume that considerations of the foregoing type provided the first estimates for the relative sizes of the epicycles of the outer planets. Fortunately one finds parameters in early Greek astronomy which concern the visibility of stars and planets and which can be used to determine the radius of the epicycle by the simplest trigonometry.[7] It seems plausible that this possibility had been exploited by the time of Apollonius.

[5] Aaboe [1963], p. 8f.
[6] ACT, No. 801, Sections 4 and 5 for Saturn, No. 810, Sections 3 and 4 for Jupiter.
[7] Cf. below p. 832.

E. Hipparchus

Non omnia possumus omnes. Desiderat Hipparchus ille Ptolemaeum, qui reliquos 5 planetas superextruat.

Kepler, Werke 14, p. 203, 38

§ 1. Introduction

Our main source for the evaluation of Hipparchus' achievements in mathematical astronomy is, of course, the Almagest. There we find sufficiently specific information to realize the differences between Hipparchian and Ptolemaic lunar theory and to define in some measure Hipparchus' dependence on Babylonian material.

Next in importance is the only preserved work of Hipparchus, his Commentary to Aratus. It provides us not only with valuable insight into Hipparchus' general attitude but also with technical details concerning the spherical astronomy of that early period. This, in turn, furnishes the necessary material for an evaluation of the relation between Ptolemy's Catalogue of Stars [1] and its predecessor.

The remaining sources consist of scattered references in the literature, often of doubtful authority and difficult to interpret. [2] A comparatively large number of references to Hipparchus is found in Strabo, [3] but mainly geographical in character and therefore outside of our topic. The astrological literature ascribes some simple computational procedures to Hipparchus; how far this is justified is difficult to say. [4] Pliny admires Hipparchus greatly but he lacks technical competence and hence is of very little help.

Particularly disappointing are the commentaries to the Almagest by such competent scholars as Pappus and Theon. They stick so closely to the original text that we learn very little about Hipparchus beyond what is already known directly from Ptolemy. And since the Arabic literature also depends largely on Theon and the Almagest we ultimately also deal here with the same sources.

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[1] Almagest VII, 5/VIII, 1.

[2] For example the note on Hipparchus by Suidas (ed. Adler II, p. 657, No. 521) gives his time as "under the consuls" which is not only meaningless but also contradicts Suidas' way of dating (cf. Rohde, Kl. Schr. I, p. 134, no. 1). Aelian, De natura animalium VII, 8 (ed. Herscher, Didot, p. 119, 20 or Teubner I, p. 175, 2) puts an anecdote about Hipparchus under "Neron the Tyrant". Following Herscher this is usually emended to "Hieron the Tyrant" (in order to find at least some motivation for the error).

[3] Cf. below I E 6, 3.

[4] Cf. below p. 823.

Very little is known about the life of Hipparchus. He was born in Nicaea in Bithynia[5] (the place where five centuries later the first oecumenical council formulated the Christian creed). Through Ptolemy's "Phaseis"[6] we know that he also observed in Bithynia. Some observations in Rhodes are attested to in the Almagest[7] but he is always called "the Nicaean," e.g. on coins,[8] or "the Bithynian,"[8a] never "the Rhodian." Strabo does not mention him among the famous men associated with Rhodes,[8b] a most inexplicable omission if he spent the major part of his life on that island. His treatise "On simultaneous risings"[9] as well as the second part of the Commentary to Aratus are based on the latitude of Rhodes.[10] There is no reason, however, for considering every observation made in Rhodes or every example which uses $\varphi = 36°$ as Hipparchian in origin.[11] Examples based on that convenient mediterranean latitude still appear in late Islamic works.

The chronological order of Hipparchus' work is not at all certain. Table 28 gives a list of observations, ranging from -161 to -126, related to Hipparchus in the Almagest. But the wording of these references rarely distinguishes between actually having made these observations or merely utilizing them. Furthermore, Alexandria as place of observation is not certain, excepting No. 4. Clearly characterized as Hipparchus' own observations seem only No. 4 (autumn equinox of -146), No. 12 (summer solstice of -134), and the three observations of the moon in $-127/126$ made in Rhodes. It also should be noted that Ptolemy reckons 265 years for the interval between "the observations of Hipparchus" and the first year of Antoninus[12] (A.D. 137/8); this would correspond to -128 for the epoch year of Hipparchus' fixed star observations.[12a]

In connection with his investigation of Hipparchus' star catalogue H. Vogt made an attempt to date the preserved Hipparchian observations by purely astronomical methods. We shall discuss later on[13] the procedure by means of which Vogt derived from the given data for 122 stars the corresponding ecliptic coordinates. He found[14] that these coordinates would be without a systematic error for the year -138. Restricting himself to right ascensions and declinations

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[5] Suidas, ed. Adler II, p. 657, No. 521; cf. also the preceding note 2.

[6] Ptolemy, Opera II, p. 67, 10 and 16 to 18 (ed. Heiberg). Cf. below p. 928.

[7] Cf. Table 28, below p. 276.

[8] E.g. Zeitschr. f. Numismatik 9 (1882), p. 127f. Coins with the picture of Hipparchus are known from the reigns of Antoninus (138 to 161), Commodus (180 to 192), Marinus (217), Alexander Severus (222 to 235), Gallus (251 to 253).

[8a] Maass, Aratea, p. 121.

[8b] Geogr. 14, 2, 13 (Loeb VI, p. 279/281. Hipparchus is mentioned, of course, among the learned men of Bithynia (Geogr. 12, 4, 9; Loeb V, p. 467).

[9] Cf. below p. 301.

[10] Hipparchus, Arat. Comm. ed. Manitius, p. 184/5; cf. also p. 292, note 3.

[11] The inscription of Keskinto, e.g., shows that other astronomers had worked at Rhodes (cf. below p. 698).

[12] Alm. VII, 2 and 3 (Manitius II, p. 15, 9 and 20, 21).

[12a] A marginal note to the Royal Canon of the "Handy Tables" (in a version of the 9th cent.) assigns the lifetime of Hipparchus to the reign of Euergetes II, equated with the years 179 to 207 of the era Philip (i.e. $-145/4$ to $-117/6$); cf. Monumenta 13, 3, p. 451, 9.

[13] Below p. 281.

[14] Vogt [1925], col. 25. Cf. below p. 284.

Table 28

No.	Date	Place	Subject	Alm.	Manitius I, p.
1	−161 Sept. 27	Alexandria	autumn equinox	III, 1	134, 11
2	−158 Sept. 27	Alexandria	autumn equinox	III, 1	134, 13
3	−157 Sept. 27	Alexandria	autumn equinox	III, 1	134, 17
4	−146 Sept. 26/27	Alexandria	autumn equinox	III, 1	134, 21; 142, 2
5	−145 March 24	Alexandria	vernal equinox	III, 1	135, 1; 142, 22
6	−145 April 21	?	lunar eclipse	III, 1	137, 11
7	−145 Sept. 27	Alexandria	autumn equinox	III, 1	134, 26
8	−142 Sept. 26	Alexandria	autumn equinox	III, 1	134, 29
9	−140 Jan. 27	Rhodes	lunar eclipse	VI, 5, 9	351, 8; 394, 25
10	−134 March 21	?	lunar eclipse	III, 1	137, 11
11	−134 March 23	Alexandria	vernal equinox	III, 1	135, 11
12	−134 June	?	summer solstice	III, 1	145, 6
13	−127 March 23	Alexandria	vernal equinox	III, 1	135, 18
14	−127 June	?	summer solstice	VII, 2	Man. II, p. 14, 22
15	−127 Aug. 5	Rhodes	moon in quadrat.	V, 3	266, 1
16	−126 May 2	Rhodes	moon in octant	V, 5	271, 1
17	−126 July 7	Rhodes	moon in octant	V, 5	274, 26

of 77 stars near the ecliptic or the equator, described by Hipparchus in the second part of his Commentary to Aratus[15] which deals with risings and settings for the horizon of Rhodes, Vogt established a date near −150. The last section,[16] however, and the declinations cited by Ptolemy,[17] lead to about −130. Hence all accessible data for Hipparchian fixed star observations suggest the decades from −150 to −130. This does not support the assumption that Hipparchus was active in Alexandria between −160 to −140 (cf. the equinox observations Nos. 1 to 8 in Table 28).[18]

Delambre assumed that Hipparchus checked the Alexandrian observations by observations made by himself at Rhodes.[19] This is a rather implausible hypothesis since Hipparchus (who is quoted verbatim by Ptolemy) does not say a word about such a duplication which only by a miracle could have led to exactly the same dates as the Alexandrian set. Fotheringham, on the other hand, exclaims[20] "Of which of his predecessors would so severe a critic as Hipparchus have said that his observations were made with the greatest possible accuracy?" Such moralizing historiography tends only to obscure the issue.

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[15] Hipparchus, Arat. Comm., Manitius, p. 182 to 270.
[16] Hipparchus, Arat. Comm., Manitius, p. 270 to 280. Cf. also below p. 279, note 22.
[17] Alm. VII, 3 (Manitius II, p. 18 to 20).
[18] Rome [1937], p. 217 quotes a passage by Theon (Comm. Alm. III, 1 ed. Rome, p. 817, 11 f.) in which he refers to the equinox observations as made by Hipparchus. But Theon's source is obviously only the Almagest and hence not an unambiguous new witness.
[19] Delambre HAA I, p. XXII to XXIV.
[20] Fotheringham [1918], p. 408.

Nothing is known about the time when the observations in Bithynia were made nor is it at all certain that the Nova Scorpii of −133 caused Hipparchus to compile a catalogue of stars. [21]

Fortunately little depends on these chronological details. For us the main problem consists in the clarification of the position of Hipparchus with respect to his predecessors and followers. He himself names Archimedes as preceding him in the investigation of the length of the year. [22] We know that Apollonius had reached full mastery of the cinematics of eccenter and epicyclic motion; we also know how much Ptolemy contributed to the lunar and planetary theory and we are aware of important data utilized by Hipparchus which originated in Mesopotamia. Obviously only by a careful analysis of technical details may one hope to obtain a valid picture of the astronomy of Hipparchus and his time. To call him the "father of astronomy" does not solve the problem.

§ 2. Fixed Stars. The Length of the Year

That much of Hipparchus' work concentrated on the fixed stars is well attested. Ptolemy's "Phaseis" contain more than 60 weather predictions on the authority of Hipparchus, related in the traditional fashion of Greek calendars to the rising and setting of stars and constellations. [1] Concerned with spherical astronomy must have been the "Treatise on simultaneous risings" [2] and the "On the rising of the 12 zodiacal signs," [3] which might be, however, only a section in the same treatise. Unfortunately also lost is the "Catalogue of stars," [4] at least as an independent work. It will be shown presently, [5] however, that very likely a substantial part of it is preserved for us in the stellar coordinates of the Commentary to Aratus. Some additional fragments can be extracted from different ancient and mediaeval sources. [6] Hence the situation is by no means so hopeless, nor so trivial, as usually depicted.

1. Stellar Coordinates. Catalogue of Stars

A. Stellar Coordinates

From the Commentary to Aratus it is quite obvious that at Hipparchus' time a definite system of spherical coordinates for stellar positions did not yet exist. We know that different numerical values were associated with the solstices and

[21] Cf. below p. 284.
[22] Alm. III, 1 (Manitius I, p. 133, 32).
[1] Ptolemy, Opera II, p. 1 to 67. Cf. also below V B 8, 1 B.
[2] Cf. below p. 301, n. 1.
[3] Cf. below p. 301, n. 2.
[4] The Greek title is not certain; cf. Rehm in RE 8, 2, col. 1670, 58. Ptolemy, Alm. VII, 1 (Heiberg, p. 3, 9) quotes "On the fixed stars", Suidas (ed. Adler II, p. 657) "On the arrangement of the stars and the Catasterism (?)."
[5] Below p. 283.
[6] Below pp. 285 ff.

equinoxes. We shall see[1] that 10° and 8° were the Babylonian longitudes for the equinoxes, and that particularly the latter norm also found widespread acceptance in the west.[2] Hipparchus informs us that Eudoxus placed the midpoints (15°) of the signs at these points[3] whereas he himself, following "most of the old mathematicians"[4] (and Aratus), reckoned the seasons from the beginning of the signs.[5]

Babylonian mathematical astronomy consistently used ecliptic coordinates and so later on did the Almagest; the Commentary to Aratus, however, contains beside other determinations which we shall discuss presently, many declinations[6] (or distances from the poles[7]) but no longitudes or latitudes.

In the whole of Hipparchus' Commentary to Aratus a certain primitivity of technical terminology is unmistakable. This is not surprising if one sees how far from exact mathematical definitions the astronomers were shortly before and still at his time (he mentions explicitly Eudoxus and Attalus[8]) when they, e.g., assumed a certain "width" for the fundamental circles of spherical astronomy, equator, ecliptic, and tropics.[9] Also Hipparchus' own terminology is neither consistent nor clear. The term "zodiacal sign" he uses for any 30-degree arc. He says, e.g., that a star is located "16° to the north of the tropic, that is more than half of one zodiacal sign."[10] Still stranger is the division of the equator and of its parallels in "zodiacal signs" which are denoted by the same names as the signs in the ecliptic. An example is the following passage[11]: "The head of the Little Bear occupies on the parallel to the equator the end of Scorpio; when (this degree) culminates then culminates of the ecliptic the 3rd degree of Sagittarius; and when (this degree) culminates the 17th degree of Aquarius rises ... (at a place) where the longest day is 14 1/2 equinoctial hours (long)."

In order to interpret this passage correctly one has first to explain Hipparchus' terminology[12], which is, however, not the same in the two parts into which the Commentary to Aratus is divided. In the first, polemical, part[13] he uses the expression a star "occupies" the n-th degree, or "lies" in the n-th degree,[14] meaning that its position is $n°$ in the respective 30° section. Only in such context

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[1] Cf. below p. 368.

[2] Cf., e.g., IV A 4, 2 A and 2 B.

[3] E.g. Aratus Comm. ed. Manitius, p. 48, 8; 128, 25; 132, 10, etc., e.g. p. 56, 15: "18° of Pisces, or, as Eudoxus divides the zodiacal circle, at 3° of Aries."

[4] Aratus Comm., p. 132, 7.

[5] E.g. Aratus Comm., p. 48, 5 to 7.

[6] E.g. Aratus Comm., p. 98, 19: "the bright star in the middle of the body of Perseus lies 40° to the north of the equator."

[7] E.g. Aratus Comm., p. 82, 24: "Arcturus is 59° distant from the northern pole while the bright star in the middle of the Altar is 46° distant from the southern pole." Cf. also below p. 283.

[8] Aratus Comm., p. 89.

[9] This is motivated by variations in the positions of the solstices, supposedly observed by Eudoxus, according to a passage in his "Enoptron" quoted by Hipparchus (Aratus Comm., p. 88, 19).

[10] Aratus Comm., p. 98, 21. Similarly p. 98, 2; 102, 9; 120, 16; 150, 26, etc. Similarly, for the summer tropic "1/2 and 1/12 of one zodiacal sign" (i.e. 17;30°) below the horizon (quoted by Strabo, Geogr. II 5, 42; Loeb I, p. 514/5).

[11] Arat. Comm., p. 68, 20ff. ed. Manitius. Cf. also Vogt [1925], col. 29.

[12] This has been done by Manitius, p. 288f. of his edition, but ignored in his translation. For a clear formulation cf. Vogt [1925], col. 27 to 29. Cf. also below (p. 596, n. 19).

[13] Aratus Comm., p. 8 to 182, ed. Manitius.

[14] ἐπέχει or κεῖται, respectively.

does the notation *μοῖρα α'* "1 degree" occur; 30° (or 0°) in our notation is called "end" of the sign.

In the second, systematic, part,[15] however, the term "first (or 1) degree" never occurs but only "beginning, 2nd, 3rd, ..., 30th degree" which must be interpreted as 0°, 1°, 2°, ..., 29°. This is the terminology applied in the above passage which, consequently, is to be understood as follows: when the right ascension of the star Σ (cf. Fig. 274) is 240° then the meridian NZΣC intersects the ecliptic at ♐ 2° while ♒ 16° rises for the horizon of Rhodes.[16]

This way of determining the position of a star is convenient both for readings on a globe and for graphic construction or plane trigonometric computation based on stereographic projection (cf. Fig. 275), assuming that the latter was known to Hipparchus.[17] Whatever the case may be the result is a stellar coordinate which is also used in Indian astronomy, the "*polar longitude*" of a star,[18] i.e. the arc m of the ecliptic from the origin to the point of intersection with the ecliptic of the meridian through the star (cf. Fig. 276). On the other hand Hipparchus occasionally expresses distances of single stars from the meridian in cubits,[19] or in half- or 2/3-cubits,[20] a notation which is probably due to Babylonian influence.[21]

Reckoning with equinoctial hours is familiar to Hipparchus as is evident from many passages in the Commentary to Aratus. The concluding chapter gives a list of stars which are located on 24 meridians, beginning with the meridian which contains the summer solstice.[22]

Since it is not always possible to find sufficiently characteristic stars located exactly on one of these 24 meridians Hipparchus also selected stars which precede or follow the circle in question by a small angle. These deviations are always expressed in fractions of hours, as is natural for the problem at hand: either $1/10^h$, or $1/20^h$, or $1/30^h$ (=0;30°). Otherwise angular distances between stars are measured in cubits or half-cubits.

On one occasion an arc on a parallel circle of fixed declination is measured in "such sections as the whole circle contains 24."[23] For minutes no special term occurs in the Commentary to Aratus beyond the ordinary one for fractional parts. Here too a certain primitivity of terminology reveals itself.

In general we know very little about the early history of the astronomical coordinate systems. Ptolemy cites observations of Timocharis and Aristyllus in terms of declinations.[24] If Pliny's story is correct that the appearance of a "new

[15] Aratus Comm., p. 182 to 280.

[16] This result agrees with the tables in Alm. II, 8.

[17] Cf. for this problem below p. 868 f.

[18] Cf. below p. 1081 and Fig. 12 there. The corresponding second coordinate, the "*polar latitude*" b, seems not to be attested as such in the writings of Hipparchus (cf. the statistics of coordinates given given below p. 283). Instead he seems to prefer to define the position of a star by its declination or by its distance from the pole (cf., e.g., below I E 2, 1 C 1).

[19] *πῆχυς*, (Arat. Comm., p. 272, 1); *πηχυαῖον διάστημα* (p. 190, 10).

[20] *ἡμιπήχιον* (Arat. Comm., p. 186, 11; 190, 8, 26 etc.); *δύο μέρη πήχεως* (p. 254, 11, 25; 268, 8, etc.).

[21] Cf. below p. 591; also p. 304.

[22] Manitius, p. 270 to 281. For the high accuracy of these hour-circles cf. Schjellerup [1881], p. 38f. The solstitial meridian is correct for the year −140. Cf. also above p. 276.

[23] Manitius, p. 150, 2f.; cf. below p. 299.

[24] Almagest VII, 3 (Manitius II, p. 18 to 20).

star" caused Hipparchus to draw up an inventory of the fixed stars [25] any method of uniquely localizing the position of a star would have served the purpose. For the study of precession ecliptic coordinates for a few main stars would suffice once the invariance of all relative positions between fixed stars is established. Of particular significance is Ptolemy's demonstration in Alm. VII, 3 that the motion of precession progresses about the poles of the ecliptic. To this end he lists for 18 stars the declinations as observed by Timocharis and Aristyllus, by Hipparchus, and by himself and declares that the declinations change in the way required by a rotation around the pole of the ecliptic. Arguing this way would be absurd if the positions of the stars in Hipparchus' catalogue had been given in ecliptic coordinates. Hence all we know seems to indicate that the positions of the stars in the catalogue were defined more or less in the same fashion as in the Commentary to Aratus. This does not exclude ecliptic coordinates for a few stars. Two cases are known from the Almagest [26] and a few more longitudes are preserved in the Liber Hermetis Trismegisti,[27] but not a single Hipparchian latitude is recorded and nowhere in Greek astronomy before the catalogue of stars in the Almagest is it attested that orthogonal spherical coordinates are used to determine stellar positions. In this context it may be significant that the orthogonal spherical coordinates in geography, geographical longitude and latitude, were also first introduced by Ptolemy.[28]

B. Hipparchus' and Ptolemy's Catalogue of Stars

The fact that Hipparchus' Catalogue of Stars is not preserved made it easy for modern historians to simplify matters by assuming that the catalogue of stars in the Almagest is nothing but a trivial replica of its predecessor. Supposedly Ptolemy kept all Hipparchian latitudes unchanged while he added 2;40° to all longitudes in order to account for precession during the intervening 265 years (in the amount required by his estimate of 1° increase per century).

This procedure was assumed (and then criticized) without proof or further discussion by Tycho Brahe.[1] It was indeed a plausible hypothesis in view of the fact that during all the Middle Ages, in the east as well as in the west, the catalogues of stars were derived by exactly this process either from the Almagest or from some other predecessor nearer at hand. Copernicus, only a few decades before Brahe, had obtained his catalogue admittedly[2] by this method from Ptolemy's.

This hypothesis found general support until the beginning of the 20th century, mainly due to Delambre's preference for any theory which would detract from Ptolemy's merits.[3] In spite of increasing evidence that things could not have been as simple as Brahe had assumed it was not until 1925 that it was convincingly demonstrated that Ptolemy's assertion that his catalogue

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[25] Cf. below p. 284.
[26] Cf. below p. 283, note 13.
[27] Cf. below p. 286. With a few exceptions all these stars are near the ecliptic.
[28] Cf. below p. 934.
[1] Remarked by Dreyer [1918], p. 348/9. Cf. Brahe, Opera II, p. 151, 10f.; p. 281, 11ff.; III, p. 335, 31ff.
[2] Copernicus, De revol. II, 14 (Gesamtausgabe II, p. 102, Thorn, p. 115).
[3] Cf., e.g., the accusations of dishonesty in HAA I, p. XXXI. Cf. also Vogt [1925], col. 33.

was based on his own observations was actually true. The proof was furnished by H. Vogt in an excellent paper in the Astronomische Nachrichten[4] in which he made use of all available Hipparchian determinations of stellar positions, mainly from the Commentary to Aratus, transforming them to the equivalent ecliptic coordinates. This made possible a direct comparison with the coordinates given in the Almagest and it became clear that only in a few exceptional cases could a schematic transformation have taken place. Furthermore the differences found allow us to understand how Ptolemy came to the conclusion that the constant of precession should be about 1° per century.

We shall first give in the following a summary of Vogt's procedure before we turn [5] to historical considerations which alone would not suffice to obtain accurate numerical results for the relation between the two catalogues. Vogt reached his conclusions without any historical hypothesis except taking it for granted that the large number of observational data recorded by Hipparchus in the second, systematic, part of his Commentary to Aratus[6] also constitute the basis for his catalogue of stars. This assumption is not only plausible in itself but would leave us with only a rather absurd alternative. Suppose Hipparchus' catalogue were obtainable by subtracting 2;40° from the longitudes given by Ptolemy, keeping the latitudes unchanged. Then it would nevertheless be legitimate to determine the ecliptic coordinates for all stars mentioned in the Commentary to Aratus. Vogt's computations show that the result would be that almost all coordinates in the Commentary disagree with the coordinates in the schematically reconstructed catalogue. Hence, instead of concluding that Hipparchus and Ptolemy observed independently, obtaining, of course, divergent results, one would have to suppose that Hipparchus discarded all his earlier data on which he had founded his criticism of Eudoxus, Aratus, and Attalus, and replaced them soon afterwards by an entirely new set of positions which show no agreement with his earlier, and better, observations.

We now can turn to the description of Vogt's method. We may consider to be given the value $\varepsilon \approx 23;50°$ for the obliquity of the ecliptic which we know was used by Hipparchus[7] and $\varphi = 36°$ for the geographical latitude.[8]

Let Σ be the star under consideration (cf. Fig. 277) and let us assume that we are given the longitude of the setting point Δ and of the culminating point M of the ecliptic when Σ is setting. We look at the spherical triangle AΔW and solve it first by using the given arc ΔA and the angles ε and $\bar{\varphi} = 90 - \varphi$. This provides us with a first set of values for the sides WΔ, WA, and the angle η.

The information about the culminating point M allows us to solve the same triangle a second time and thus to check the previously obtained data. Indeed, in the right triangle ACM we are given AM and ε; thus we can find AC. Because CW = 90 we also have the side WA = 90 − AC in the triangle AΔW. With ε and $\bar{\varphi}$ this gives us the second set of values for ΔA, WΔ, and η. If all data were accurate

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[4] No. 5354–55 (1925).

[5] In section C, below p. 284.

[6] Hipparchus, Arat. Comm., p. 186 to 270 ed. Manitius.

[7] Almagest I, 12 (Manitius, p. 44).

[8] Hipparchus, Arat. Comm., p. 184, 1: longest day = 14 1/2 hours. This is the equivalent of $\varphi = 36°$ according to Arat. Comm., p. 72, 23f.

it would be identical with the first set. In cases where small deviations appear Vogt takes the arithmetical mean of the corresponding values.

If we have similar information for the same star Σ when it is rising (cf. Fig. 278), i.e. if we are given the longitudes of the simultaneously rising and culminating points H and M of the ecliptic, we can again twice solve the triangle AEH, first by using the given arc AH with ε and $\bar{\varphi}$, secondly by determining the side $AE = 90 - AC$ where AC can be found in the right triangle ACM from the given arc AM.

Hence we have two values for each side in each of the triangles $A\Delta W$ and AEH, respectively and also two values for each of the angles η and η' between the ecliptic and the horizon.

We can now combine the two cases, Σ setting and Σ rising, in one schematic diagram (cf. Fig. 279) by drawing through one point Σ the two characteristic positions of the horizon, $\Sigma W \Delta$ and ΣHE. Since a star rises at the same azimuth from the point E as it sets from the point W we know that $E\Sigma = \Sigma W$. Since ΣR is perpendicular to EW we see that R is the midpoint of $EW = WA + AE$ which is known since we have found WA and AE separately. Hence the right ascension α of Σ can be found from

$$180 - \alpha = AR = 1/2\,WE - WA.$$

The declination $\delta = \Sigma R$ can be found either in the right triangle ΣRW or ΣRE in which we know $WR = RE = 1/2\,WE$ and the angle $\bar{\varphi}$.

In order to find the ecliptic coordinates of Σ one can compute in the right triangles ΣRW or ΣRE the sides ΣW and $E\Sigma$. Previously we found $W\Delta$ and EH. Hence the sides $\Sigma\Delta = \Sigma W + W\Delta$ and $H\Sigma = E\Sigma - EH$ are known. Using the angles η and η' found before one can compute $\beta = \Sigma L$ in either one of the two right triangles $\Sigma L\Delta$ and ΣLH. Similarly ΔL and LH can be found and with it

$$180 - \lambda = AL = \Delta L - \Delta A = AH - LH$$

by means of the previously determined arcs ΔA and AH.

It should be emphasized that we have assumed so far the optimal set of information, namely the location of the simultaneously rising, setting, and culminating point of the ecliptic for the moment when the given star is rising and when it is setting. We have seen that this overdetermines the coordinates of the star, allowing us not only to check the compatibility and accuracy of Hipparchus' data but also to compute the desired ecliptic coordinates as mean values resulting from all available observations.[9]

Ordinarily Hipparchus does not furnish all this information for a single star. In fact much less suffices to find the coordinates λ and β. As we have seen from Fig. 277 the knowledge of Δ alone, or of M alone, determines all parts in the triangle $A\Delta W$. Fig. 279 shows that the knowledge of either H or M in the situation depicted in Fig. 278 suffices to find α, δ and λ, β of Σ. If, however, we do not have a rising *and* a setting element at our disposal we need something else to determine the necessary parts in Fig. 279. Fortunately Hipparchus in some cases gives the

[9] From the numerical examples given by Vogt one sees that the discrepancies between the alternative possibilities are usually very small or zero, reaching only in a few cases 10 or 15 minutes.

declination $\delta = \Sigma R$ (or the distance $\bar{\delta} = 90 - \delta$ from the pole). Hence we can find from δ and $\bar{\varphi}$ in the right triangle ΣRW the side ΣW and this, in combination with any information which allows us to solve either the triangle $A\Delta W$ or AHE, completes Fig. 279.

Similarly a datum about the right ascension α of Σ, i.e. the knowledge of the arc AR, restores Fig. 279 as soon as even one of the triangles $A\Delta W$ or AHE is known. This is also the case when we are not given the right ascension directly but the point T where ΣR intersects the ecliptic,[10] since AT and ε suffice to find AR.

Vogt utilized 881 numerically given data concerning a total of 374 stars, extracted from the Commentary to Aratus, from the Almagest and from Ptolemy's Geography,[11] and from Strabo.[12] This material provided him with two or more coordinates for 122 stars, with only one coordinate for 252 stars.

The distribution of the 881 numerical data is of interest as an illustration of Hipparchus' ways to determine positions of stars:

408 data for simultaneous risings, settings, culminations
340 polar longitudes (i.e. arc AT in Fig. 279)
67 right ascensions
64 declinations
2 longitudes; no latitudes.

Among these 881 data only 473 are spherical coordinates in the modern sense (even including the 340 polar longitudes). And with the exception of 2 cases[13] none of these coordinates are ecliptic coordinates.

For the 122 stars for which Vogt could determine both coordinates λ and β he made a direct comparison a) with Ptolemy's coordinates and b) with the actual positions according to modern knowledge.[14] The first comparison had to answer the question whether or not Ptolemy's coordinates show constant differences against Hipparchus'; the second would give an estimate for the accuracy of Hipparchus' observations.

If we call λ_H and β_H the coordinates of a star according to Hipparchus, λ_P and β_P the corresponding values in Ptolemy's catalogue, and $\Delta\lambda = \lambda_P - \lambda_H$, $\Delta\beta = \beta_P - \beta_H$, then a schematic application of Ptolemy's theory of precession for the interval of 265 years[15] would mean $\Delta\lambda = 2;40°$ $\Delta\beta = 0$. In fact, however, the data obtained for 122 stars show that neither $\Delta\lambda$ nor $\Delta\beta$ is constant, but that $\Delta\lambda$ varies between $+4°$ and $0°$ (ignoring a few cases with even greater variations), $\Delta\beta$ between $+2°$ and $-2°$ (cf. Fig. 280[16]). This proves conclusively the independence of Ptolemy's catalogue.

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[10] I.e. the "polar longitude" μ of Σ. Cf. above p. 279.

[11] Alm. VII, 3 (Manitius II, p. 18 to 20); Geography I, 7, 4 (ed. Nobbe, p. 15, 6).

[12] Strabo, Geogr. II, 5, 41 (ed. Meineke, p. 181, 21 to 25).

[13] These two exceptions (Alm. VII, 2 Manitius II, p. 12, 26–28 and p. 15, 1–3) are positions of Regulus and Spica, observed by Hipparchus in connection with the problem of determining the constant of precession. In Alm. VII, 3 (Man. II, p. 16, 27–17, 3) the permanency of the latitude of Spica ($\beta = -2$) is stressed, quoting Hipparchus.

[14] In 17 cases Vogt had to replace the identifications by Manitius in his translation of the catalogue of stars in the Almagest by the identifications given in Peters-Knobel, Catal.

[15] Cf. above p. 275.

[16] Taken from Vogt [1925], Tables III and IV where the deviations for the single stars are arranged in decreasing order of $\Delta\lambda$ and $\Delta\beta$.

At the same time Vogt's results permit us to compare the relative accuracy of the two catalogues. Fig. 281 shows the number of stars as function of the deviation from modern values, P for Ptolemy, H for Hipparchus.[17] In particular for latitudes Ptolemy's observations are clearly better than Hipparchus' since Vogt found for the individual errors:

	Latitude	Longitude
Hipparchus:	$\pm 0.71°$	$\pm 0.88°$
Ptolemy:	$\pm 0.31°$	$\pm 0.58°$.

The fact that the distribution of errors is quite different for H and P underlines once more the independence of Ptolemy's measurements. Only for a small group of stars (5 out of 122) did Vogt find data which appear to be the result of a schematic change of Hipparchian values.

There exist, however, systematic errors in comparison with modern data. For the latitudes it turns out that these errors are negligible[18]; for the longitudes, however, Hipparchus' error is $+0.16°$ ($\pm 0.08°$), Ptolemy's $+1.23°$ ($\pm 0.05°$). This shows that in both cases the vernal point ($\lambda = 0°$) was incorrectly determined, in Hipparchus' case such that his longitudes would be correct about 11 years earlier; Ptolemy's, however, about 89 years earlier.[19] This is reflected in Ptolemy's correction for precession which would be valid for an interval $89 - 11 = 78$ years shorter than the 265 years which separate Ptolemy from Hipparchus. We shall discuss presently the cause of so large an error in Ptolemy's zero point for longitudes.[20]

C. The Catalogue of Stars. Continued

It is an often repeated story that Pliny ascribes the appearance of a Nova as having given the impetus to Hipparchus to compile a catalogue of stars.[1] Édouard Biot published in 1843[2] an "extrait du livre 294 de la grande collection de Ma-touan-lin" (a Chinese historian of the 13th century) concerning "étoiles extraordinaires," the first of which is dated to July -133[3] and located in the constellation Fang, identified with β, π, ρ Scorpii. Herschel, in his Outlines of Astronomy (1849)[4] suggested a Nova "which may possibly have been Hipparchus' star" and this has become the generally accepted opinion.[5]

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[17] Cf. for details Vogt [1925], col. 23 to 26.

[18] Vogt [1925], col. 23: Hipparchus $-0.06°$ ($\pm 0.065°$), Ptolemy $+0.01°$ ($\pm 0.03°$).

[19] I.e. around -138 and $+48$, respectively.

[20] Cf. below p. 294.

[1] Pliny NH II, 95 (Ian-Mayhoff I, p. 159, 12–14; trsl. Loeb Class. Libr. I, p. 239; trsl. Collect. Budé II, p. 41, p. 180f.).

[2] Biot [1843], p. 61.

[3] This date is generally accepted by Chinese scholars; cf. Yoke [1962], p. 145, No. 41 or Hsi [1958], p. 114, No. 6.

[4] First edition p. 563, 4th edition (1851), p. 474. Also referred to by Humboldt, Kosmos III (1850), p. 221.

[5] The Chinese sources seem not to exclude the possibility that the "Nova of -133" was only a comet. Cf. Needham SCC III, p. 425f.

All would be well if Pliny did not speak of a displacement of the new star:

Text	Loeb	Budé
novam stellam (et aliam)[6] in aevo suo genitam deprehendit eiusque motu qua fulsit ad dubitationem est adductus ...	detected a new star that came into existence during his lifetime; the movement of this star in its line of radiance led him to wonder ...	découvrit une étoile nouvelle, différente des comètes, née de son temps; constatant que le point où elle brillait se deplaçait, il fut amené à se demander ...

The French rendering of the dubious "*et aliam*" by "différente des comètes" has no basis in the text; the "movement ... in its line of radiance" in the English translation makes no sense at all. Pliny, in the continuation of this passage leaves open all possibilities of interpretation since he speaks as well of motions of the so-called fixed stars as of stars which are being born and die. What remains certain is only the fact that the "Nova" of −133 provides at best a very insecure element for the dating of Hipparchus' catalogue of stars.

Since no copy of this work has survived it is of interest that at least some excerpts, in Greek as well as in Latin, seem to exist.[7] These are simple lists of constellations, some mentioning Hipparchus as the author, some anonymous. One group of these excerpts gives only the names of the constellations, the other group adds the number of stars within each constellation. This second type may be illustrated by the following passages[8]: "Northern constellations: the Great Bear 24 stars; the Little Bear 7; Draco between the Bears 15; Bootes 19" etc.

Rehm and Boll[9] gave good reasons supporting the authorship of Hipparchus of the original from which these lists were taken. Boll then argued that also the numbers of stars should be taken as belonging to Hipparchus' Catalogue of stars. There are some lacunae and doubtful variants in the extant excerpts but it is safe to conclude that the total could not have exceeded about 850 stars.[10] Since Ptolemy's catalogue contains 1022[11] stars it is clear that he must have added some 170 new positions to the list of his predecessor.

This is not the only point of deviation between the catalogues of Ptolemy and Hipparchus. The excerpts agree, if not exactly but in all major points of arrangement, with the list of constellations found in Hipparchus' Commentary to

[6] Variant: *uel aliam.* The text is probably corrupt. Ph. H. Külb in Balss, Ant. Astr., p. 140/141 deletes whatever is found between *stellam* and *in aevo*; on the other hand he inserts *die* between *qua* and *fulsit.* Consequently he translates "... entdeckte auch einen neuen, zu seiner Zeit entstandenen Stern und wurde durch dessen Bewegung an dem Tage selbst, an dem er zum Leuchten kam, zu dem Zweifel veranlaßt ..."

[7] First discovery: Maass, Aratea (1892), p. 377; republished in Maass, Comm. Ar. rel., p. 134. Most recent edition by Weinstock, CCAG 9, 1, p. 189f. I count a total of 9 Greek and 2 Latin MSS. Cf. also Rehm [1899] and Boll [1901].

[8] From CCAG 9, 1, p. 189 (ignoring variants).

[9] Rehm [1899], Boll [1901].

[10] Boll [1901], p. 192ff.

[11] Almagest VIII, 1 (Manitius II, p. 64).

Aratus.[12] The discrepancies with Ptolemy, however, are far greater. It may suffice to remark that the Commentary to Aratus and the excerpts place the zodiacal constellations, starting with Cancer, after the northern and the southern ones. Ptolemy, however, brings the zodiac in its natural position between northern and southern constellations and begins with Aries. Ptolemy counts 48 distinct configurations; the count in the Commentary to Aratus is not unambiguous but 45 to 47 seems fairly certain. Ptolemy states that he had repeatedly[13] changed the boundaries of constellations and quotes some examples where he deviates from Hipparchus. In all these aspects the independence of Ptolemy's catalogue is evident.

As usual Pliny causes more trouble than he is helpful. In II, 110 of his Natural History he reports that the "experts" *(periti)* distinguish 72 constellations *(signa)* with 1 600 stars. A plausible explanation of the number 72 was suggested by Boll[14]: if one counts 36 decans for the ecliptic one is left with 36 extrazodiacal constellations which is still Ptolemy's number. But the 1 600 stars cannot be reconciled with any known list from antiquity or the Middle Ages. An equally inexplicable number of 1080 stars is ascribed to Hipparchus by an anonymous commentary to Aratus.[15] Amusing is Delambre's approach to these problems[16]: "Quant à la difficulté du travail, Pline paraît le mettre dans le nombre des étoiles, et cependant Hipparque n'en avait observée que 1080. Ce serait l'affaire de quelques belles nuits."

An unexpected discovery happened in 1936 when W. Gundel found in a Latin "Liber Hermetis Trismegisti" a list of 68 stars[17] with longitudes (all but one[18] in integer degrees) with a few exceptions about 2° or 3° less than the corresponding Ptolemaic longitudes. If one assumes that simple truncation of minutes produced the integer values found in the Liber Hermetis, a procedure commonly encountered in mediaeval treatises, one finds that these numbers fit the time of Hipparchus or a little later.[19] Also the terminology agrees with Hipparchus' Commentary to Aratus.[20] Finally one may note that the list in the Liber Hermetis begins with a little star (Asinorum septemtrionalis[21]) of longitude ♋ 8° which is the point of the summer solstice in the Babylonian "System B."[22] Perhaps also

[12] Hipparchus, Arat. Comm., p. 186 to 271 ed. Manitius. Also the number of stars within the single constellations (using the index Manitius, p. 364 to 372) shows the expected relation: in 7 cases the numbers are equal, in 2 cases the Commentary to Aratus has 2 more stars than the excerpts, in one case one more. For the remaining 35 constellations the totals in the excerpts exceed the number of stars mentioned in the Commentary; the latter was not intended to enumerate all stars in each constellation.

[13] Heiberg II, p. 37, 15: πολλαχῆ.

[14] RE 6, 2, col. 2417. 1. Another suggestion was made by Dreyer [1917], p. 529 note (counting the external groups of stars as constellations, resulting in a total of 70) which seems to me less plausible.

[15] Maass, Comm. Ar. rel., p. 128 (No. 12). The same formulation also in CCAG 8, 4, p. 94, fol. 10.

[16] HAA I, p. 290.

[17] Gundel, HT. His assumption (p. 135, p. 142, note 1) that the original number of stars must have been 72 seems to me unfounded.

[18] The one exceptional case (Gundel HT, p. 25, 8 and p. 152, No. 63) gives 30 minutes beyond integer degrees.

[19] From about 130 to 60 B.C.; cf. for details my Exact Sciences (2), p. 68f.; also below p. 287, n. 30. Gundel based his hypothesis of partly pre-Hipparchian origin on the comparison of rounded with not rounded numbers and dealing with the resulting differences as if they were exact.

[20] Gundel, HT, p. 127ff.

[21] γ Canc; Almagest: $\lambda = ♋10;20$, $\beta = 2;40$.

[22] We also known that the Babylonian division of seasons took the summer solstice as the point of departure (Neugebauer [1948]).

Hipparchus' Catalogue of stars is related to Babylonian norms as we know in much greater detail for his lunar theory.[23]

It is of interest to see that the majority of stars in the Hermetic catalogue is defined by longitudes (no latitudes are given), except for a few cases which preserved the "polar longitudes" known from the Commentary to Aratus.[24] We also have evidence for the definition of stellar positions with respect to "alignments"[25] e.g. by means of the location of 3 or more stars on a "straight line". Ptolemy quotes[26] from Hipparchus a list of such alignments with respect to all twelve zodiacal constellations (again beginning with Cancer), probably from the treatise "On the Fixed Stars" which one usually identifies with the "Catalogue of Stars".[27] In other words one is led to the conclusion that the Catalogue of stars showed the same complexity of stellar coordinates as we know from the Commentary to Aratus.[28]

Nobody doubts that long before Hipparchus there existed descriptions of constellations in which the relative positions of the stars were described in one way or another. Hipparchus' own references to the writings of Eudoxus are proof enough. When Ptolemy says that he had redefined the association of many single stars with respect to the traditional configurations he adds the remark that his predecessors did not act differently.[29] But the arrangement of stars in pictures is something other than a catalogue of stars where the main problem consists in giving accurate positions for the single stars, regardless of their grouping. Here, it seems, that Ptolemy had only one predecessor, namely Hipparchus, who in turn, according to Ptolemy, had only very few earlier observations at his disposal, mainly those by Aristyllus and Timocharis, which were unfortunately not of very reliable quality.[30]

From the modern literature one may get the impression that there had existed, about a century before Hipparchus, a catalogue of stars, being part of the "Catasterisms" ascribed to Eratosthenes. We shall discuss in other context[31] some of the intricate problems connected with this work. For the moment it will suffice to say that there exist three (and only three) passages which contain numerical data[32] and which were assumed to belong to the original work of Eratosthenes. Rehm then conjectured from these three cases (without attempting to bring any astronomical sense into the obviously corrupt Latin text) that Eratosthenes had assigned numerical data to *all* constellations.[33]

[23] Cf. below p. 309 f.
[24] Cf. above p. 279.
[25] σχηματισμοί.
[26] Alm. VII, 1, Manitius II, p. 5 to 8.
[27] Cf. above p. 277, note 4.
[28] Cf. above p. 283.
[29] Alm. VII, 4 (Manitius II, p. 31 f.).
[30] Alm. VII, 1 (Manitius II, p. 4). The longitude of Spica, observed by Timocharis as ♍22;20 (in −293) and as ♍22;30 (in −282) and referred to by Hipparchus as "about ♍22 in the time of Timocharis" (Alm. VII, 3 and VII, 2, respectively) appears also with ♍22 in Gundel's Hermes Trismegistos (p. 149, No. 15).
[31] Below p. 577.
[32] First remarked by Boehme [1887], p. 298. Cf. for the text Maass, Comm. Ar. rel., p. 183, 186, 189. Translation and commentary below p. 288 ff.
[33] Rehm [1899], p. 265.

In fact, however, the passages in question concern the three circumpolar constellations, Great and Little Bear and Draco, and determine their boundaries by means of maximal and minimal distances from the north pole and, in the east-west direction, by means of the extremal polar longitudes (cf. Fig. 282), i.e. by simultaneous culminations. The numerical data (as far as they make any sense at all) agree only with the time of Hipparchus. In the Commentary to Aratus he discusses constellations insofar as they rise and set. Now we also see how he handled the circumpolar cases. There remains no trace of a catalogue of stars by Eratosthenes who obviously belonged, to use Hipparchus' words[34] to the "old ones" who did not associate accurate numerical data with their description of constellations.

Turning to the time after Hipparchus but before Ptolemy one has to mention a conjecture by aṣ-Ṣūfī (10th century) who thought that Ptolemy had obtained his stellar coordinates by adding 0;25° to the longitudes found by Menelaos 41 years earlier. Aṣ-Ṣūfī based this theory on a conjecture by al-Battānī (end of ninth century) who knew through the Almagest[35] of Menelaos' observations of Spica and of β Scorpii, made in A.D. 98. Björnbo [1901] took aṣ-Ṣūfī seriously, assumed the existence of a catalogue of stars by Menelaos, although there is no trace of it in any ancient source, and combined the theory of schematic borrowing from Hipparchus by Ptolemy with the assumption of similar borrowings from Menelaos. All these speculations are untenable as has been shown by Nallino, by Dreyer, and by Vogt.[36]

Hence, to the best of our knowledge, the first catalogue of stars, though not yet based on orthogonal ecliptic coordinates, is the Hipparchian, the second one is Ptolemy's catalogue in the Almagest. The latter is separated by 740 years from the first Islamic catalogue, by al-Battānī,[37] i.e. by an interval almost three times as long as between Ptolemy and Hipparchus; this made possible an essential improvement in the determination of the constant of precession.

1. Appendix. Hipparchian Boundaries for Circumpolar Constellations. The following is a tentative translation of three passages from a Latin Aratus text, published by Maass, Comm. Ar. rel. p. 183, 196, 189. The text is corrupt, the Latin barbaric, hence much in the following is only guesswork.[1] Nevertheless about half of the coordinates under consideration can be associated with known stars and show agreement only with the time of Hipparchus.[2]

A. (1) The Great Bear, located in the north, occupies in length ... 365 (variants: 265 and 75) equinoctial degrees (i.e.?) parts of a circle.

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[34] Hipparchus, Comm. Arat., p. 184, 23 ed. Manitius.

[35] Alm. VII, 3 (Manitius II, p. 26 and p. 28, respectively).

[36] Nallino, Battānī I, p. 124, p. 292; II, p. 269ff.; Dreyer [1917], [1918]; Vogt [1925], col. 37f. Cf. also Knobel [1877], p. 3f.

[37] Nallino, Battānī II, p. 144 to 186. Much shorter lists of fixed star positions are found in earlier zījes, e.g. in the Mumtaḥan zīj (about A.D. 830) for 24 stars. Cf. Kennedy, Survey, p. 146.

[1] The division in sentences is mine, made for easier reference.

[2] For the determination of stellar coordinates I have used α and δ whenever given in P. V. Neugebauer, Sterntafeln, interpolated for -130. Then $\bar{\delta}=90-\delta$ and for μ, the "polar longitude," $\tan\mu=\tan\alpha/\cos\varepsilon$. In all other cases I took λ and β from Peters-Knobel, Catal. (listed for -130) and computed α and δ.

(2) From Gemini ... equinox ... to the Vergiliae 5 degrees.
(3) In width it extends 23 degrees, namely from 18 1/2 from the north pole to 40 degrees.
(4) ... which precedes a little ... above it the bright star in the throat is 18 1/2 degrees distant from Gemini.
(5) The last one towards east is 30 degrees distant (from ...?).
(6) And the northern(most) is the one in the shoulder, 18 1/2 degrees distant from the north pole.
(7) The southern(most) (is) the one in the two hind feet, 40 degrees distant from the north pole.
(8) There remain 112 degrees and a fraction.

Comments

(1) Probably contamination of the dimension in length, now lost, and a gloss explaining the division of the circle in 360 (of course not 365) degrees.

(2) For ♊ cf. (4). Equinox: perhaps reference to ♎ when the Vergiliae (= Pleiades) rise at sunset at the time of the autumnal equinox.[3]

(3) $40-18\ 1/2=21\ 1/2\approx 22$ not 23. Cf. (6) and (7); also (8).

(4) The name suggests the star Uma τ but for it $\mu\approx$ ♊ 26 not 18 1/2. It is also unlikely that a star "in the throat" should mark the extremal position. The star "in the muzzle" (o) would be much better with $\mu\approx$ ♊ 20 1/2 (Hipparchus, Comm. Arat., p. 211). Both stars are only of 4th magnitude.

(5) This should be the star Uma η but its $\mu\approx$ ♎4;45 $\bar{\delta}=29;18$. From (5) and (4) would follow the "length" $\Delta\mu\approx 98;45°$.

(6) Uma α with $\bar{\delta}=18;31$. Cf. (3).

(7) Uma μ with $\bar{\delta}=39;35\approx 40$. Cf. (3). According to Hipparchus, Comm. Ar., p. 46, 7 neither the star μ nor τ (cf. (4)) were part of the Great Bear for the "old" astronomers.

(8) Perhaps corrupted from $40-18\ 1/2=21\ 1/2$; cf. (3) and below B (9).

B. (1) The Little Bear, located in the north, has the length of the fourth part of the equinoctial circle and 7 degrees.
(2) As width it has 1 1/2 degrees.
(3) ... which are less than 8 degrees.
(4) ... twelve (degrees) and two fifth degrees.
(5) There precede 12 stars which lead it.
(6) These located in the shoulder occupy the beginning of Sagittarius.
(7) The last to the east lies at the top and occupies Pisces 17 degrees.
(8) And the northern(most) one of the two in the shoulder is 40 degrees distant from the north pole.
(9) There remain ... equinoctial degrees.

[3] Cf., e.g., Geminus, ed. Manitius, p. 216, 11.

(10) The southern(most) one is (the one) in the tail.

(11) The remaining two are called Circenses because they move (always) in the (same) circle.

Comments

(1) "Equinoctial circle" serves only to define the "degrees" as "equinoctial". $\Delta\mu = 97°$: from (6) and (7) one finds 107°. Hence emend 7 to 17.

(2) The extremal distances from the pole (cf. (3) and (4)) give $\Delta\delta = 4\ 1/2°$. The error 1 instead of 4 suggests a Greek original. Cf. also (8) and below **C**(1).

(3) and (4) give the values which lead to (2): northernmost Umi β with $\bar{\delta} = 7;56°$ (text: less than 8°), southernmost Umi α (cf. (10)) with $\bar{\delta} = 12;29°$ (text 12;24°), hence $\Delta\delta = 4;33 \approx 4\ 1/2$ (cf. (2) and (4)).

(4) Also Ptolemy, Geogr. I, 7, 4 (ed. Nobbe, p. 15, 6) quotes (from Marinus) 12 2/5° as Hipparchian value for $\bar{\delta}$ of the southernmost star (α) of the Little Bear. Cf. Mžik, Ptol. Erdkunde, p. 85/86; also Strabo, Geogr. II 5, 35 (Loeb I, p. 507; ed. Aly, p. 162, 26–28).

(5) This interpretation was suggested to me by Mr. G. Toomer. Cf. also Robert, Erat. Cat., p. 59a, 10: "above (it, the Little Bear) 10 other (stars) which lead (it)". No such extraneous stars are known in the ordinary iconography.

(6) For the stars Umi β and γ one finds $\mu = ♐\ 3;4$ and 3;30, respectively. Hipparchus (cf. above p. 279) gives for the head (not the shoulder, but cf. (8)) $\mu = ♐\ 2$. Cf. also Hipparchus, Comm. Arat., p. 274, 23–26 where the stars β and γ are listed as belonging to "about" the beginning of ♐ (*περὶ τὴν ἀρχήν*).

(7) For Umi α I find $\mu = ♓\ 16;36$. Text: ♓ 17. Hipparchus, Comm. Arat., p. 56, 14 and Gundel, HT, p. 151, No. 41: ♓ 18.

(8) For Umi β (cf. (6)) one has $\bar{\delta} \approx 8°$ (cf. (3)) not 40. This error suggests a Greek original (μ for η); cf. (2) and below **C**(1).

(9) Cf. above A(8). The text probably ends here; what follows are glosses.

(10) Umi α; cf. (4).

(11) Probably the stars η and γ with $\bar{\delta} \approx 11;30$ and 11;15, respectively. Cf. also Robert, Erat. Cat., p. 58/59.

C. (1) Serpens, which is located in the north, occupies in length (the arc) of the equinoctial circle from the fourth degree of Leo to the 28th of Capricorn.

(2) In width it has 26 degrees and two thirds.

(3) In the part which belongs to Leo (the distance) from the pole is 5 degrees and two thirds;

(4) in the other part, which belongs to Sagittarius, 35 degrees.

(5) The preceding star, which is at the top of the tail, occupies one degree of Leo.

(6) The easternmost one is the northern one on the side.

(7) The one on its neck … has a distance of 27 degrees from the pole.

(8) And the southernmost one in the region of Leo, the one in the tail, has a distance of 10 degrees from this pole.

Comments

(1) Note the old terminology Serpens (Ὄφις) for Draco; cf. Rehm [1899], p. 262. Here ♌ 4 instead of ♌ 1 as given correctly in (5). This error suggests a Greek original; cf. above **B**(2).

(2) It would follow from (3) and (4) that $\Delta\delta = 35 - 5;40 = 29;20$ instead of 26;40. Cf. also (7).

(4) Probably the star β Dra with $\bar{\delta} = 34;59°$ $\mu =$ ♐ 12;38. Also Hipparchus, Comm. Arat. p. 34, 13: $\bar{\delta} = 35°$.

(5) Indeed λ Dra has $\mu =$ ♌ 1;2. Cf. (8).

(7) A star with $\bar{\delta} = 27°$ is neither northernmost nor southernmost. Hipparchus, Comm. Arat., p. 112, 26 gives for the southernmost star (γ Dra) $\bar{\delta} = 37°$; same p. 34, 17.

(8) For λ Dra $\bar{\delta} \approx 10;20°$. "Southernmost" is wrong (cf. (4)) and perhaps error for northernmost, although this would contradict (2) and (3).

D. Stellar Magnitudes

Ptolemy's catalogue of stars assigns to each star a "magnitude" between 1 and 6, excepting 5 stars which are called "nebulous" and 9 "faint" ones. The magnitudes themselves are not always integers but occasionally modified by "greater" or "less", indicating a brightness between two magnitudes.[1] Ptolemy's classification remained essentially valid until the telescope made more accurate and objective definitions possible.[2]

We do not know where and when the concept of six stellar magnitudes originated. Again Hipparchus' Commentary to Aratus is the almost only source from which one can hope to obtain information about the terminology at his time and before. There is no evidence from this early period for numerical classes of stellar magnitudes; only qualitative distinctions are made like "bright" and "small" stars,[3] etc. Similar terms are also found in the Latin Hermes Trismegistus.[4] Everything points to merely qualitative distinctions, often only with reference to the stars within a specific constellation (e.g. "the brightest of all" in the constellation[5]). Hipparchus also discusses terms like "weak" applied to a whole constellation by

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[1] This is the case for 154 stars among 1008. For the distribution of the different magnitudes see the summaries at the end of each section (Manitius II, pp. 43, 45, 64) and the total at the end of the catalogue.

[2] For a comparison with modern standards cf. Peters-Knobel, Ptol. Cat., p. 120f. and the literature quoted there.

[3] Manitius, p. 293f. in his edition of the Commentary to Aratus, gave a list of all occurrences of these terms in relation to the individual stars. He came to the conclusion that "bright" (λαμπροί) are the stars of the first three magnitudes. This result is, however, not too well founded because the term "bright" (and, repeatedly, for the same star "very bright") occurs about 5 times as frequently than all four remaining terms together. What is really made evident, it seems to me, by Manitius' statistics is the absence of an accurate terminology.

[4] Gundel HT, p. 133 and p. 134. Only one star (ϑ Erid.) is called "magnitudinis primae" (in agreement with the Almagest). In the Commentary to Aratus this star is called "very bright." Also Servius (around A.D. 400) denotes in his Commentary to Vergil's Georgics I, 137 (ed. Thilo, p. 164) a terminology as Hipparchian which is similar to the one in the Hermes Trismegistus. Cf. note 7.

[5] Hipparchus, Comm. Ar., p. 238, 31.

Aratus and Attalus[6]; a passage in Servius[7] shows that he himself "writing on constellations" (i.e. in the Catalogue of stars?[8]) followed the same practice. This shows again how much importance was placed on the constellations in contrast to the characteristics of individual stars; the role of "alignments" versus coordinates is only another aspect of the same general situation.

It is not surprising to find that Hipparchus uses only rarely the term "magnitude" when he speaks about the brightness of stars.[9] We cannot tell what Pliny had in mind (if anything) when he wrote[10] that Hipparchus had "constructed instruments to indicate positions and magnitudes" of the stars. But the Latin version of the excerpts which are probably based on Hipparchus' Catalogue of stars[11] also give as title "De magnitudine et positione ...".[12] All this seems to indicate that the Catalogue of stars contained more about stellar magnitudes than we know from the Commentary to Aratus.

The earliest extant evidence for a classification of stars in six orders of magnitude is found in Manilius' Astronomicon, in the fragmentary section which now concludes Book V. Here the poet speaks about a "tertia forma" for the Pleiades[13] and a "quartum quintumque[14] genus", plus a remaining (i.e. sixth) class of stars[15]. In spite of the lack of precision it seems clear from these passages that the concept of six stellar magnitudes existed at the beginning of our era.

2. The Length of the Year. Precession

Hipparchus' most famous achievement is unquestionably the discovery of precession. It requires the existence of earlier records of fixed star distances with respect to equinoxes or solstices. Some such records, about 170 years old,[1] were available to Hipparchus. It also requires sufficiently reliable data for the moments of equinoxes and solstices, i.e. for the length of the tropical year. We do not know in what order Hipparchus' work proceeded but we know that he wrote at least two treatises which are closely related to these problems, one "On the variation of solstices and equinoxes",[2] the other "On the length of the year".[3] As a natural

[6] E.g. p. 42 to 45.

[7] Servius, Comm. in Verg. Georg. I, 137 (ed. Thilo, p. 164): "nam Hipparchus scripsit de signis et commemoravit etiam, unumquodque signum quot claras, quot secundae lucis, quot obscuras stellas habeat."

[8] "De signis"; for the title of the "Catalogue of Stars" cf. above p. 277, note 4.

[9] Only twice in the Commentary to Aratus; once more in a quotation from Attalus.

[10] NH II, 95.

[11] Cf. above p. 285.

[12] Maass, Comm. Ar. rel., p. 137.

[13] In the Almagest three stars of the Pleiades are considered to be of the 5th magnitude, only one of the 4th.

[14] Changed by Housman arbitrarily to sextumque which makes no sense.

[15] Manilius V, 710–717 ed. Housman (V, p. 89–91), 711–719 ed. Breiter (I, p. 148, II, p. 178f.).

[1] From observations made by Timocharis in Alexandria in the years −294/−282 (Alm. VII, 3, Manitius II, p. 22 to 27).

[2] Alm. VII, 2 (Heib. II, p. 12, 21): περὶ τῆς μεταπτώσεως τῶν τροπικῶν καὶ ἰσημερινῶν σημείων.

[3] According to a quotation from Hipparchus (Alm. III, 1, Heib., p. 207, 20) the title was περὶ τοῦ ἐνιαυσίου χρόνου βιβλίον ἕν. Ptolemy quotes it as περὶ ἐνιαυσίου μεγέθους (Alm. III, 1, Heib. I, p. 206, 24; VII, 2 and 3, Heib. II, p. 15, 18 and 17, 21). From the last quoted references we know that this work was written when Hipparchus was aware of the existence of precession.

consequence of these investigations may be considered a book "On intercalary months and days"[4] of which we have perhaps a fragment preserved in Censorinus.[5]

A. Tropical and Sidereal Year

Through an explicit quotation by Ptolemy from Hipparchus we know that he defined as "year" the tropical year, i.e. the interval between returns of the sun to the same solstice or equinox,[6] a practice later on followed by Ptolemy[7] but by no means generally accepted.[8]

In the same passage from Hipparchus it is also stated that he found the length of the (tropical) year to be 365 1/4 days minus 1/300 of one day (i.e. 365;14,48 days), a value which Ptolemy confirmed and consistently used in his own computations.[9]

In the work "On the length of the year" Hipparchus came to the conclusion that the equinoctial points move at least 1° per century in a direction opposite to the order of the zodiacal signs.[10] This is the famous constant of precession which was considered valid until the 9th century when Muslim astronomers began to make independent observations.

Whenever a value for the constant of precession is known one can compute the length of the sidereal year from the tropical, or vice versa. Assume that the sun has to move in one century 1° beyond complete tropical rotations in order to return to the same fixed star. Since the mean velocity of the sun is $0;59,8^{\circ/d}$ this requires about $1;0,53^d$. Hence we know that 100 sidereal years are this amount longer than 100 tropical years, i.e. $1;0,53^d$ longer than $100(365\ 1/4 - 1/300) = 100 \cdot 365\ 1/4 - 0;20^d$. Therefore 100 sidereal years amount to $100 \cdot 365\ 1/4 + 0;40,53^d$. Hence 1 sidereal year $= 365\ 1/4 + 0;0,24,32^d \approx 365\ 1/4 + 0;0,25^d = 365\ 1/4\ 1/144^d$.

A value for the (sidereal) year of Hipparchus is mentioned by Galen, the famous physician of the second century A.D. In his treatise "On seven-month children"[11] he tells us that according to Hipparchus half of a "year" amounts to 182 days 15 hours" and a little fraction of about 1/24 of one hour". This gives for one year the length of $365\ 1/4\ 1/288^d$. The fraction $1/288^d = 5$ minutes instead of $1/144^d = 10$ minutes suggests an error caused by repeated halving. This emendation is supported by a passage in another author of the second century, Vettius Valens, who says[12] that the Babylonians had a year of $365\ 1/4\ 1/144^d$. It would not be surprising to see this value for the sidereal year, rightly or wrongly,[13] considered to be of Babylonian origin when found in writings of Hipparchus.

[4] Cf. below p. 308.
[5] Cf. below p. 296.
[6] Alm. III, 1 (Manitius, p. 145) from the work on the length of the year.
[7] Alm. III, 1 (Manitius, p. 132).
[8] Astrological computations, e.g., are commonly based on sidereal coordinates. Cf., e.g., for the second century A.D., Vettius Valens (Neugebauer-Van Hoesen, Gr. Hor., p. 172, p. 180). The astrologers of the 5th century use, in general, more sophisticated astronomical methods and hence adopt with the tables of Ptolemy and Theon tropical coordinates.
[9] Alm. III, 1 (Manitius, p. 146). The same value is also found in the Romaka Siddhânta (Pañca-Siddhântikâ I, 15 and VIII, 1; Neugebauer-Pingree I, p. 31, p. 85; II, p. 11, p. 59).
[10] Alm. VII, 2 (Manitius II, p. 15).
[11] Preserved only in Arabic; published by Walzer [1935]. Cf. also Neugebauer [1949, 1].
[12] Anthol., ed. Kroll, p. 353, 12f.; cf. below p. 601.
[13] The value 365 1/4 1/144 is, however, not attested in cuneiform sources. This is not very significant since we know only little about the Babylonian solar year. Cf. below II B 8.

Table 29

Autumnal Equinoxes

No.	3. Call. Per. year	Date	Moment	No. in Table 28, p. 276
1	17	Mesore 30 (−161 Sept. 27)	sunset	1
2	20	Epag. 1 (−158 Sept. 27)	morning	2
3	21	Epag. 1 (−157 Sept. 27)	noon	3
4	32	Epag. 3/4 (−146 Sept. 26/27)	midnight	4
5	33	Epag. 4 (−145 Sept. 27)	morning	7
6	36	Epag. 4 (−142 Sept. 26)	evening	8

Vernal Equinoxes

No.	3. Call. Per. year	Date	Moment	No. in Table 28, p. 276
1	32	Mekhir 27 (−145 March 24)	morning (?)	5
1a	until y. 37	agreement with 365 1/4		
2	43	Mekhir 29/30 (−134 March 23)	after midn.	10
2a	until y. 50	agreement with 365 1/4		
3	50	Phamen. 1 (−127 March 23)	sunset	13

For the determination of the parameters of the motion of precession Hipparchus had at his disposal only few observations, neither very old nor very accurate.[14] One can therefore well understand that he formulated his results very cautiously and in a preliminary form.[15] For the length of the tropical year he had observations of solstices, made by "the school of Meton and Euctemon" in −431 and by Aristarchus in −279, hence preceding his own observations[16] by about 300 to 150 years, intervals well suited to establish a mean value for the length of the tropical year.

Nevertheless observations of autumnal equinoxes between −161 and −141 and of vernal equinoxes between −145 and −128 showed such contradictory results that Hipparchus could not convince himself of the constancy of the tropical year. Fig. 283 illustrates these data which are also listed in Table 29.[17] The autumnal equinoxes reach deviations up to 3/4 of one day from the interval of 365 1/4 days, whereas the vernal equinoxes agree with a 365 1/4 days' interval for the whole stretch of 28 years.

Hipparchus realized, of course, that instrumental errors could easily account for errors of about 1/4 day[18] (e.g. an error of 0;6° in the solar declination) but he seems to have found that real variations in the length of the tropical year must be admitted.[19] We shall come back to this conclusion in other context.[20] Obviously

[14] Alm. VII, 1 (Manitius II, p. 4).

[15] Cf., e.g., his doubts whether or not the poles of the ecliptic are really the center of motion (Alm. VII, 3 Manitius II, p. 17).

[16] Alm. III, 1 (Manitius, p. 143ff.).

[17] The diagonals in Fig. 283 represent intervals of exactly 365 $1/4^d$.

[18] Alm. III, 1 (Manitius, p. 135).

[19] Cf., e.g., the statement about the variations found with the permanently mounted ring in Alexandria (Alm. III, 1 Manitius, p. 134, 1 to 8).

[20] Below p. 295.

it was not before Ptolemy that the constance of the tropical year was firmly established.

A particular instance of Ptolemy's arguing against Hipparchus' acceptance of variations in the length of the year[21] is also of interest from the viewpoint of observational techniques which are in a large measure responsible for the incorrect value of the Hipparchian-Ptolemaic constant of precession. Ptolemy describes[22] Hipparchus' procedure of determining the longitude of Spica (α Virg) which is located near the autumnal equinox. Let t_0 be the observed moment of the vernal equinox, t_1 the moment of the middle of a lunar eclipse near t_0, then one can compute from the time difference $t_1 - t_0$ the true longitude $\lambda_\odot$ of the sun at t_1 and hence also of the moon $\lambda_☾ = \lambda_\odot + 180$. Correcting $\lambda_☾$ for parallax we know the distance from the autumn equinox of the moon at the eclipse middle. The observation of the elongation $\lambda - \lambda'_☾$ of the star at this moment furnishes the longitude λ of the star.

Hipparchus applied this method in the case of two eclipses, 11 years apart,[23] and finds for the corresponding longitudes of Spica a difference of 1;15°. Since such a displacement is excluded for the star Hipparchus concluded that the interval between the equinoxes cannot have been as regular as originally assumed (cf. Fig. 283, p. 1309). Ptolemy rightly criticizes this argument as circular because the longitudes were found under the assumption of correctly determined moments t_0 for the equinoxes. For us it is of interest to see how large the accumulated error of ancient computations carried out on the basis of the best available theoretical and empirical data can be.

An example for the selection of eclipses in relation to equinoxes and solstices seems to have been preserved in the discussion of Hipparchus' attempts to determine the eccentricity of the lunar orbit or the radius of the epicycle from triples of eclipses (Alm. IV, 11). A look at the corresponding configuration (cf. below Fig. 288, p. 1311) shows that the eclipses R and S are very near to the solstitial diameter whereas U and V belong to the equinoxes. In particular V is valuable not only in relation to the equinoxes but again for the determination of the longitude of Spica (at about ♍ 22) since, according to Ptolemy[24] the mean moon had the longitude ♎ 1;7, the true moon ♍ 26;16. Unfortunately this aspect of Hipparchus' choice is not discussed in our sources.[25] Since we know, however, from Ptolemy

[21] Cf. Alm. VII 2.

[22] Alm. III, 1 (Manitius, p. 137f.).

[23] The vernal equinoxes are the Nos. 1 and 2 listed in Table 29, p. 294 (cf. also Nos. 5 and 10 in Table 28, p. 276). The lunar eclipses are both total (−145 Apr. 21 and −134 March 21). An additional detail is given by Theon in his commentary to this passage (cf. Rome CA III, p. 828, note (2)) where he tells us that (at least at the second of the two lunar eclipses) a star emerged from occultation by the moon at the moment of first contact (determined by Rome as *h* Virg).

[24] Alm. IV, 11, Manitius I, p. 252.

[25] A trace of the association of the eclipses R, S, T with the solstices can perhaps be seen in the peculiar description of the moon as "eclipsed from the summer solstice rising(s)" (ἀπὸ θερινῆς ἀνατολῆς in R and S, ἀπὸ θερινῶν ἀνατολῶν in T). Manitius always takes this to mean "von Nordost her"; but this is only possible for R and T while the eclipse S begins at the south-east rim of the moon. No similar terminology appears with U, V, W; note, that the occultation of Spica by the moon, observed in −293 by Timocharis (Alm. VII, 3, Heib. II, p. 28, 14f.) is said to have begun at the moon's rim "toward the equinoctial rising" (πρὸς ἰσημερινὴν ἀνατολήν), i.e. "exactly east."

that his data were marred by serious errors[26] we may assume that also these eclipses did not help to clarify the problem of stability of the tropical year.[27]

The character of the motion of precession could not immediately become clear to Hipparchus, even after he had realized that one must distinguish between the return of the sun to equinoxes and solstices and the return to the same fixed star. We know from a remark in the Almagest[28] that at first he considered the possibility that only the longitudes of stars near the ecliptic show a slow increase with time. In other words there could have existed beside the known planets still other celestial bodies with a very slow motion in longitude. Here the invariance of the alignments of the constellations[29] must have shown the incorrectness of this preliminary hypothesis. But, as we have mentioned before,[30] Hipparchus could not find certain proof from his very limited empirical material that the pole of the ecliptic was the center of the motion of precession. Ptolemy, 2 1/2 centuries later, no longer doubted it.

B. Intercalation Cycles

Investigations about the length of the solar year and the lunar month are naturally related to the question of constructing convenient cycles of intercalations which equate, with sufficiently good approximation, an integer number of years and an integer number of months.

Such a cycle is also associated with the name of Hipparchus by Censorinus in his De die natali (written in A.D. 238[1]) in the form that 112 years among 304 should be intercalary, i.e., should contain 13 months.[2]

It is not difficult to see the reasoning which underlies this cycle. First, both 304 and 112 are divisible by 4; hence the Hipparchian cycle is 4 times the Callippic cycle[3] of 76 years among which 28 are intercalary. This cycle, in turn, is 4 times the "Metonic" cycle of 19 years with 7 intercalations, well-known as providing the calendaric frame for the Babylonian ephemerides though only approximating the more accurate values for the length of the mean synodic month and the (sidereal) year known in Babylonian astronomy.[4]

The Callippic cycle of 76 years is a modification of the Babylonian (or "Metonic") 19-year cycle, designed to obtain an integer number of "years" of 365 1/4 days each. In modern terminology: the Callippic cycle equates 76 julian years with 940 synodic months.[5]

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[26] Cf. below p. 318.
[27] Ptolemy's corrected data lead to very satisfactory dates for the solstices in −382/381 (from R and S) and for the equinoxes in −200/199 (from U and V).
[28] Alm. VII, 1 (Manitius II, p. 4).
[29] Cf. above p. 280, p. 287.
[30] Above p. 294, note 15.
[1] ed. Jahn, Proleg., p. V.
[2] De die natali 18, 9, ed. Hultsch, p. 38, 18f. One may assume that we have here a fragment from the work "On intercalary months and days" mentioned above p. 293.
[3] Cf. below p. 624.
[4] Cf. below II Intr. 3, 1.
[5] The resulting length of the synodic month would be 29;31,51,3,49, ... days.

The Hipparchian modification of the Callippic cycle is based on his result that the tropical year is shorter than the julian year by about 1 day in 300 years.[6] The multiple of 19 nearest to 300 is $304 = 76 \cdot 4$. Hence we know that a cycle of 304 years should contain not only $304 \cdot 12 + 112 = 3760$ synodic months but also the number of $304 \cdot 365\ 1/4 - 1 = 111\,035$ days.[7] Hence one year would obtain the length of 365;14,48,9,... days, only slightly more than the estimate $365\ 1/4 - 1/300 =$ 365;14,48. For the mean synodic month one also finds a good approximation 29;31,50,6,22,... of the Babylonian value 29;31,50,8,20 which was otherwise adopted by Hipparchus[8] and which is considerably better than in the Callippic cycle.[9]

Ginzel stated,[10] in my opinion correctly, that the Hipparchian cycle was never practically employed. Dinsmoor[11] tried to demonstrate that not only traces of actual use can be found but also that its chronological applications could be almost completely restored, beginning with −144 July 21. All this requires such a mass of complicated and unprovable hypotheses[12] that I can only consider them as a confirmation of Ginzel's conclusion.

C. *Constant of Precession; Trepidation*

In a preceding section[1] we reviewed the considerations which led us to ascribe to Hipparchus the estimate

$$365\ 1/4\ 1/144 \text{ days} = 365;15,25^{d} \tag{1}$$

for the length of the sidereal year. This number is arrived at by assuming 1° per century as constant of precession, in combination with the accepted length of $365;14,48^{d}$ for the tropical year. From a remark in the Almagest[2] we know, however, that Hipparchus considered a precession of 1° per century only as a lower limit and therefore it is quite permissible to look for a higher estimate for the precession in Hipparchus' work.

Evidence in this direction has indeed turned up.[3] Among the numerical relations underlying the Hipparchian lunar theory Ptolemy mentions[4]

$$126007^{d}\ 1^{h} = 345 \text{ sid. rot. of the sun } -7;30° \tag{2}$$

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[6] Cf. above p. 293.

[7] The insight that the Hipparchian cycle is motivated by his estimate for the length of the tropical year is due to Ideler, Chron. I, p. 352; also Ginzel, Hdb. II, p. 390.

[8] Cf. below p. 310.

[9] Cf. note 5.

[10] Ginzel, Hdb. II, p. 391. Similar already Ideler, Chron. I, p. 353.

[11] Dinsmoor, Archons, p. 410–423.

[12] E.g. Archons, p. 414: "Therefore we may assume that the authorities ... called in a specialist, namely, Hipparchos."

[1] Above p. 293.

[2] Alm. VII, 2 (Man. II, p. 15).

[3] The following is due to Viggo M. Petersen [1966]. Mr. Toomer drew my attention to the fact that exactly the same conclusions had been reached by L. Am. Sédillot in 1840 (cf. his Matériaux ... des sciences mathématiques chez les grecs et les orientaux, Paris 1845, p. 11–14). Van der Waerden [1970, 2] accepting Petersen's result as of absolute numerical accuracy expanded its consequences to all related parameters, in my opinion much too rigorously.

[4] Alm. IV, 2 (Man. I, p. 196); cf. below p. 310 (5).

or, sexagesimally written

$$35,0,7;2,30^{d} = 5,44;58,45 \text{ sid. rot.} \qquad (2a)$$

Thus one finds by simple division

$$1 \text{ sid. rot.} = 365;15,35,29,28,\ldots^{d} \approx 365\ 1/4\ 1/100 \qquad (3)$$

for the length of the sidereal year.

The corresponding difference between sidereal and tropical year is therefore

$$\Delta t = 365;15,35,29 - 365;14,48 = 0;0,47,29^{d}$$

requiring a solar motion of

$$0;0,47,29 \cdot 0;59,8 = 0;0,46,47,51°.$$

Hence (2) implies

$$\text{precession per year: } 0;0,46,48° \text{ or } 1° \text{ precession in 77 Eg. y.} \qquad (4)$$

It seems hardly possible to assume that Hipparchus in his investigations of the differences between sidereal and tropical years could have overlooked such a direct consequence of some of his basic parameters. Hence one must conclude from the coexistence of (4) and of the previously mentioned limit of 1° precession per century that Hipparchus did not exclude a priori the possibility of variations in the lengths of the years, either sidereal, or tropical, or both. This conclusion is fully supported by Ptolemy's discussion of Hipparchus' methodology.[5]

Variations in the length of the year can equally well be expressed as variations in the value of the constant of precession. It is also plausible to assume that any such variation would have been considered to be periodic. In fact we know the existence in pre-Ptolemy times of a theory of a periodic change in the motion of precession, recorded by Theon,[6] conventionally named "trepidation of the equinoxes". This early theory assumes

$$\text{precession per year: } 0;0,45° \text{ or } 1° \text{ precession in 80 Eg. y.} \qquad (5)$$

Not only is this parameter practically identical with (4) but the periodic variation in the sidereal longitude λ^* of the vernal point is assumed to follow a linear zigzag function with the maximum $\lambda^* = 8°$ in the year Augustus −127 (i.e. −157), i.e. in a time exactly coinciding with Hipparchus' investigations of the equinoxes.[7] All this seems to suggest Hipparchus as the inventor of the theory of trepidation, a theory which Ptolemy preferred to disregard in silence.

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[5] Above p. 295.

[6] Cf. for the details below IV B 2, 3.

[7] Cf. above Table 28, p. 276.

§ 3. Trigonometry and Spherical Astronomy

1. Plane Trigonometry; Table of Chords

It is a frequently repeated story told about Hipparchus that he wrote a work on chords in 12 books.[1] This number is obvious nonsense since 13 books sufficed for the whole of the Almagest or of Euclid's "Elements" and none of Archimedes' works occupies more than two books. Ptolemy's discussion of the mathematical basis of the computation of chords covers one short chapter, followed by the tables which require only 8 triple columns (including the difference columns). Whatever clumsiness one may ascribe to Hipparchus in the presentation of the computation of chords,[2] 12 books cannot have been filled. Even if one included the whole of trigonometry in the largest sense one should remember that Ptolemy's spherical trigonometry with all preliminary material and all astronomical applications and tables fills only two books. The only conclusion left is the usual one: we know even less about the details of Hipparchus' work than one might assume at first sight.

There are certain indications of a rather primitive level of Hipparchus' plane trigonometry. In the Commentary to Aratus he measures arcs in "sections" (*τμήματα*) of 1/24 of the circumference,[3] units which are well known in certain sources of a more practical or elementary level and named "steps" (*βαθμοί*).[4] These units of 15° and their parts seem to have been the units on which Hipparchus' trigonometric tables were built and which also underlie the Indian sine tables, known in the west by the name "kardaga" (derived from a Sanskrit term meaning "half-chord").[5] Additional evidence for a connection between Indian trigonometry and Hipparchus' astronomy can be found in the stellar coordinates known as "polar longitudes" and "polar latitudes", or in the strange terminology of expressing arc lengths on any circle in terms of "zodiacal signs".[6]

The decisive step in proving that the Indian table of sines was derived from the Hipparchian table of chords was made by G. J. Toomer who showed[7] that the two tables not only agree in the steps of the argument but also in the basic radius of 3438 "minutes". Ptolemy, in describing the method for determining the radius r of the lunar epicycle (or the eccentricity e) not only gives his own numerical results but he also reports the values found by Hipparchus for the

[1] E.g. Rehm in R.E. 8, 2, col. 1669, 11 ff. who then discusses the "Lebensperiode" into which this work must have fallen. He takes from Theon's commentary to Alm. I, 10 (ed. Rome CA II, p. 451, 4 f.) as title *Περὶ τῆς πραγματείας τῶν ἐν κύκλῳ εὐθειῶν βιβλία ιβ'*. But Rome [1933], p. 178 has pointed out that the sentence in question does not contain a book title but has to be rendered as "a study on the chords was also made by Hipparchus in 12 books and so by Menelaos in 6." Cf. also Toomer [1973], p. 19/20.

[2] Theon indeed admires the conciseness of Ptolemy's derivations (ed. Rome CA II, p. 451).

[3] Aratus Comm., Manitius, p. 150, 2; cf. above p. 279.

[4] Cf. below IV B 5.

[5] Burgess, Sûr. Siddh., p. 64; cf. also Nallino, Scritti V, p. 220 f.

[6] Cf. above p. 278.

[7] Toomer [1973].

ratios R/r and R/e, R being the radius of the deferent:[8]

$$R/r = 3122\ 1/2/247\ 1/2 \qquad R/e = 3144/327\ 2/3. \tag{1}$$

Toomer found that these peculiar ratios are numerically explicable as the consequence of using in all trigonometric computations tables of chords based on the above mentioned radius 3438.

The reason for choosing this basic radius is, of course, well known: the unit of length on the radius is made the same as on the circumference c, such that the latter measures 360 (in other words one also measures the radius in "degrees" and "minutes"). Then, with $\pi \approx 3;8,30,$[8a] one finds for the radius

$$3,0/3;8,30 \approx 57;17,39,54 \approx 57;18° = 3438'. \tag{2}$$

The resulting tables of chords, in sexagesimal and decimal form, is given in VI C 5, 2, p. 1132.

It was also pointed out by Toomer that the computation of the Hipparchian table in steps of 7;30° requires fewer mathematical tools than Ptolemy's table. Indeed, one can reach all entries of Hipparchus' table from Crd 90° and Crd 60° as soon as one can find Crd $(180-\alpha)$ and Crd $\alpha/2$ from Crd α.[8b] The first requires only the Pythagorean theorem, while Archimedes had developed the necessary relations for the second,[8c] most likely used by Hipparchus. What is not needed, however, are formulae for Crd $(\alpha \pm \beta)$, necessary for Ptolemy to reach Crd 1°. It seems significant that Ptolemy uses for Crd $(\alpha \pm \beta)$ the theorem based on a quadrilateral of chords whereas he keeps the Archimedean procedure for Crd $\alpha/2$.

Obviously Ptolemy still had the Hipparchian tables of chords at his disposal, but this seems not the case for Heron, a century earlier. In the first book of his "Metrica"[9] he determined the area of the regular polygons from $n=3$ to $n=12$ under the assumption that in all cases the side has the length 10. Had he known a table of chords he would have kept the radius fixed and used the same formula in all cases. Furthermore he refers in two cases ($n=9$ and $n=11$) to a work "on the straight lines in a circle"[10] when he needs the ratio of the side of the polygon to the diameter of the circumscribed circle. These are exactly the two cases where he does not operate with the triangle formed by the side of the polygon and two radii but with the right triangle that has the diameter as its hypotenuse and the side of the polygon as its smaller side. Obviously the work in question is a treatise on polygons which is not based on a table of chords.[11] Perhaps it is on formal mathematical grounds that Heron avoids existing numerical tables and referred instead only to data obtained by Archimedes.[12]

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[8] Cf. below p. 315.
[8a] Cf. above p. 140, n. 3.
[8b] Cf. the diagram Toomer [1973], p. 19, Table II.
[8c] Cf. above p. 23.
[9] Metrica I, 17–25, Heron, Opera III, ed. Schöne, p. 46, 23–64, 31.
[10] Opera III, p. 58, 19; p. 62, 17–18. This need not to be understood as an exact title.
[11] This is a conclusion first clearly established by A. Rome [1933].
[12] Opera III, p. 66, 6–68, 5.

2. Spherical Astronomy

The titles of two works (if actually different) by Hipparchus on spherical astronomy are preserved: one "Treatise on simultaneous risings" from his own reference in the Commentary to Aratus,[1] the other "On the ascensions of the 12 zodiacal signs" from Pappus.[2] Through the latter we even know a few details from the contents: he demonstrated "by arithmetical methods" (δι' ἀριθμῶν) the symmetry of the oblique ascensions with respect to the equinoxes and he spoke about the existence of geographical regions for which an ecliptic arc, beginning at ♋0°, has a shorter rising time than an arc of equal length that ends at ♎0°.

One can hardly doubt that these "arithmetical methods" mean an extension of the Babylonian schemes of rising times to a sequence of geographical locations characterized by a linear sequence of longest daylights M. Since the rising times of a given sign (or arc) form a linear function of M, both for the pattern of System A and B,[3] the relation $\rho(♋)<\rho(♍)$ is valid for all $M(>12^h)$. Thus the statement ascribed to Hipparchus is factually correct but trivial, unless one assumes that Pappus implied that the extension of the Babylonian schemes to a sequence of latitudes was a novel feature due to Hipparchus.

Hipparchus had no spherical trigonometry in the mathematical sense of the term, as is evident from everything we know about Greek "Spherics" and Menelaus' role in it.[4] There are two ways, however, to overcome this difficulty. The first has been followed by Babylonian astronomy and consists in the use of arithmetical sequences; the second operates with geometrical constructions, either based on procedures known as "analemmata" or on stereographic projection.

That Hipparchus made use of both the arithmetical and geometrical methods can hardly be doubted. For the arithmetical approach we have not only Pappus' remark but we shall also be able to establish the use of difference sequences in Hipparchus' mathematical geography.[5] Questions concerning Hipparchus' familiarity with stereographic projection will be discussed in a later section.[6] The much less sophisticated analemma methods can safely be assumed known at an early period, e.g., from the theory of sun dials. Analemma methods also dominate Indian spherical trigonometry[7] and thus suggest one more parallel with Hipparchian astronomy.

A case in which the use of analemma methods can probably be assumed is found in the Aratus Commentary where Hipparchus wishes to determine the distance between a star S of given declination ($\delta=27\ 1/3°$), located in the western

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[1] Ar. Comm., p. 129, 6 and p. 148, 20f. (ed. Manitius): Ἡ τῶν συνανατολῶν πραγματεία.

[2] Pappus, Coll. II, p. 600, 10 (ed. Hultsch); trsl. Ver Eecke II, p. 458: Περὶ τῆς τῶν ιβ̄ ζῳδίων ἀναφορᾶς; cf. also Björnbo, Menelaos, p. 70.

[3] Cf. below p. 714.

[4] Above I A 2, 1. Cf. also Pappus' remarks concerning Menelaus (Hultsch, p. 600, 25–602, 1; Ver Eecke II, p. 459).

[5] Cf. below p. 304f.

[6] V B 3, 7 B.

[7] Cf., e.g., Varâhamihira, Pañcasiddhântikâ IV, 41 to 44; Neugebauer-Pingree, II, p. 41–44.

horizon, and the point T of the same declination located at the same moment in the meridian, the distance being measured on the parallel of declination δ. Hipparchus declares that he found "by rigorous methods,"[8] explained in a larger work,[9] that the arc above the horizon for the given star S is of the length of about $15-\frac{1}{20}$ of twenty-fourths of a circle (i.e. "steps" of 15° length) hence about 224;15°.[10]

In order to discover the basis for such a statement we simply apply the standard method of an analemma construction (cf. Fig. 284). In the plane of the meridian let OM represent the equator, δ the given declination. Thus $\mathrm{TR}=r_\mathrm{d}$ is the radius of a circle on which the star S travels during its daily rotation[11]; hence

$$\mathrm{VT}=2r_\mathrm{d}=\mathrm{crd}\,(180-2\delta). \tag{1}$$

S represents the point of the star in the horizon, T in the meridian, and we wish to find the length of the arc $\mathrm{SRT}=90°+n$. But from

$$\mathrm{UR}=1/2\,r_\mathrm{d}\,\mathrm{crd}\,2n=e \tag{2}$$

and $\mathrm{RO}=1/2\,\mathrm{crd}\,2\delta$ we have

$$\frac{\mathrm{UR}}{\mathrm{RO}}=\frac{\mathrm{crd}\,2\varphi}{\mathrm{crd}\,(180-2\varphi)}=\frac{r_\mathrm{d}\,\mathrm{crd}\,2n}{\mathrm{crd}\,2\delta}$$

and thus

$$\mathrm{crd}\,2n=\frac{2\,\mathrm{crd}\,2\varphi\,\mathrm{crd}\,2\delta}{\mathrm{crd}\,(180-2\varphi)\,\mathrm{crd}\,(180-2\delta)}. \tag{3}$$

Using the tables of the Almagest (I, 11) one finds with $\varphi=36$ and $\delta=27;20$

$$\begin{aligned}
&\mathrm{crd}\,2\varphi=\mathrm{crd}\,72=1{,}10;32\\
&\mathrm{crd}\,(180-2\varphi)=\mathrm{crd}\,108=1{,}37;5\\
&\mathrm{crd}\,2\delta=\mathrm{crd}\,54;40=55;6\\
&\mathrm{crd}\,(180-2\delta)=\mathrm{crd}\,125;20=1{,}46;36
\end{aligned}$$

and thus

$$\mathrm{crd}\,2n=\frac{2{,}9{,}32{,}52}{2{,}52{,}29}\approx 45;2{,}26$$

hence $2n\approx 44;6$ and therefore the day arc $180°+2n=224;6°$, i.e. practically the same as Hipparchus' result $\approx 224;15°$.

This result could suggest the conclusion that Hipparchus must have had a table of essentially the density of Ptolemy's table of chords, i.e. steps of 1/2°. In order to test such a hypothesis I repeated the whole computation assuming only

[8] For this meaning of διὰ τῶν γραμμῶν cf. below p. 771, n. 1 (in contrast to the above-mentioned δι' ἀριθμῶν).

[9] Presumably the work on simultaneous risings (cf. above p. 301).

[10] Ar. Comm., p. 150, 1–3. Manitius' translation (p. 151) and commentary (p. 297 f.) are incorrect since he assumes that 1/20 means 3 minutes instead 1/20 of 15°. This error furthermore forces him to assume as underlying geographical latitude $\varphi=36;29°$ (instead of simply 36°) in order to obtain agreement with modern computation, as if this were of any interest.

[11] This is the "day-radius" of Indian astronomy.

a "kardaga" table with 7;30° as smallest interval[12] and linear interpolation for all other values. Then one finds[13] $180° + 2n \approx 224;0°$, i.e. again a result near to Hipparchus'. Considering the unavoidable approximations of square roots and the roundings of divisions beyond our control it is clear that one cannot reconstruct from these data the type of the underlying tables.

In the Aratus Commentary one also finds the remark that the solstitial circles are about 10/11 of the equator,[14] i.e. $\text{crd}\, 2\varepsilon \approx 20/11$. This formulation is not accurate enough, however, to say more than that Hipparchus most likely assumed $\varepsilon = 24°$. This is confirmed, e.g., by remarks like "40° north of the equator, thus 16° north of the tropic."[15]

It is a well attested fact[16] that Hipparchus dealt with the oblique ascensions of the zodiacal signs, and from his remarks in the Aratus Commentary one may perhaps conclude that he had tabulated rising times for a sequence of geographical latitudes.[17] In the Aratus Commentary, however, he restricts himself to the longest daylight $M = 14\ 1/2^{\text{h}}$ (i.e. to the latitude of Rhodes) and means by the 12 "ζώδια" not the 12 zodiacal signs but the zodiacal constellations of unequal size which gave their names to the signs. Hence we cannot determine from these data[18] the scheme of oblique ascensions he may have developed. That he did not operate with the crudely incorrect assumption of equal rising times nobody would doubt, even without his explicit criticism of Attalus and Aratus[19]; but all details of his own approach remain unknown.

As we have seen (p. 302) the "analemma" Fig. 284 provides us directly with three quantities: the "day-radius"

$$r_{\text{d}} = 1/2\, \text{crd}\,(180 - 2\delta) \qquad \text{i.e. } r_{\text{d}} = \cos\delta, \tag{4}$$

the "earth-sine"[20]

$$e = 1/2\, r_{\text{d}}\, \text{crd}\, 2n \qquad \text{i.e. } e = r_{\text{d}} \sin n \tag{5}$$

and the "ascensional difference"[21] n, to be found from

$$\text{crd}\, 2n = 2 \frac{\text{crd}\, 2\varphi}{\text{crd}\,(180 - 2\varphi)} \cdot \frac{\text{crd}\, 2\delta}{\text{crd}\,(180 - 2\delta)} \qquad \text{i.e. } \sin n = \tan\varphi \tan\delta. \tag{6}$$

All these quantities are known for a given locality of geographical latitude φ as soon as the solar declination δ is known. Although solar declinations are easily obtainable by direct observations it is useful to show that it is again a simple analemma which provides δ to a given solar longitude λ. Indeed (cf. Fig. 285)

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[12] This is the equivalent of the Indian limit of 3;45° in a table of sines; cf., e.g., Pc.-Sk. IV, 1.

[13] The respective values are: $\text{crd}\, 2\varphi = 1,10;19,26$ $\text{crd}\,(180 - 2\varphi) = 1,37;1,55$ $\text{crd}\, 2\delta = 55;4,31$ $\text{crd}\,(180 - 2\delta) = 1,46;33,20$ thus $\text{crd}\, 2n = 44;57,3$.

[14] Ar. Comm., p. 96, 11.

[15] Ar Comm., p. 98, 20f.; also Theon Smyrn., p. 202, 19ff. (Hiller); Dupuis, p. 327.

[16] Cf. above p. 301, note 2.

[17] Ar. Comm., p. 182-185.

[18] Given in Ar. Comm., p. 244-271 (Manitius).

[19] Ar. Comm., p. 124/5.

[20] This is again a term of Indian astronomy (cf., e.g., Pc.-Sk. IV, 27 and 28).

[21] Cf., e.g., above I A 4, 3. According to Varāhamihira's introduction to the Bṛhat-Saṃhitā the determination of the day-radius and of the ascensional differences belongs to the topics which must be mastered by the astrologer (cf. Kern, Verspr. Geschr. I, p. 175).

let SMCE be the meridian, OM the intersection of its plane with the equator, OC with the ecliptic when the equinoctial points lie in the horizon. Turning the ecliptic 90° about OC as axis brings the vernal point into E and shows the given solar longitude λ undistorted as the angle EOΣ. Thus Σ' is the orthogonal projection of the sun upon the plane of the meridian and DΣ'F (parallel to MO) is the intersection with the sun's day-circle. Therefore the angle DOM is the solar declination δ which belongs to the given λ.[22]

Table 30

	Hipparchus			Alm. II, 6		
M	a at W.-Solst.	Δa	a	$a=\bar{\varphi}-\varepsilon$	Δa	φ
16^h	9 cubits		18°	17;37°		48;32°
		3 cub. = 6°			5;29°	
17	6		12	12; 8		54; 1
		2 c. = 4°			4	
18	4		8	8; 8		58
		1 c. = 2°			3	
19	3		6	5; 8		61

We now can turn to the use of arithmetical methods in problems of spherical astronomy and geography. From Strabo[23] we know of values given by Hipparchus for the noon altitude of the sun at the winter solstice for northerly geographical locations, i.e. for latitudes where the longest daylight (M) ranges between 16^h and 19^h. Hipparchus expresses the solar altitudes (a) in "cubits" (of 2° each) which form an arithmetical progression of the second order (cf. Table 30). The trigonometrically accurate data from the geographical sections in Alm. II, 6 show to what extent the arithmetical data deviate from the correct values. It is clear that the results obtained by means of a simple arithmetical device give a perfectly satisfactory answer to the problem in question. The same can be said for another application of the same methodology to the connection between the geographical "climata" and the length of daylight.

Strabo's excerpts from Hipparchus' geographical writing[24] do not give geographical latitudes expressed in degrees but terrestrial distances along a meridian measured in stades. We shall return in a later chapter[25] to Hipparchus' latitudinal scheme as described by Strabo. In the present context only a partial set of data needs to be considered.

Let S denote distances measured on a meridian in units of 100 stades. In Table 31 are listed the values of S[26] (beginning with $S=0$ at the equator) for the parallels that correspond to integer values of the longest daylight M, from $M=14^h$ to $M=19^h$. Since we know that Hipparchus followed Eratosthenes in assuming

[22] Fig. 285 allows one to read off the relation $\sin\delta=\sin\varepsilon\sin\lambda$, valid in the spherical triangle which relates δ to λ.

[23] Geogr. II 1, 18 and 5, 42 (Loeb I, p. 282/3; p. 514/517).

[24] Presumably the treatise against the Geography of Eratosthenes; Strabo, Geogr. II 1, 18 (Loeb I, p. 280–285) and II 5, 34–43 (Loeb I, p. 502–521).

[25] Below I E 6, 3 A, in particular Fig. 291, p. 1313.

[26] Not given explicitly but easily obtainable by adding intervals.

that 700st = 1° of a great circle[27] we can convert distances into estimates of latitudes φ. The last two columns in Table 31 give a comparison of these estimates $\varphi \approx S/7$ with the trigonometrically computed values from the Almagest.[28] The agreement is in general good enough[29] with two glaring exceptions at $M = 17^h$ and 18^h: not only do we find a deviation of about 2° from the Almagest but also the differences ΔS show an impossible change of trend.

Table 31

M	Localities	S	ΔS	φ	
				$S/7 \approx$	Alm.
14^h	400st south of Alexandria	214		30;34°	30;22°
15	between Rome and Naples	288	74	41; 8	40;56
16	Borysthenes	341	53	48;43	48;22
17	6300st north of Byzantium	366	25	52;17	54; 1
18	9100st north of Massalia	394	28	56;17	58
19	12500st north of Massalia	428	34	61; 8	61

This error in the extant text can be easily repaired, however, by counting the given intervals not from the parallel $M = 15\ 1/4^h$ (with $S = 303$) but from $M = 15\ 1/2^h$ (with $S = 317$); then the corresponding interval of 1400st accounts exactly for the missing 2°.

Having thus restored the correct sequence we look once more at the values of S and ΔS, replacing, however the parallel $M = 14^h$ by the parallel of Alexandria, 400st to the north. This gives us Table 32 which shows that the Hipparchian system of stades constitutes a difference sequence of the second order, similar to the pattern found before for the solar altitudes.[30] In other words we see once more a problem of spherical geometry, the relationship between longest daylight and geographical latitude, solved by applying to it an arithmetical scheme, in this case for the terrestrial distances reckoned in stades.

Table 32

M	S	ΔS	$\Delta\Delta S$
Alex.	218		
15^h	288	70	
16	341	53	−17
17	[380]	39	−14
18	[408]	28	−11
19	428	20	− 8

[27] Strictly speaking we know only for certain that Hipparchus followed Eratosthenes in assuming that the circumference c of the earth measures 252000 stades (Strabo, Geogr. II, 5, 7 and II, 5, 34), but we are not sure whether it was Hipparchus who first introduced the division of c into 360 degrees or Eratosthenes (who perhaps remained at a strictly sexagesimal division of the circle; cf. below p. 590).

[28] Cf. above Table 2, p. 44 (from Alm. II, 6).

[29] This also holds for the intermediate values at $1/4^h$ and $1/2^h$, not listed in Table 31.

[30] Cf. above p. 304.

Furthermore: the starting point of this pattern is the parallel of Alexandria, exactly as in the arithmetical schemes for the length of daylight which also begin at Alexandria.[31] The basic importance of Alexandria also explains another peculiarity in the Hipparchian list of stades: the only case in which the intervals are not multiples of 100st is the distance of 3640st between Alexandria and Rhodes, leading to an exact equivalent of degrees and minutes ($\Delta\varphi = 3640/700 = 5;12°$). For all subsequent intervals these 40st are again ignored. On the other hand it is clear that the data, quoted by Strabo, for the regions between $M = 14^h$ and the equator are independent of the scheme that begins at Alexandria.

It is, of course, impossible to say whether this arithmetical pattern for the meridian distances from Alexandria to $M = 19^h$ was first constructed by Hipparchus or taken over by him from an earlier hellenistic source.[32] For our evaluation of early mathematical geography or spherical astronomy the result remains the same.

It may seem implausible that one should be able to construct arithmetical patterns that fit intricate astronomical relations fairly well. In fact, however, this is not at all surprising and occurs in many areas of ancient astronomy outside of Greek astronomy, e.g. in Babylonia or India. Common to these devices is the process of interpolation between given extrema. Linear interpolation leads to the Babylonian zigzag functions, the use of $c \sin \alpha$ is a standard Indian method.[33] The study of arithmetical progressions is well attested in Babylonian and hellenistic mathematics,[34] and we know from Babylonian astronomy the use of parabolic interpolations which fit given boundary conditions.[35] In our present case there exist many astronomical possibilities to estimate the extremal geographical latitudes φ in the area under consideration and thus the distances S from the equator. Hence one knows fairly well the interval ΔS to be bridged, in the problem in question we have neatly $\Delta S = 210 = 30°$ (cf. Table 32), and the number of steps ($n = 5$). This suffices to construct the proper sequence.

§ 4. Solar Theory

The much quoted passage in which Vettius Valens tells us that he used Hipparchus for the sun, Sudines, Cidenas, and Apollonius for the moon,[1] can hardly imply anything except the existence (in the second century A.D.) of tables for solar and lunar positions, compiled by these authors (or at least ascribed to them). Since Ptolemy did not modify Hipparchus' parameters for the motion of the sun[2] one is led to the further conclusion that the tables in Alm. III, 2 and III, 6 for the solar mean motion, and for the equation, must be essentially the same as

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[31] Cf. below IV D 3.

[32] Eratosthenes, however, can probably be excluded since his distances deviate from the Hipparchian (cf. Fig. 291, p. 1313).

[33] Cf. also below p. 1014.

[34] Cf., e.g., the treatise by Hypsicles (below IV D 1, 2 A) who is about contemporary with Hipparchus.

[35] E.g. column J of the lunar theory of System B. Cf. below II B 3, 5 B.

[1] Vettius Valens, Anthol. IX, 11 (ed. Kroll, p. 354, 4–6); also above p. 263.

[2] Cf. above p. 58.

the Hipparchian tables. A. Rome even argued[3] that the arrangement of the mean motion tables in 45 lines per column[4] goes back to Hipparchus (as well as the use of completed years of the era Nabonassar), eventually to be greatly modified by Ptolemy in his "Handy Tables" (current years of the era Philip and 25-year groups[5]). Let us hope that all this is correct.

If one could put any trust in the philosophical palaver of Theon of Smyrna one might perhaps conclude that Hipparchus preferred the epicyclic over the eccentric arrangement for his solar model.[6] Ptolemy derived the parameters of the solar model from the eccenter arrangement but all steps are numerically identical for both versions[7] since the angles at O and M are proportional to the same time intervals.

Since Ptolemy did not change Hipparchus' intervals between the equinoxes and solstices, his apsidal line had to remain at the same distance from the equinoxes. In other words the assumption of a tropically fixed solar apogee is a necessary consequence of retaining the Hipparchian seasons. On the other hand, if one changes the length of the tropical year from Hipparchus' round value ($365\ 1/4^d$) to a more accurate one and if one furthermore modifies the intervals between the equinoxes and solstices one is forced to accept a change in the position of the apsidal line. It is in this fashion that the Islamic astronomers discovered the motion of the solar apogee. Table 33 illustrates these effects by a comparison of Hipparchus' and Battānī's parameters,[8] about 1000 years apart. It is in this indirect way that one arrived at a determination of the position of the solar apogee, and never by direct observation of the solar motion. Comparatively large errors are caused by the sensitivity of the Hipparchian procedure to changes in the seasonal parameters which, in turn, are very difficult to establish accurately, a fact of which the ancients were well aware.[9]

Table 33

	Hipparch. = Ptol.	Battânî
Vernal Equin. → S. Solst.	$94\ 1/2^d$	$93^d\ 14^h = 93;35^d$
S. Solst. → Autumn. Equin.	92 1/2	$93^d\ 3/4^h = 93;\ 1,52,30^d$
Autumn. Equin. → Vernal Equ.	178 1/4	$178^d\ 14\ 1/2^h = 178;37,15^d$
Year	365 1/4	$365^d\ 5\ 1/4^h = 365;13,\ 7,30^d$
λ_A	♓ 5;30°	♓ 22;17°
e	2;30	2;4,45

[3] Rome [1950], p. 214f.
[4] Cf. above p. 55.
[5] Cf. below p. 971, n. 21.
[6] ed. Hiller p. 188, 15 (trsl. Dupuis, p. 305); what Theon has to say otherwise about Hipparchus and the equivalence of eccenters and epicycles does not inspire confidence (Hiller, p. 166, 6; p. 185, 17; trsl. Dupuis, p. 269, p. 299). Cf. also above p. 264, n. 3.
[7] Cf. Fig. 286 as compared with Fig. 53 (p. 1221).
[8] Nallino, Batt. I, p. 43f.
[9] Cf., e.g., Geminus VI, 28-33 (Manitius, p. 78/81); also Aaboe-Price [1964], p. 6-10. The use of the octants by Thabit b. Qurra is motivated by the difficulty of accurately observing the solstices (cf. Neugebauer [1962, 2], p. 274/5).

Ptolemy, in describing Hipparchus' method of determining the eccentricity of the solar orbit,[10] remarks that the lengths of two seasons suffice to reach the desired result. In other words the characteristic parameters of the solar orbit can be found without making very definite assumptions about the length of the year, although the tables of the solar mean motions contain implicitly such an assumption. Indeed, assuming $365;15^d$ for the length of the year one obtains for the daily mean motion $0;59,8,15,16°$ instead of Ptolemy's $0;59,8,17,13°$. If one then continues to compute with either one of the parameters one finds in both cases

$$R \sin \delta_2 = 1;1,55 \qquad R \sin \delta_1 = 2;15,26$$

as compared with 1;2 and 2;16, respectively, used by Ptolemy. Hence we see that Ptolemy's roundings have a much greater effect than the change from a julian to a tropical year.

When one finds durations of all four seasons ascribed to Hipparchus,[11] totalling 365 1/4 days, one is certainly not dealing with independent observations but with conveniently rounded numbers, derived from the solar model.[12]

We know from Alm. IV, 2[13] that Hipparchus assumed that the sun completes 345 sidereal rotations $-7;30°$ in 126007 days $+1^h$. Consequently the sidereal progress of the sun would be

$$\frac{34,29,52;30°}{35,0,7;2,30^d} = 0;59,8,9,31,51,30^{\circ/d}.$$

If Hipparchus, as is plausible, constructed his solar table on the basis of sidereal coordinates they would not be identical with the tables in Alm. III which are based on a mean motion of $0;59,8,17,13,12,31^{\circ/d}$ with respect to the vernal point. The difference of $0;0,0,7,41,21^{\circ/d}$ would amount to nearly $1;18°$ in 100 years. Hence Ptolemy's solar tables are not derived from a Hipparchian table of sidereal mean motions, corrected for 1° per century of precession[14].

§ 5. The Theory of the Moon

We know from Galen that Hipparchus had written "a whole book" on the length of the (synodic) months and also that he dealt with the problem of intercalations.[1] The title of the latter work is given in the Almagest: "On intercalary months and days" and it seems likely that both of Galen's references concern the

[10] Alm. III, 4 (Manitius I, p. 166); cf. Fig. 53, p. 1221.

[11] Alm. III, 4 (Manitius I, p. 170); also Theon of Smyrna (Dupuis, p. 218/219), etc.

[12] The parameters in Alm. III, 4 also determine the two remaining seasons, because the mean motion $\bar{\alpha}_3$ is given by $90-(\delta_1+\delta_2)=86;51°$ and thus $\bar{\alpha}_4=88;49°$. This, then, gives for the corresponding seasons $s_3 \approx 88\ 1/8^d$ and $s_4 \approx 90\ 1/8^d$, with a total of $365\ 1/4^d$ for the year. Again it is impossible to distinguish between Ptolemy's and Hipparchus' parameters on the basis of these round numbers.

[13] Cf. above p. 297 (2) or below p. 310 (5).

[14] Cf. also above p. 55, note 1.

[1] Cf. below p. 339, n. 10 and Galen's commentary to Hippocrates' "On epidemics" (Galen, Opera XVII, 1 ed. Kühn, p. 23); cf. also above p. 296.

same work.[2] A treatise on the draconitic month was called "On the monthly motion of the moon in latitude"[3] while writings on eclipses[4] are probably the main source of what is known about Hipparchus through the Almagest.

The inadequacy of the ancient tradition became evident, however, when Kugler discovered the fact that essential parameters in Hipparchus' lunar theory had originated in Babylonian astronomy.[5] No mention of this dependence was made by Ptolemy when he described in the Almagest the steps which supposedly lead to the Hipparchian parameters. Only now can we realize that Ptolemy's story is only his own reconstruction of the events, obviously without a factual basis; a numerical discrepancy therefore escaped Ptolemy and found its simple explanation only after the underlying Babylonian data had become known.[6]

From the little we know it seems as if Hipparchus were the first to introduce Babylonian parameters into Greek mathematical astronomy. At any rate there is no such evidence for Apollonius whose astronomy appears only to be concerned with geometric-cinematic problems.[7] Judging from the Almagest it was Hipparchus who first associated these cinematic models as accurately as possible with numerical data, in part undoubtedly of Babylonian origin, and in part based on systematic observations of his own.

It is a familiar experience that refinements of empirical data create difficulties for the original theoretical picture. This apparently also happened to Hipparchus, both with the theory of the motion of the moon and with planetary theory. We know from the Almagest[8] that Hipparchus became aware of discrepancies between the simple lunar theory and positions of the moon observed by him in the quadratures and octants. Yet it was not until Ptolemy that lunar longitudes could be predicted satisfactorily for all elongations. Essentially the same situation must have prevailed in the planetary theory. One may perhaps say that the role of Apollonius, Hipparchus, and Ptolemy has a parallel in the positions of Copernicus, Brahe, and Kepler.

1. The Fundamental Parameters

A. Period Relations

We know from Alm. IV, 2[1] that Hipparchus accepted in his lunar theory the following mean values, known since Kugler's discoveries to belong to "System B"

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[2] Alm. III, 1 (Heiberg, p. 207, 7/8; Manitius, p. 145). No title of a work by Hipparchus on the length of the synodic month is ever mentioned in the extant sources; Rehm's Περὶ μηνιαίου χρόνου (RE 8, 2, col. 1670, 20) is a pure conjecture. For the Arabic tradition cf. Walzer [1935], p. 347 (110/75), based on Galen's treatise "On seven-month children"; cf. above p. 293.

[3] From Suidas, ed. Adler II, p. 657, 27f.

[4] Cf. below I E 5, 2 A.

[5] Kugler, Mondrechnung (1900), p. 111; cf. p. 348 ff.

[6] Cf. below p. 310.

[7] Cf. above p. 271.

[8] Alm. V, 2 and V, 3; cf. above p. 84 and p. 89.

[1] Manitus I, p. 197f.; cf. above p. 69 (1) to (3).

of the Babylonian lunar ephemerides[2]:

$$1 \text{ syn.m.} = 29;31,50,8,20 \text{ days} \tag{1}$$

$$251\ (=4,11) \text{ syn.m.} = 269\ (=4,29) \text{ anom.m.} \tag{2}$$

$$5458\ (=1,30,58) \text{ syn.m.} = 5923\ (=1,38,43) \text{ drac.m.} \tag{3}$$

The value (1) for the length of the mean synodic month appears over and over again in ancient and medieval astronomy,[3] first mentioned in our Greek sources a century before Ptolemy in Geminus' "Isagoge," but there without reference to Hipparchus.[4]

Ptolemy ascribes[5] to "still earlier observers" (i.e. earlier than the "old mathematicians") another set of relations,[6] also known to us to be of Babylonian origin, and conventionally called "Saros":

$$\begin{aligned} &6585\ 1/3\ (=1,49,45;20) \text{ days} = 223\ (=3,43) \text{ syn.m.} \\ &= 239\ (=3,59) \text{ anom.m.} = 242\ (=4,2) \text{ drac.m.} = 241\ (=4,1) \text{ rot.} + 10;40^\circ. \end{aligned} \tag{4}$$

Multiplication by 3 eliminates all fractions and produces the equivalent relations known as "exeligmos"[7]

$$\begin{aligned} &19756\ (=5,29,16) \text{ days} = 669\ (=11,9) \text{ syn.m.} \\ &= 717\ (=11,57) \text{ anom.m.} = 726\ (=12,6) \text{ drac.m.} = 723\ (=12,3) \text{ rot.} + 32^\circ. \end{aligned} \tag{4a}$$

According to Ptolemy Hipparchus replaced (4) by the improved parameters

$$\begin{aligned} &126007\ (=35,0,7) \text{ days } 1 \text{ hour} = 4267\ (=1,11,7) \text{ syn.m.} \\ &= 4573\ (=1,16,13) \text{ anom.m.} = 4612\ (=1,16,52) \text{ sid. rot. of the moon } -7;30^\circ \\ &= 345\ (=5,45) \text{ sid.rot. of the sun } -7;30^\circ. \end{aligned} \tag{5}$$

The first of the equations in (5) is supposedly the source of the value (1) for the length of the mean synodic month. As far as I know Copernicus was the first to check Ptolemy's account[8] and to find that in fact (5) leads to

$$1 \text{ syn.m.} \approx 29;31,50,8,9,20^d$$

and not to the Babylonian parameter ...,8,20 as stated in (1).

What is generally absent in the period relations deducible from the cuneiform sources are the number of days and of degrees in excess or default of complete rotations. Only for the first equation in (4) do we have a Babylonian parallel[9]: 1,49,45;19,20^d for the length of the 18-year eclipse cycle (the "Saros"), as compared with Ptolemy's 1,49,45;20^d.

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[2] Cf. below p. 483 (3); p. 478 (2c); p. 523 (2c) or ACT, p. 75 (20).

[3] Galen in his treatise "On Seven-Month Children" ascribes this value to Hipparchus in a form which is equivalent to saying "1 syn.m. = 29;31,50,8^d and a little." Cf. for the details Neugebauer [1949].

[4] Cf. below p. 585 (6).

[5] Alm. IV, 2 (Manitius I, p. 195).

[6] A fragment from an anonymous commentary (probably from the third century A.D.; cf. below p. 321, note 3) gives consistently, but wrongly, 235 instead of 239 for the number of anomalistic months (CCAG 8, 2, p. 127, lines 12, 16, 17).

[7] Alm. IV, 2 (Manitius I, p. 196); cf. also Geminus, below p. 586.

[8] Copernicus, De Revol. IV, 4, silently correcting Ptolemy's value (Gesamtausg., p. 215, 31 ff.).

[9] ACT I, p. 272.

The relation contained in (5)

$$4267 \text{ syn. m.} = 4573 \text{ anom. m.} \qquad (6)$$

is not new since it is the equivalent of (2) multiplied by 17. Thus neither (5) nor (2) constitute a Hipparchian improvement of the less accurate relations (4), but (5) as well as (2) were taken by Hipparchus from Babylonian astronomy.

In fact one can derive (5) from the Babylonian parameters (2) and (1) without making use of any new observations[10] and thus obtain intervals for the recurrence of eclipses. One has only to take into account another relation which is of importance in various parts of Babylonian astronomy,[11] i.e. the relation which connects the mean synodic month and the (sidereal) year

$$1 \text{ year} = 12;22,8 \text{ syn. m.} \qquad (7)$$

Combining (7) with (2) and (3) one obtains

$$4,11 \text{ syn. m.} = \frac{4,11 \cdot 1,38,43}{1,30,58} \text{ drac. m.} = 4,32;23,2,29,\ldots \text{ drac. m.} \qquad (8)$$

and

$$4,11 \text{ syn. m.} = \frac{4,11}{12;22,8} \text{ years} = 20;17,34,15,\ldots \text{ years.} \qquad (9)$$

A recurrence of eclipses requires intervals of either complete or of half draconitic months (i.e. either identical or opposite nodes) and also the return to the same lunar and solar anomaly. The restitution of the lunar anomaly is secured, because of (2), by the use of 4,11 synodic months. The solar anomaly will be the same when we deal with intervals of complete, or nearly complete, years. Hence one must look for multiples of 4,11 synodic months, such that the fractional parts for the draconitic months are either 0 or 0;30 while the years are integers. Fortunately these conditions need not be satisfied exactly. Since 0;1 drac. m. $\approx$ 0;27,30^d the moon moves during this time about 6°; but eclipses are possible when the moon is about 12° distant from the nodes.[12] Hence the fractions in the multiples of (8) may range within $\pm$0;2 or 0;30$\pm$0;2 draconitic months. The solar equation changes at most 0;14° in 6^d $\approx$ 0;1^y. Hence the influence on the solar longitude will remain below 1° if the fractions in the multiples of (9) remain within $\pm$0;4 years.

A simple table of multiplications will show for which multiples *n* these conditions for the fractions in (8) and (9) are satisfied (indicated by an asterisk):

n	drac.m.	years
1	0;23,2,30	0;17,34,15
4	*1;32,10	1;10,17
10	3;50,25	*2;55,42,30
13	*4;59,32,30	3;48,25,15
17	*6;31,42,30	*5;58,42,15

[10] This was pointed out by A. Aaboe [1955].

[11] Cf. below p. 378 (15b), p. 396 (5b), and p. 496 (20)

[12] Cf., e.g., above p. 125f.

Hence the factor 17 is the smallest common multiple which restores eclipses; thus (6) is a consequence of (2) and (3), using (7).

If we now multiply the number $4{,}11 \cdot 17 = 1{,}11{,}7$ of synodic months by the number of days given in (1) we find

$$\begin{aligned} 1{,}11{,}7 \text{ syn. m.} &= 35{,}0{,}7;2{,}42{,}38{,}20^{d} \\ &= 126007^{d} + 0;2{,}30^{d} + 0;0{,}12{,}38{,}20^{d} \\ &= 126007^{d} + 1^{h} + 0;5{,}3{,}20^{h}. \end{aligned}$$

In other words, the Hipparchian relation (5) disregards 5;3,20 minutes of time.

If we multiply the right-hand side of (9) by 17 we find

$$5{,}44;58{,}42{,}23 = 5{,}45 - 0;1{,}17{,}37 \text{ years.}$$

Knowing that this interval also contains an integer number of mean synodic months we see that sun and moon have completed an integer number of sidereal rotations minus 0;1,17,37 rotations, i.e. $-7;45{,}42°$. According to the Almagest Hipparchus rounded this deficit to 7 1/2°. An anonymous commentary, probably of the third century A.D.[13] ascribes to Hipparchus the correction of $-8°$ which is a slightly better approximation of $-7;45{,}42°$.

B. The Draconitic Month

By accepting the parameters of the "exeligmos" Hipparchus also had the consequences at his disposal that can be derived from it by purely arithmetical operations. For the latitude of the moon this means that the relations[1]

$$1{,}49{,}45;20^{d} = 4{,}2 \text{ drac. m.} = 4{,}1 \text{ sid. rot.} + 10;40° \qquad (1)$$

imply for the length of the draconitic month

$$1{,}49{,}45;20^{d}/4{,}2 \approx 27;12{,}43^{d} \qquad (2)$$

and for the sidereal mean motion

$$24{,}6{,}10;40°/1{,}49{,}45;20^{d} \approx 13;10{,}34{,}51, \ldots {}^{\circ/d}. \qquad (3)$$

The moon's progress during one draconitic month is therefore

$$27;12{,}43 \cdot 13;10{,}34{,}51 \approx 5{,}58;33{,}14{,}30 = 6{,}0 - 1;26{,}45{,}30°.$$

This shows that the nodal line recedes with a daily velocity

$$-1;26{,}45{,}30/27;12{,}43 \approx -0;3{,}11^{\circ/d}. \qquad (4)$$

We do not know the degree of arithmetical accuracy which satisfied Hipparchus for his numerical conclusions. Ptolemy, in the introductory section of Alm. IV, 3 proceeds to six digits of sexagesimal fractions, in keeping with the range of his tables of corrected mean motions in IV, 4.[2] On the other hand one may well assume that Hipparchus accepted different roundings for different purposes,

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[13] Cf. above p. 310, n. 6.
[1] Cf. above p. 310 (4).
[2] Cf. below p. 314.

just as the Babylonian astronomers adapted their parameters to the practical requirements of their tables and ephemerides.

Since it seems likely that Hipparchus compiled tables of mean motions[3] he also must have faced the problem of determining the epoch values for his variables, presumably for the era Nabonassar. Thanks to Ptolemy's introduction to Alm. IV, 9 we have a general idea of how Hipparchus proceeded to solve this problem in the case of the nodal motion.[4] He assumed[5] that the apparent diameter $d_{☾}$ of the moon at mean distance is the 650th part of the lunar orbit, thus (in modern terms)

$$d_{☾} = \frac{6,0}{10,50} = \frac{7;12}{13} \approx 0;33,13,51° \approx 0;33,14°. \tag{5}$$

He furthermore assumed that the diameter $2u$ of the earth's shadow at the mean distance of the moon has to $d_{☾}$ the ratio

$$2u = 2;30\, d_{☾} \tag{6}$$

hence

$$2u \approx 1;23°. \tag{6a}$$

[The "Canobic Inscription" expresses the same parameters as fractions of a quadrant[6]:

$$d_{☾} = 90/162 \qquad 2u = 90/65 \tag{6b}$$

from which we obtain

$$d_{☾} = 5/9 = 0;33,20° \qquad 2u = 18/13 \approx 1;23°.] \tag{6c}$$

The two data (5) and (6) suffice for the determination of the distance of the center of the moon from the nearest node at the middle of an eclipse of given magnitude m. Indeed, let us suppose m to be given in angular measurements (always obtainable by means of (5)) then the distance ω of the center B of the true moon from the node N can be computed from

$$\omega = \Sigma\mathrm{B}/\tan i \tag{7}$$

(cf. Fig. 287). For the moment in question we can also find the lunar anomaly (assuming again tables for mean motions since epoch) and hence the distance of the mean moon from N. Thus we know for the given time the longitude of N and can from it derive its position for the date chosen as epoch.

Ptolemy, in using this method for determining intervals $\Delta\omega$ for pairs of eclipses of known time intervals Δt did not obtain constant ratios $\Delta\omega/\Delta t$ as one should expect for the motion of the nodes. This is, of course, not surprising because the initial assumption of fixed values for $d_{☾}$ and u is incorrect. This caused Ptolemy to develop a new method (described above in I B 3, 6 B) which is independent of any determination of apparent radii of moon or shadow.

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[3] Cf. above p. 306.

[4] The proper understanding of this passage in relation to Ptolemy's method is due to Olaf Schmidt [1937].

[5] We shall discuss these parameters later on (below p. 325).

[6] Ptolem. Opera II, p. 153, 18–20. The ratio (6) is also used in Tamil eclipse computations; cf. Neugebauer [1952], p. 272 (3).

Hipparchus' disregard for the influence of the moon's distance on the apparent diameters also caused some inaccuracy in the determination of the mean value of the daily increment of the argument of latitude as determined from two widely spaced lunar eclipses, -719 March 8 and -140 Jan. 27.[7] The two eclipses are supposedly[8] of equal magnitude (1/4 of the diameter), but at the earlier one the moon was located at the apogee of the epicycle, at the later one at the perigee, causing mean and true moon to coincide. Ptolemy analyses the consequences of this Hipparchian procedure.[9] First he remarks that accurate computation shows that both eclipses deviate from the apsidal line of the epicycle, the first one by $+1°$, the second by $+1/8°$. Hence the argument of latitude was $7/8 = 0;52,30°$ short of a complete rotation. Secondly, equal magnitude does not necessarily imply equal distance from the node. In the case in question the geocentric distances of the moon differ by the greatest possible amount; the tables in Alm. VI, 8 show that $m = 3''$ corresponds at the apogee to $\omega' = 279;18°$, at the perigee to 280;30°. Thus the nodal distance increased between the two eclipses by 1;12°. Fortunately the two errors compensate one another almost completely, leaving an error of only $1;12 - 0;52,30 \approx 0;20°$. Ptolemy considers it possible that Hipparchus had some inkling of the compensating effects in this situation.

If one would wish to estimate the correction which should eliminate this error one had to divide 0;20° by the time between the two eclipses, i.e. by $211439 = 58,43,59^d$. Obviously the result is about $0;0,0,0,20^{°/d}$, to be added to the mean motion in latitude. Ptolemy,[10] however, did not use these two eclipses for his correction but a much greater interval (between -490 Apr. 25 and $+125$ Apr. 5[11]) of $224609 = 1,2,23,29^d$. For these two eclipses he finds an excess of 0;9° over the nodal motion computed with Hipparchus' parameter, hence a correction of $0;0,0,0,8,39,18^{°/d}$. The resulting lunar motion provides the basis for the tables in Alm. IV, 4 which give for the argument of latitude and for the mean motion in longitude

$$\begin{aligned} \omega&: \ 13;13,45,39,48,56,37^{°/d} \\ \bar{\lambda}&: \ 13;10,34,58,33,30,30 \end{aligned} \tag{8}$$

thus for the nodes

$$-0;3,10,41,15,26,7^{°/d}. \tag{8a}$$

The uncorrected, supposedly Hipparchian, values are therefore

$$\omega: \ 13;13,45,39,40,17,19^{°/d}, \tag{9}$$

hence for the nodes

$$-0;3,10,41,6,46,49^{°/d}. \tag{9a}$$

We do not know whether Hipparchus had his table extended to this number of digits nor can we say how far the change from sidereal to tropical longitudes

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[7] Both eclipses have been discussed before: the earlier one belongs to a triple (recorded in Babylon) used for the determination of the radius of the lunar epicycle (above p. 77), the later one is one of a pair that served to find the apparent diameter of the moon (above p. 104).

[8] Actually the magnitudes are 1.5 and 2.8, respectively, (P. V. Neugebauer, Kanon d. Mondf.).

[9] Alm. VI, 9 (Manitius I, p. 394–396).

[10] Alm. IV, 9 (Manitius I, p. 238–241).

[11] Cf. above p. 81.

was taken into account. Hence (9) and (9a) can only be taken as plausible estimates for Hipparchus' nodal motions.

C. *The Epicycle Radius*

It is well-known from Ptolemy's discussion of the theory of lunar motion (Alm. IV and V) that Hipparchus operated with a simple epicyclic (or the equivalent eccentric) model.[1] We also know from Alm. IV, 11[2] that he used two triples of lunar eclipses for the determination of the eccentricity e, or the radius r of the epicycle, and he found, because of computational errors, as pointed out by Ptolemy, different values for e and r in relation to the radius R of the deferent:

$$\frac{R}{e}=\frac{3144}{327\,2/3}=\frac{60}{6;15} \tag{1}$$

$$\frac{R}{r}=\frac{3122\,1/2}{247\,1/2}=\frac{60}{4;46}. \tag{2}$$

We shall discuss presently the empirical data from which these ratios were derived.[3] At the moment it suffices to remark that Ptolemy's sexagesimal equivalents as given in (1) and (2) are not quite accurate since

$$\frac{3144}{327\,2/3}=\frac{2,37,12}{16,23}\approx 9;35,42,\ldots \qquad \frac{1,0}{6;15}=9;36 \tag{1a}$$

$$\frac{3122\,1/2}{247\,1/2}=\frac{20,49}{1,39}\approx 12;36,58,\ldots \qquad \frac{1,0}{4;46}\approx 12;35,14,\ldots.^{4} \tag{2a}$$

Much more serious, however, is the question of how Hipparchus could have accepted drastically different values for e and r in view of Apollonius' theorem of equivalence of epicyclic and eccentric models. The only answer I can suggest is the assumption that Hipparchus was not convinced of the constancy of the radius of the epicycle (or the equivalent eccentricity). After all, we know from Ptolemy[5] that Hipparchus had found discrepancies, e.g. in the quadratures, between observations and equations as derived from the simple lunar model which was based, of course, on fixed values for r (or e). We also know from Ptolemy's discussion[6] of the parameters (1) and (2) that they were obtained from two triples of eclipses, each of which consisted of consecutive eclipses (i.e. six months apart), a choice which one could interpret as an attempt to remain as close as possible to a fixed diameter of the epicycle. Furthermore the two diameters for the two triples are chosen at practically right angles,[7] very near to the four cardinal points of the ecliptic. This again may have to do with the

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[1] Cf. above p. 84.
[2] Manitius I, p. 245f.; cf. also Man. I, p. 212, 25.
[3] Below p. 318.
[4] A denominator 4;45 would have been slightly better because $1,0/4;45\approx 12;37,53,\ldots$.
[5] Cf. above p. 84.
[6] In Alm. IV, 11; cf. below p. 316.
[7] Cf. Fig. 288 below p. 1311.

possibility that perpendicular directions might produce extremal variations. Such tentative hypotheses of Hipparchus would also explain why Ptolemy goes to great length to show that Hipparchus had operated with erroneous data and that no contradiction to his own assumption of a fixed radius of the lunar epicycle can be derived from Hipparchus' eclipses, an assumption of crucial importance for Ptolemy's analysis of the observational data which led him to the discovery of the second lunar inequality.

Before turning to a detailed discussion of the two eclipse triples which led Hipparchus to the ratios (1) and (2), respectively, we shall present a conclusion by G. Toomer who showed that we have one more possibility to inform us about Hipparchus' estimates of the size of the lunar epicycle. The point of departure is a remark in Pappus' Commentary to Alm. V, 14[8] about absolute distances of the moon, assumed by Hipparchus, which allows us to determine the underlying ratio R/r. Indeed if m, μ, and M represent the smallest, mean, and greatest distance of the moon, respectively (measured in earth radii r_e) one has

$$\frac{R}{r}=\frac{\mu}{\mu-m}=\frac{\mu}{M-\mu}=\frac{M+m}{M-m}. \tag{3}$$

Pappus gives two sets of values for m, μ, and M, one (I) deduced from a solar eclipse which was total at the Hellespont but of magnitude 4/5 in Alexandria[9]; the other set (II) was arrived at by Hipparchus "from many considerations" in Book II of his work "On Size and Distances (of sun and moon)," [10] suggesting (II) as his final results. The values given by Pappus are

$$\begin{array}{llll} \text{I:} & m=71\,r_e & \text{II:} & m=62\,r_e \\ & \mu=77 & & \mu=67\;1/3 \\ & M=83 & & M=72\;2/3. \end{array} \tag{4}$$

From this one obtains with (3) the following values (exactly)

$$\text{I:}\quad R/r=12;50 \qquad \text{II:}\quad R/r=12;37,30 \tag{5}$$

or, for $R=60$

$$\text{I:}\quad r\approx 4;41 \qquad \text{II:}\quad r\approx 4;45. \tag{5a}$$

These values, and in particular II, are so near to the ratio (2)=(2a) found by Hipparchus according to Ptolemy (above p. 315) that it can hardly be doubted that Hipparchus did eventually accept (2) as the correct ratio, hence (using Ptolemy's norm)

$$r\approx 4;46 \qquad R=60. \tag{6}$$

[8] Rome, CA I, p. 68; translation Toomer [1967], p. 147.

[9] This eclipse is also discussed by Cleomedes (II, 3 ed. Ziegler, p. 172, 20–174, 15) who is about a generation younger than Pappus (cf. below V C 2, 5 A and p. 963). Hultsch ([1900], p. 198 f.) suggested its identification with the eclipse of −128 Nov. 20, P. V. Neugebauer (Astron. Chron. I, p. 132 and p. 113) with the "Agathocles" eclipse of −309 Aug. 15. In analyzing Hipparchus' procedure in his determination of the effect of a measurable or not measurable solar parallax on the moon's distance G. Toomer has shown [1974, 2] that only the eclipse of −189 March 14 satisfies the conditions imposed by Hipparchus' method.

[10] Cf. below I E 5, 4 B.

This shows that the radius of his lunar epicycle was about $5-0;15$ in contrast to Ptolemy's $5+0;15$.[11]

We can now turn to Ptolemy's explanation of the discrepancy between the value of e and r, obtained by Hipparchus, as the result of incorrect time intervals assumed in the two sets of lunar eclipses. In other words the variation discovered by Hipparchus is not real but is caused by an erroneous determination of parameters which have a great influence on the geometric configurations which define e or r.

The first triple of lunar eclipses used by Hipparchus is of Babylonian origin, the second was observed in Alexandria[12]:

R:	−382 Dec. 23	U:	−200 Sept. 22[13]	
S:	−381 June 18	V:	−199 March 19/20	(7)
T:	−381 Dec. 12/13	W:	−199 Sept. 12.	

Ptolemy gives us for all six eclipses the circumstances as reported by Hipparchus (cf. Fig. 288[14]) and then determines as closely as possible the eclipse middle in terms of Alexandrian equinoctial time.

Two facts can be noted: none of these eclipses are recorded with any degree of accuracy such that its midpoint could be securely defined; secondly: Ptolemy's analysis uses finesses (e.g. equation of time) which make little sense in relation to the crude estimates for duration or first and last contact. Ptolemy then computes with his solar and lunar tables the true longitudes of sun and moon for the time of mid-eclipse he has decided upon. Obviously the result should be exactly $\lambda_{\odot}=\lambda_{☾}+180°$. In fact his results never deviate more than $\pm 0;2°$ from this relation; consequently the errors in elongation reach at most 4 minutes in time.[15] Ptolemy therefore accepts the empirical data as established with sufficient accuracy. What he then shows is that Hipparchus derived from these data incorrect intervals of time and longitude between the eclipses; it is these parameters that are crucial for the determination of e and r.

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[11] Cf. above p. 76. Toomer also pointed out ([1967], p. 149) that this difference excludes Hipparchus (and suggests Ptolemy) as the ultimate source for the epicyclic radius r in Āryabhaṭa's lunar theory ($\approx$ A.D. 500). There one finds (Āryabh. I, 8; trsl. Clark, p. 18)

$$R:r=360:7\cdot 4\,1/2=60:5;15$$

i.e. exactly Ptolemy's parameter. For the sun, however, Āryabhaṭa has

$$R:r=360:3\cdot 4\,1/2=60:2;15$$

against Ptolemy's (and thus Hipparchus') $e=2;30$ (cf. I B 1, 3 B).

[12] These observations antedate Hipparchus' lifetime.

[13] For a Babylonian record of this eclipse cf. Schaumberger, Erg., p. 368, note 1.

[14] Fig. 288 is drawn to scale with Hipparchus' values for e and r. Black dots denote mean positions, white dots true positions of the moon; O = observer, M = center of eccenter.

[15] As usual, checking these results by carefully using Ptolemy's tables, one finds slightly different deviations, but still very modest in comparison with the crude initial data:

	R	S	T	U	V	W
Ptolemy:	+0;1°	−0;2	+0;2	−0;1	+0;1	−0;1
accurate:	+0;2°	−0;6	+0;2	−0;2	−0;3	−0;3

Ptolemy does not give the details of the computations of e and r but since we have his basic parameters and since we know the mathematical procedures[16] we can reconstruct all single steps, without knowing, of course, the unavoidable inaccuracies of interpolations or roundings, etc. Nevertheless, in computing both e and r with Ptolemy's parameters and using the tables in the Almagest I find excellent agreement with the result $e=r=5;15$ that Ptolemy must have expected.[17] From the first triple I obtained $e \approx 5;16$, from the second $r \approx 5;13$.

It is clear from the discussion in the Almagest that Hipparchus used the same mathematical procedure which Ptolemy explained for the radius of the lunar epicycle or for the eccentricity of the deferent of the outer planets. Hence the cause of the error can only lie in the numerical execution. The intervals which Ptolemy found and which agree, as we have just seen, with his own results, differ indeed greatly from Hipparchus' parameters as reported by Ptolemy, shown in Table 34.

Table 34

	Ptolemy		Hipparchus	
	Δt	$\Delta\lambda_{\odot}$	Δt	$\Delta\lambda_{\odot}$
R→S	$177^d\,13;36^h$	173;28°	$177^d\,13;45^h$	173 − 1/8°
S→T	$177^d\;2^h$	175;44°	$177^d\;1;40^h$	175 + 1/8°
[thus R→T]		349°		348°
U→V	$178^d\;6;50^h$	180;11°	$178^d\;6^h$	180;20°
V→W	$176^d\;0;24^h$	168;55°	$176^d\;0;20^h$	168;33°
[thus U→W]	$354^d\;7^h$		$354^d\;6^h$	

Accepting the Hipparchian intervals as given by Ptolemy we can directly find the basic parameters of Hipparchus' model. The time intervals Δt give us the corresponding mean motions $\Delta\bar{\lambda}_{☾}$ and $\Delta\alpha$, for such short stretches the tables of Hipparchus will not differ from the Almagest. In this way we find the differences shown in Table 35. The discrepancies in $\Delta\lambda_{☾}-\Delta\bar{\lambda}_{☾}$ produce the essential effect on the final results.

Table 35

				$\Delta\lambda_{☾}-\Delta\bar{\lambda}_{☾}$	
	$\Delta\lambda_{\odot}=\Delta\lambda_{☾}$	$\Delta\bar{\lambda}_{☾}$	$\Delta\alpha$	Hipp.	Ptol.
R→S	172;52,30°	179;46°	159;59°	−6;54°	−6;10°
S→T	175; 7,30	173; 8	153;25	+2; 0	+2;21
T→R	12; 0	7; 6	46;36	+4;54	+3;49
U→V	180;20	188;41	168;50	−8;21	−8;56
V→W	168;33	159;14	139;37	+9;19	+9;38
W→U	11; 7	12; 5	51;33	−0;58	−0;42

[16] The method for finding r is described in Alm. IV, 6 (cf. above I B 3, 4 A). For the eccenter model Prolemy refers briefly to the arrangement obtainable by a transformation with reciprocal radii, known since Apollonius (cf. above p. 265).

[17] Cf. above p. 76. The corrections mentioned in note 15 have practically no influence on the final results; the eccentricity, e.g. changes only from 5;16,22 to 5;15,33.

We can again compute with this set of Hipparchian parameters the values of e and r. The results do not agree very closely with the reported data in (1) and (2), p. 315, although they show the expected direction for the deviations from $e=r=5;15$. One finds

$$\frac{R}{e}=\frac{56,13;32}{6,1;53} \quad \text{thus} \quad e=6;26 \qquad \left(\text{instead } \frac{52,24}{5,27;40} \text{ thus } e=6;15\right)^{18} \tag{8}$$

and

$$\frac{R}{r}=\frac{1,3,29;48}{5,21;50} \quad \text{thus} \quad r=5;4 \qquad \left(\text{instead } \frac{52,2;30}{4,7;30} \text{ thus } r=4;46\right). \tag{9}$$

Simple computing errors which Ptolemy seems to have found in Hipparchus' work [19] may have contributed to the deviations.[20]

Supplementary Note. In order to show that Hipparchus worked with a table of chords, based on a circle of radius 3438, G. J. Toomer [21] carried out in all numerical details the computations which should lead from the data (7), p. 317 to the ratios (1) and (2), p. 315. In this way he was not only able to explain the peculiar numbers in (1) and (2) but he also discovered an essential error in the geometrical configuration assumed by Hipparchus in his determination of the ratio R/r. Using our terminology, introduced in the discussion of Ptolemy's procedure,[22] we can describe Hipparchus' error simply by saying that he assumed the wrong sector for the location of the observer in relation to the epicycle. The resulting error of sign produces a discrepancy which combines with the inaccuracies of the observational data as discussed by Ptolemy.

2. Eclipses

A. Tables

In a famous passage Pliny tells us[1] that Hipparchus had predicted the eclipses of sun and moon for 600 years–a truly gigantic achievement, were it not obvious that the story, as it stands, makes no sense.

First purely historically: Hipparchus, to whom as the "father of Greek astronomy" one ascribes all first steps in theoretical and observational methodology should have found nothing better to do than to compute eclipses for centuries ahead, dwarfing the range of predictions by his next successor, Th. Ritter von Oppolzer, who, in 1885, went not quite three centuries beyond his time. Nor does Ptolemy ever mention a Hipparchian prediction in comparison with an observed eclipse from the intervening three centuries.

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[18] Ptolemy says (Manitius I, p. 246, 7f.) that the corresponding maximum equation amounts to 5;49° and Theon repeats this figure (Rome, CA III, p. 1084, 6). In fact, however, $1/2 \text{ arc crd } 2e=5;59°$.

[19] Manitius I, p. 247, 5.

[20] Taking the equation of time into consideration does not help matters.

[21] Toomer [1973], p. 9–16. Cf. also above p. 299f.

[22] Above p. 74f. and Figs. 65 to 67 there.

[1] Pliny NH II, 53 (Ian-Mayhoff I, p. 143; Budé II, p. 24; Loeb I, p. 203). About five centuries later Lydus improved on this story by mentioning only solar eclipses (Lydus, De ost., p. 15, 2f. ed. Wachsmuth).

But even worse, such an enterprise would not be to the credit of Hipparchus' scientific methodology. We know that he was fully conscious of the fact that many of the parameters as well as the theoretical models at his disposal were only approximations in need of refinement by future generations. But then it makes no sense to compute some thousands of eclipses on the basis of an admittedly preliminary lunar theory.

When Pliny mentioned "both luminaries" he obviously was not aware of the fact that the prediction of solar eclipses is an affair totally different from the computation of lunar eclipses. The investigation of long sequences of syzygies in order to find out whether or not (and in what magnitude) a solar eclipse would be visible at a given place would not only require an enormous effort but would yield only results of very little interest. Only a small fraction of cases would (after much computing) indicate visibility of the eclipse at a given place, e.g. Rhodes, but these would then be valueless for Alexandria or Rome. One needs only to have worked through[2] an ancient solar eclipse computation once to be convinced that nobody in his sound mind would do that hundreds of times for the benefit of future residents of the same locality.

For lunar eclipses we know from Babylonian ephemerides that it is possible to predict relatively simply and with reasonable accuracy the magnitudes and nearer circumstances.[3] But even so the coverage of a century or two with such ephemerides, assuring correctness of the continued arithmetical operations, requires a great deal of labor, senseless to invest for six centuries ahead. Truly, Pliny for all his admiration of Hipparchus did not understand what he was talking about.[4]

This does not mean that one should not try to bring some sense into the tradition, garbled by Pliny. On the contrary this seems quite simple when one remembers that Ptolemy tells us that the ancient observations were preserved, by and large, from the reign of Nabonassar onward and that this motivated the use of the Era Nabonassar for his tables.[5] But Hipparchus' own observations all belong to the years 600 of Nabonassar such that Pliny's six centuries may well concern the time before Hipparchus for which he had Babylonian eclipse reports at his disposal. We know the crucial role which ancient lunar eclipses played for the determination of the parameters of the lunar theory.[6] Not any arbitrary eclipse could be used in this problem and the elements at its occurrence must be known,[7] data not explicitly given in the Babylonian records, nor could they have been given in a form useful for the Apollonius-Hipparchian cinematic models. Hence Hipparchus faced for his program of determining the lunar parameters

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[2] Cf., e.g., Rome [1950].

[3] Below II B 6 and II B 7.

[4] D. R. Dicks, in his Hipparchus, p. 51 (H), seems to think that the passage becomes more acceptable by using a variant reading which he translates "Hipparchus foretold the course of both the sun and moon for hundreds of years." In fact this makes even less sense. For mean positions one needs for all times nothing but a few tables (e.g. Alm. VI, 3). Hence one must assume true positions. But 600 years contain almost 15000 syzygies which Hipparchus should have undertaken to compute with no useful purpose at all.

[5] Alm. III, (Man. I, p. 183, 5).

[6] Cf. above p. 73ff.

[7] Cf. e.g., above p. 72; p. 77.

the necessity of analyzing the material at his disposal. His great achievement was not the prediction of future eclipses for 600 years but the arrangement and classification of the material at his disposal from the past 600 years. This meant the laying of a solid foundation for theoretical astronomy and making it possible for Ptolemy to take full advantage of the past in relation to his own observations. It must still have meant a colossal amount of work to determine the needed data (e.g. anomaly and nodal distance), from the Babylonian records in whatever form they were available to Hipparchus. But without this work one could never have hoped to predict eclipses with reasonable accuracy and to test the foundations of theoretical astronomy.

From a remark by Achilles (3rd cent.) it has been inferred that Hipparchus also wrote a treatise on "Solar eclipses for the seven climata."[8] To speculate here again about a canon of eclipses makes no sense, not only because of the gigantic size and practical uselessness of such an enterprise, but also for the simple reason that ancient and medieval astronomy was not capable of determining the path of solar eclipses but could only discuss the special circumstances for a given locality. We shall see presently,[9] however, that Achilles' remark can be understood perfectly well within another context.

B. Eclipse Cycles and Intervals

We have seen[1] that the Hipparchian parameters of the lunar theory, ascribed to him by Ptolemy, can be derived from Babylonian data among which the relation

$$5458 \text{ syn. m.} = 5923 \text{ drac. m.} \tag{1}$$

plays an important role.

An anonymous fragment of some commentary,[2] probably written in the third century A.D.[3] describes methods of the "early astronomers," including Hipparchus. In particular (1) is related to the empirical rules that intervals between eclipses are either 5 or 6 months[4] and it is said[5] that (1) represents an excellent eclipse period, containing 122 five-month intervals and 808 six-month intervals (which gives the proper total of 5458 months).

Both numbers are even. One should therefore expect to find also half these intervals associated with eclipses. This is indeed the case. Plutarch (first cent. A.D.) in "De Facie"[6] says that "465 periods of ecliptic full moons contain 404 six-month (intervals) and the rest five-month (intervals)."

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[8] E.g. Manitius in Hipparchus, Aratus Comm., p. 286. Rehm (RE 8,2, col. 1668, 63–1669, 1) rightly objected against constructing a title of a treatise from this note in which Achilles names four astronomers as occupied with the same topic (cf. below p. 666).

[9] Below p. 322.

[1] Above p. 310 (3).

[2] Published in CCAG 8, 2, p. 126 to 134.

[3] Rome [1931, 2] made it plausible that this commentary was not written before A.D. 213; cf. also Rome [1931, 1], p. 97, note 2. The terminology shows parallels with Proclus' Hypotyposis (e.g. the use of μηκικός).

[4] Strictly speaking one should say that intervals between lunar eclipses are always of the form $5m+6n$ months, where m and n are non-negative integers.

[5] CCAG 8, 2, p. 126, 21 to 28.

[6] Plutarch, De facie in orbe lunae, 20, 933 E (Loeb XII, p. 131).

It is easy to see how this leads to the equation (1). Let us assume that we begin counting with a total lunar eclipse at an ascending node. The opposition 6 months later takes place at a descending node. Therefore each 6-month interval between eclipses corresponds to 13 half-periods of the lunar latitude; a 5-month interval contains 11 half-periods of the latitude. Consequently 404 six-month intervals plus 61 five-month intervals, i.e. 2729 months, contain $404 \cdot 13 + 61 \cdot 11 = 5923$ half-periods of latitude. Doubling this relation in order to obtain whole draconitic months gives the equation (1). Thus Plutarch's rule for eclipse intervals is the equivalent of (1). The anonymous commentary, mentioned above, implies that Hipparchus was familiar with both aspects of the cycle (1).

We know from Pliny[7] that Hipparchus had made discoveries about eclipse intervals, e.g. that lunar eclipses can occur at 5-month intervals, while solar eclipses can be as little as one month and as much as 7 months apart, the shorter interval, however, being excluded for the same locality. As we have seen[8] this problem of eclipse intervals is discussed at length by Ptolemy in Alm. VI, 6. The facts mentioned by Pliny as Hipparchus' discoveries correspond to the items **B**, **E**, and **F** in Ptolemy's chapter. No doubt the remaining facts were also known to Hipparchus, in part, of course, inherited from Babylonian astronomy. Indeed, not only is (1) used in the ephemerides of System B[9] but eclipse lists, e.g. the famous "Saros Canon",[10] expressly note the insertion of a 5-month interval between the regular 6-month steps.

An often cited (and then "emended") remark by Achilles ($\approx$A.D. 250) mentions, among others, Ptolemy and Hipparchus as having written on solar eclipses in relation to the seven climata.[11] I see no reason for declaring this passage as erroneous; the question of intervals between solar eclipses hinges on parallax and thus on geographical latitude, as plainly discussed by Ptolemy. Even if we do not learn anything new from Achilles about the theory we may at least take his remark as a confirmation of Pliny.

3. Parallax

Who introduced the concept "parallax" into Greek astronomy we do not know, but it is clear that it played an important role in the work of Hipparchus. His attempts to determine the distances of sun and moon are intimately related to this topic[1] and in his use of the moon as an object of accurately known longitude[2] the effect of parallax had to be taken into consideration.

We know from the Almagest that Hipparchus had written a work entitled "Parallaxes" in (at least) two books.[3] It is, however, only little that Ptolemy

[7] NH II, 57 (Budé II, p. 25f.).
[8] Above I B 6, 4.
[9] Below p. 523 (2c).
[10] Below p. 549; cf. also Neugebauer [1973, 3], p. 248ff. or Aaboe [1972], p. 114. The emphasis on the 5-month intervals is a common feature in Babylonian eclipse texts.
[11] Maass, Comm. Ar. rel., p. 47, 13; also below p. 666.
[1] Cf. below I E 5, 4 B.
[2] Cf. for this method, e.g., above p. 295.
[3] Alm. V, 19, Heiberg I, p. 450, 1 and 4 (παραλλακτικά).

quotes from its contents and that little is not presented too clearly. Pappus refers to this passage in his commentary to Alm. V, 19 and presents us with a lengthy discussion.[4] Unfortunately it is quite clear that he had nothing significant to contribute from Hipparchus.

In fact I know of only one single passage which gives us any factual information about Hipparchian parallaxes: in connection with lunar observations made in Rhodes in the year −126 May 2 we have explicit values for the longitudinal parallax component.[5] As we have shown before[6] the agreement between Hipparchus' figure and the parallax found by using Ptolemy's tables is relatively close, which seems to imply for Hipparchus the correct determination of zenith distances and angles corresponding to the tabulation in Alm. II, 13.[7]

This leads to the question of how Hipparchus could have determined parameters of this type. Obviously he did not have at his disposal the methods of spherical trigonometry used by Ptolemy in constructing the tables in Alm. II, 13.[8] One could think of arithmetical sequences[9] but to compile multi-entry tables equivalent to Ptolemy's tables would be extremely complicated, going far beyond attested methods. Hence it seems necessary to refrain from the idea of a tabulation of zenith distances and angles for any larger range of the relevant variables. This conclusion is supported by a remark in Ptolemy's criticism of Hipparchus' handling of parallaxes, saying that Hipparchus restricted himself to one single case[10] instead of determining the parameters for a proper sequence of nodal distances.

What specific case Hipparchus dealt with Ptolemy does not say. Pappus, however, in his commentary to Alm. V, uses as an example[11] a lunar position in the beginning of Pisces with a latitude of +2;30°, the nearby descending node being at ♈0°. These are elements in round numbers which, however, are similar to the situation in the above-mentioned Hipparchian observation in −126 May 2 when the moon had a longitude of ♓21;40. Using the tables in Alm. IV, 4 one finds a distance of 26;30° from the descending node and thus a latitude of about +2;15°. For the given observation one finds the moon at 2;45^h east of the meridian whereas Pappus assumes for his example 3^h east of the meridian. All this could indicate that Hipparchus discussed in his treatise only the specific observation of −126 or a nearby situation for the endpoints of zodiacal signs.

In concluding his criticism Ptolemy refers to the crude simplifications made by Hipparchus,[12] e.g. by using the zenith distance of the point L and the angle γ (cf. Fig. 289) unchanged for the point L′ as well.

[4] Rome CA I, p. 150, 20–155, 27.

[5] Alm. V, 5 (Manitius I, p. 271, 6–8), above p. 89(1). A second observation (−126 July 7) is of no interest for our present problem because the moon is so near to the highest point of the ecliptic that obviously $p_\lambda \approx 0$.

[6] Above p. 90.

[7] Cf. above I A 5, 5.

[8] In principle one could reduce the problem to plane trigonometry by means of stereographic projection. This, however, would require the knowledge of conformality of this mapping, a property unknown in antiquity (cf. below p. 860).

[9] Cf. above p. 304 ff.

[10] Alm. V, 19, Manitius I, p. 329, 24–29.

[11] Cf. below p. 324.

[12] Actually Ptolemy's own methods are just as crude (cf. above I B 5, 6).

It is this, least interesting, part of Ptolemy's criticism on which Pappus elaborates. Before describing the procedure followed by Pappus we observe that the basic parameters are simply taken from the Almagest and cannot be considered to be authentic Hipparchian. Indeed Pappus himself says that the zenith distance at ♓0° (i.e. $ZL'=65;48°$) is taken "from the table of the angles" (i.e. Alm. II, 13). In fact this number is an error for 63;48° that is only found in the version D of the Almagest manuscripts,[13] which has been shown to be the version used by Pappus.[14] It seems absurd to postulate that such a scribal error could go back to Hipparchus and then be perpetuated in just one branch of the Almagest manuscripts, against the correct value in the remaining versions. The angle $\theta=71;26$ at L′ is simply taken from Alm. II, 13.

Turning now to the details of Pappus' commentary I cannot consider it as a faithful reproduction of Hipparchus' procedure. On the contrary, the single steps are in part obviously meaningless and in my opinion only introduced in order to obtain some procedure which Ptolemy could have rightly criticized. In particular I think that the "primitivity" of the trigonometry displayed by Pappus is his own reconstruction of how badly Hipparchus may have argued. In fact, I think, Pappus knew as little about the original Hipparchian text as we do.

Pappus' first step consists in determining the arc ML′ on the meridian that passes through the moon (cf. Fig. 289). Instead of using in the (plane) triangle AML′ the sine theorem[15]

$$ML'=L'A\ \sin i/\sin v=30\cdot 5{,}13{,}46/54{,}59{,}45\approx 2;51$$

he divides the triangle by the altitude $L'\Xi$ into two right triangles. In the first one $L'\Xi$ is found from the absurd proportion

$$AL'/180=L'\Xi/10 \qquad (180=2\cdot 90°,\ 10=2i)$$

which gives (with $AL'=30$) $L'\Xi=1;40$ (instead of $30\sin i\approx 2;37$). In the second triangle a similar proportion is postulated

$$ML'/180=L'\Xi/132;52 \qquad (180=2\cdot 90°,\ 132;52=2v)$$

which would give (with $L'\Xi=1;40$) $ML'\approx 2;15$. Actually, however, the value $ML'=1;44$ is accepted which is near the trigonometric solution

$$ML'=L'\Xi/\sin v=1;40/0;55\approx 1;49.$$

Consequently the zenith distance of the moon is taken to be

$$ZM=ZL'-ML'=65;48-1;44=64;4$$

(where 65;48 is the erroneous tabular value discussed above), not greatly different from $ZM=65;48-2;57\approx 63$, what a correct trigonometric procedure would have yielded. $ZM=64;4$ is supposedly the result obtained by Hipparchus.[16]

In the next step Pappus gives another solution which he seems to consider an improvement over Hipparchus'. He assumes that for small latitudes β the angles

[13] Cf. the apparatus to Alm. II, 13 in Heiberg, p. 181, 28.
[14] Rome, CA I, p. 152, note (2); similarly p. 168, note (1).
[15] With $i=5°$, $v=\theta-i=66;26°$, $\theta=71;26°$ from Alm. II, 13.
[16] Rome, CA I, p. 153, 3.

between ecliptic and meridians are the same at L and L′, thus $\gamma \approx \theta = 71;26$. Assuming furthermore[17]

$$\beta = \text{ML} = 2;30^\circ$$

he finds from

$$\beta/180 = \text{LN}/142;52 \qquad (180 = 2 \cdot 90^\circ,\ 142;52 = 2\theta)$$

$\text{LN} \approx 1;59$ and from

$$\beta/180 = \text{MN}/37;8 \qquad (180^\circ = 2 \cdot 90^\circ,\ 37;8 = 2\delta = 2(90-\theta))$$

$\text{MN} \approx 0;31$.

Now $\text{ZN} = \text{ZL} - \text{LN} \approx \text{ZL}' - \text{LN} = 65;48 - 1;59 = 63;49$. Instead of saying that obviously $\text{ZM} \approx \text{ZN} = 63;49$ Pappus computes laboriously

$$\text{ZM}^2 = \text{ZN}^2 + \text{MN}^2 \approx 63;49^2 + 0;16$$

and then indeed finds that $\text{ZM} \approx 63;49 = \text{ZN}$. This he considers an improvement over Hipparchus' 64;4.

Finally Pappus derives some numerical contradictions (e.g. $\text{ML}' \approx 1;59^\circ < \beta = 2;30^\circ$) which are not worth discussing since they are based on making contradictory assumptions like $\text{ZL} = \text{ZL}'$ while actually $\text{ZL} < \text{ZL}'$. It seems to me obvious that all this has nothing in common with Hipparchus' treatise.

4. Size and Distance of Sun and Moon

A. Distance of the Sun

All actual dimensions in this chapter will be reckoned in earth radii, i.e. we shall use the norm $r_e = 1$.

A widely accepted result of a paper by F. Hultsch[1] according to which Hipparchus had determined the distance of the sun as 2490 (hence better than Ptolemy's $R_s = 1210$) was disproved by N. Swerdlow[2] who showed that Hultsch's emendation of a figure 490, given by Pappus, to 2490 is wrong since $R_s = 490$ can be derived from parameters ascribed to Hipparchus by Ptolemy in Alm. IV, 9, combined with lunar distances quoted by Pappus in his commentary to Alm. V, 11 from Hipparchus' work "On Sizes and Distances."

Swerdlow's demonstration of consistency between these parameters is very direct.[3] We know from Ptolemy[4] that Hipparchus assumed at the mean distance of the moon for the apparent radii of moon and shadow

$$r_{☾} = 0;16,37^\circ \qquad u = 2;30\, r_{☾} \tag{1}$$

[17] Alm. V, 8 gives for $\text{AL}' = 30$ the latitude 2;30°. Thus the latitude at L should be greater.

[1] Hultsch [1900].

[2] Swerdlow [1969].

[3] Swerdlow [1969], p. 297/298.

[4] Cf. above p. 313. The parameters (1) are also mentioned by Pappus, Coll. VI (the passage in question is translated in Heath, Arist., p. 412). The ratio $u/r_{☾} = 2;30$ occurs also in Tamil eclipse computations; cf. Neugebauer [1952], p. 272 (3).

while Pappus quotes as mean distance of the moon

$$R_m = 67;20. \tag{2}$$

It follows from (1) and (2) for the actual size of the moon that

$$r_m = R_m \sin r_{☾} \approx 67;20 \cdot 0;0,17,20 \approx 0;19,27 \approx 0;19,30 \tag{3}$$

hence for the radius of the shadow

$$r_u = 2;30 \cdot 0;19,30 = 0;48,45. \tag{4}$$

Now we know that according to Hipparchus' construction as described by Ptolemy[5]

$$R_m = (r_m + r_u - 1) R_s \tag{5a}$$

hence here

$$R_s = \frac{R_m}{r_m + r_u - 1} = \frac{1,7;20}{0;8,15} \approx 8,9;42 \approx 490, \tag{5b}$$

q.e.d.

Before discussing the historical implication of this result we may remove an obstacle which misled Hultsch to his emendation. Theon of Smyrna says[6] that Hipparchus found the volume V_s of the sun 1880 times greater than the volume of the earth, which in turn is 27 times the volume V_m of the moon.[7] The latter statement agrees with (3) since it follows from $r_m \approx 0;20 = 1/3$ that $V_m = 1/27$. From

$$V_s = 1880 \tag{6}$$

however, it would follow that the actual solar radius r_s is about

$$r_s \approx 12;20 \tag{7}$$

hence

$$R_s = \frac{12;20}{0;20} R_m = 37\, R_m = 37 \cdot 67;20 \approx 2490 \tag{8}$$

which motivates Hultsch's emendation.

Hence the question arises how (6) can be reconciled with (5). I think the explanation lies in an ancient mistake which is very much in line with Hultsch's motivation. It follows from (5) that the actual solar radius is given by

$$r_s = R_s \sin r_{☾} \approx 8,10 \cdot 0;0,17 \approx 2;19$$

hence for the volume

$$V_s = 2;19^3 = 12;24 \approx 12;20.$$

[5] Cf. above p. 109 and Fig. 98. Calling now EM $= R_m$, ES $= R_s$, $\sigma = r_u$ we have MC $= 2 - r_u$, hence from (5) AC $=$ MC $- r_m = 2 - r_m - r_u$ and from (6) $R_m = (1 - \text{AC}) R_s = (r_m + r_u - 1) R_s$ which is our present relation (5a).

[6] Dupuis, p. 318/319; ed. Hiller, p. 197, 9.

[7] The same ratios are also mentioned by Chalcidius (4th cent.) and Proclus (5th cent.); Chalcidius gives as title of Hipparchus' work "De secessibus atque intervallis solis et lunae" (ed. Mullach, Fragm. II, p. 202b, ch. 90; ed. Wrobel, p. 161, ch. 91); Proclus, Hypot., ed. Manitius, p. 133. In CCAG 7, p. 20 n. 1 one finds a passage quoted which erroneously assumes $V_s = 1880\, V_m$.

It must have seemed utterly implausible in the time of Ptolemy that the volume of the sun should be only about 12 times the volume of the earth.[8] Thus somebody (Theon or his source) may have taken the figure $V_s \approx 12;20$ as a mistake for $r_s = 12;20$ from which one obtains

$$V_s = 12;20^3 \approx 1880.$$

There is an anonymous statement, cited by Cleomedes,[9] unrelated to all known sources, according to which Hipparchus had assumed a size (volume?) of the sun, 1050 times the size of the earth.

B. Hipparchus' Procedure

To demonstrate the consistency of historically attested Hipparchian parameters does not mean that they were originally obtained in the same fashion in which they were connected in the preceding section. In fact, as was pointed out by Swerdlow, it seems clear from our sources that Hipparchus started with assumptions about the distance of the sun and derived from it the distance of the moon. Both Ptolemy and Pappus agree that Hipparchus made three different attempts to establish the relative distances of the two luminaries. According to Ptolemy his initial data were

(a) solar parallax = least perceptible parallax
(b) from a solar eclipse, ignoring parallax
(c) from the same eclipse, assuming a perceptible parallax

while Pappus says that Hipparchus in Book I of "On Sizes and Distances" assumed

(a) negligible size of the earth with respect to the sun[1]
(b) from a solar eclipse, smallest (perceptible) parallax
(c) from the same eclipse, assuming a greater parallax.

The two sources agree in the multiplicity of Hipparchus' attempts and in his starting from the solar distance in order to find the distance of the moon.

Pappus furthermore explains that the solar eclipse in question was total in the region of the Hellespont and of magnitude about 4/5 in Alexandria.[2] Using these data Hipparchus showed in Book I that the distances of the moon vary between the following limits

$$R_m = \begin{cases} 71 \\ 77 \\ 78 \end{cases} \tag{1}$$

[8] In Alm. V, 16 he finds $V_s \approx 170$.

[9] Cf. below p. 962.

[1] This assumption is made, of course, only as a preliminary simplification of the mathematical discussion. Pappus in his "Collections" (VI, 37 ed. Hultsch, p. 554, 21 f.; p. 556, 6–10; translated in Heath, Arist., p. 413) says that Hipparchus and Ptolemy considered the earth's size negligible only with respect to the sphere of the fixed stars. Proclus (Hypot., p. 112, 15 f. ed. Manitius) ascribes the "leadership" in this question to Hipparchus; cf. also Hypot., p. 228, 19 f.

[2] Cf. for this eclipse (of −189 March 14) above p. 316, n. 9.

whereas "many considerations" led him in Book II to the result

$$R_m=\begin{cases}62\\67;20\\72;40\end{cases}\qquad R_s=490 \tag{2}$$

(of which we made use in the preceding section[3]).

The geometric procedure, described in Alm. V.15, that leads from R_m to R_s is, of course, reversible. In view of Ptolemy's and Pappus' statements we must assume that Hipparchus chose first $R_s=490$ and the derived from it $R_m=67;20$ at mean distance. The choice of R_s is explicable as the consequence of a choice for the horizontal parallax[4] of the sun

$$\sin p_0=1/R_s=1/8,10\approx 0;0,7,21$$

hence

$$p_0=0;7° \tag{3}$$

which seems a plausible estimate for the "least perceptible" parallax.

Much more difficult to answer is the following question: how did Hipparchus use the solar eclipse in question for the determination of the ratio of the distances R_m and R_s? The following discussion does not pretend to give a definitive answer to this question; it only tries to connect a specific procedure with the given data in order to see whether or not acceptable distances could be obtained in this way.

Obviously we can only proceed in a very general fashion. Thus we refrain from taking into account the influence of parallax, i.e. we assume that identical visual cones tangent to the sphere of the moon have their vertices in the observer at A and at H; similarly we assume identical distances R_m and R_s from both observers to the luminaries and identical apparent radii for sun and moon. Although the observations at A and H are not made at the same moment or in the same meridian we may nevertheless combine the diameters as seen from H with the solar disk as seen from A (cf. Fig. 290). The difference of appearance is caused by the terrestrial distance between A and H which corresponds approximately to a difference of 10° ($\approx 41-30;30$) of geographical latitude, hence to about 1/36 of the terrestrial meridian; hence $\Delta\approx 6/36=0;10$ earth radii. An apparent diameter of 1/2° of the sun means that the real diameter covers about $6R_s/720=R_s/2,0$ of the solar orbit. According to observation only 1/5 of this arc was visible from A. Hence the length of the visible part of the solar diameter is $\Sigma=R_s/10,0$ and we have the following relations

$$\frac{R_s-R_m}{R_m}=\frac{\Sigma}{\Delta}=\frac{R_s}{1,40}$$

or

$$\frac{R_s}{R_m}=1+\frac{R_s}{1,40}. \tag{4}$$

Since we know that the eclipse data resulted in the values of R_m given in (1) we can find from (4) the solar distances which would have produced R_m. In this

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[3] For the epicycle radii which result from (1) and (2) cf. above p. 316 (5).

[4] Swerdlow [1969], p. 299. For parallax cf. above I B 5.

way one finds

$$R_s = \begin{cases} 4{,}5 \approx 240 \\ 5{,}37 \approx 340 \\ 8{,}8 \approx 490 \end{cases} \quad \text{for} \quad R_m = \begin{cases} 71 \\ 77 \\ 83 \end{cases} \tag{5}$$

Hence the order of magnitude for R_s agrees well with Hipparchus' known assumption (2). The corresponding horizontal parallaxes of the sun are

$$p_0 = \begin{cases} 0;14^\circ \\ 0;10 \\ 0;7 \end{cases} \tag{6}$$

also in reasonable agreement with plausible modifications of "perceptible" parallaxes, mentioned by Ptolemy and Pappus.

Once more I wish to underline that I do not pretend to have reconstructed Hipparchus' method. Nevertheless considerations of this type are shown capable of producing preliminary estimates which might have served Hipparchus as the basis for refined investigations presented in Book II. The fundamental source of error lies, of course, in a priori assuming a perceptible solar parallax, a hypothesis rightly considered very questionable by Ptolemy.

§ 6. Additional Topics

The fact that no work of Hipparchus is preserved, except for the least important, the Commentary on Aratus, makes it impossible to give a coordinated description of his astronomy. All we can do is to regroup some of the scattered remarks found in ancient sources and hope that they are not too badly misleading.

1. The Planets

Hipparchus considered it a possibility that not only the five hallowed planets had a motion of their own but that a multitude of stars near the ecliptic participated in an eastward motion, however slow. I think this departure from the traditional viewpoint is a truly remarkable example of Hipparchus' independence of thought. When he eventually realized that the phenomenon in question was caused by the precession of the equinoxes[1] his initial hypothesis had paved the way to an insight of the highest importance.

In dealing with the traditional problems of planetary motion he undoubtedly relied for basic parameters on Babylonian results, in particular on the so-called "goalyear" periods and the corresponding mean motions.[2] Not only for the basic parameters but also for the cinematic theory the situation is similar for the planets and for the moon. In both cases Hipparchus established discrepancies between

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[1] Cf. above p. 296.

[2] Alm. IX, 3 (Man. II, p. 99). Cf. above I C 1, 4; also Neugebauer [1956], p. 295.

the existent theory and his own observational data,[3] but he felt that the time was not yet ripe for establishing definitive models,[4] a task which had to wait three centuries for the next step.

From Ptolemy's report it seems clear that Hipparchus was aware of the existence of both planetary anomalies, i.e. not only of the anomaly caused by the motion of the planet on the epicycle but also of the anomaly that corresponds to an eccentricity of the deferent. But we know nothing about the parameters of such a model, be it Hipparchian or assumed by contemporary astronomers or predecessors.

In general one would like to know Hipparchus' attitude toward contemporary planetary theory. It does not seem likely that he knew the intricacies of the Babylonian computations of ephemerides. But he surely must have known about those contemporary procedures still available to us through papyri and from the astrological literature,[5] though one cannot say how significant it is that his name is occasionally associated with these rather primitive methods.[6]

Another aspect of Hipparchus' interest in the planets has only recently become known through the discovery of the second part of Book I of Ptolemy's "Planetary Hypotheses" in an Arabic translation.[7] He attempted to estimate the size and the brightness of planets in relation to the sun and to fixed stars. For the apparent diameter of Venus he proposed

$$d_{♀}=1/10\,d_{\odot} \tag{1}$$

presumably at mean distance, though Ptolemy (who agreed with this estimate) is not explicit about this point.[8] Similar data for the remaining planets are also listed by Ptolemy but it seems more likely that they are Ptolemy's own estimates than that they come from Hipparchus.

For the diameter d_0 of the smallest fixed star Hipparchus assumed

$$d_0=1/30\,d_{\odot}. \tag{2}$$

Estimates of this kind are obviously purely speculative since (2) implies that $d_0 \approx 0;1°$, whereas an angle of about 0;7° was considered the "smallest perceptible parallax."[9] Probably also Hipparchian is an estimate for the apparent diameter d_1 of a fixed star of the first magnitude, equated with Mars:

$$d_1=1/20\,d_{\odot}=d_{♂}. \tag{3}$$

These two types, d_0 and d_1, correspond perhaps to the two classes of magnitudes distinguished by Hipparchus as "bright" and "dim."[10] Such a comparison of planets and fixed stars need not to be, however, an innovation by Hipparchus;

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[3] Perhaps to this group belongs his interest in the distance of Mercury from Spica (Alm. IX, 7, Manitius II, p. 134, 29; above p. 159).

[4] Ptolemy, Alm. IX, 2, Man. II, p. 96.

[5] Cf. below p. 823f.

[6] Below V A 1.

[7] Goldstein [1967]; cf. below V B 7.

[8] Goldstein [1967], p. 8. This is also assumed by Bar Hebraeus (L'asc. II, ch. 7, Nau, p. 194f.) who quotes from the k. al-manshūrāt (cf. Goldstein, p. 4, n. 8). Caution is nevertheless necessary since the values for the moon are based on the Ptolemaic model, not on the simple Hipparchian.

[9] Cf. above p. 327.

[10] Cf. above p. 291.

we know, e.g., from an hellenistic papyrus on weather signs that Mars and Arcturus were considered of equal brightness.[11]

Estimates for the size of Venus and for the smallest fixed stars are also given by Plutarch[12] but there is no evidence for a direct relation to Hipparchus.

2. Astrology

It is difficult to reach a secure conclusion concerning Hipparchus' attitude toward astrology which, during his lifetime, was in its first period of ascendency in the hellenistic world. Ptolemy does not mention Hipparchus in the Tetrabiblos, admittedly a weak argument. Hipparchus is mentioned by Vettius Valens but only in connection with astronomical tables or computational methods.[1]

Pliny[2] is often quoted, praising Hipparchus because "no one having done more to prove that man is related to the stars and that our souls are part of heaven, (he) detected a new star ...[3]" Even if this rather senseless non sequitur is not simply due to Pliny's concept of Stoic philosophy it certainly does not support any connection with routine astrology, as Bouché-Leclercq has rightly pointed out.[4] Nevertheless many scholars have taken the opposite view, e.g. Cumont said[5] "Hipparque, dont le nom doit être placé en tête des astrologues comme des astronomes grecs."[6]

A direct association of Hipparchus with astrological doctrine is due to Firmicus Maternus (4th cent.) who says[7] that Fronto (an otherwise unknown astrologer) followed Hipparchus in his theory of the "*antiscia*." Fortunately we do not need describe here what this astrological term means, beyond that it implies pairing opposite points of the ecliptic in directions parallel to the equinoctial or solstitial diameter.[8] I do not think it out of the question that there exists some relation to the early theory of sun dials since one of its coordinates is called "*antiskion*"[9] but I see no connection with any known work of Hipparchus.

The main source for relating Hipparchus with astrology is, however, Hephaistio (end of 4th cent.) who extensively discusses different systems of "astrological

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[11] Wessely [1900]; Neugebauer [1962, 3], p. 40, col. II, 7–10. Cf. for this text also below p. 737 (n).
[12] Cf. below p. 693.
[1] Cf. above p. 306 and below p. 823.
[2] NH II, 95, Loeb I, p. 239, Budé II, p. 41.
[3] Continued in the passage discussed above p. 285.
[4] AG, p. 543/4; also Pfeiffer, Sterngl., p. 115.
[5] Cumont [1909], p. 268; similar Ég. astrol., p. 156, etc.
[6] Similar Boll, Kl. Schr., p. 5, n. 1 (1908); Rehm in RE 8, 2, col. 1680, 29 ff. (1913); Gundel HT, p. 303/4 (1936) who makes Hipparchus responsible for the astrology of his "pupil" Serapion (a relationship which is very doubtful: cf. Neugebauer [1958, 1], p. 111, note 39). A naive blunder is committed by Dicks, Hipp., p. 3, who did not realize that the "testimony K" from CCAG 5, 1, p. 205 (also in CCAG 1, p. 80) is taken from Ptolemy's "Phaseis" (Heiberg, p. 67) and has nothing to do with astrology.
[7] Firmicus Maternus, Mathesis II, Praef. (ed. Kroll-Skutsch I, p. 40, 8 ff.; p. 41, 5 f.) I see no reason for considering "antiscia" a book title (Rehm, RE 8, 2, col. 1668, 32).",
[8] The theory of "antiscia" is described by Firmicus in II, 29 (Kroll-Skutsch, p. 77–85); cf. also Vettius Valens III, 7 (ed. Kroll, p. 142, 28) from Critodemus (1st B.C.?). For discussion cf. Bouché-Leclercq AG, p. 161 f.; p. 275, note 2.
[9] Mentioned by Ptolemy in his "Analemma"; cf. below p. 1380 and Figs. 26 and 27.

geography," i.e. associations of zodiacal signs with geographical regions. Right at the beginning[10] he gives a list of countries under the influence of Aries according to "Hipparchus and the old ones of Egypt." A second time Hipparchus is explicitly mentioned[11] for the regions ruled by Sagittarius and it seems a plausible assumption that he did not restrict himself to just these two signs, in particular since Hephaistio[12] provides us with similarly arranged groupings for the ten signs from Aries to Capricorn.[13] What is most peculiar in the system ascribed to Hipparchus is the fact that the countries in question are not simply associated with zodiacal signs but with parts of the zodiacal constellations, e.g. Babylonia with the left, Thrace with the right shoulder of the Ram, etc. This seems consistent with our suggestion that Hipparchus' "Catalogue of Stars" was arranged according to stellar configurations, not according to any uniform system of spherical coordinates, in particular not longitudes and latitudes.[14]

A small treatise on the astrological qualities of the twelve zodiacal signs is in some manuscripts assigned to Hipparchus but there is no reason to take this very seriously.[15]

3. Geography

The discovery of the sphericity of the earth has important methodological consequences: it seems natural to postulate a corresponding sphericity of the sky[1] and thus to introduce spherical coordinates both on the earth and on the "celestial sphere." The daily rotation of the latter leads to equatorial coordinates on the earth, i.e. geographical longitudes and latitudes. On the celestial sphere the importance of the inclined ecliptic results in a mixed system of equatorial and ecliptic coordinates,[2] which eventually, through the discovery of precession, gives place to strictly ecliptical coordinates for the stars and for the planets, besides a consistently equatorial system for the phenomena connected with the daily rotation.

We do not know the details of the development toward mathematically consistent spherical coordinates, a process probably complicated by Babylonian

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[10] Engelbrecht, p. 47, 20.

[11] Engelbrecht, p. 60, 30.

[12] These passages from Hephaistio are conveniently tabulated in a monograph by Karl Trüdinger, Studien zur Geschichte der griechisch-römischen Ethnographie (Basel 1918), p. 84.

[13] As Rehm has pointed out (RE 8, 2, col. 1680) Hephaistio's text has a close parallel in Vettius Valens I, 2. Since both texts mention Corinth as existent (Engelbrecht, p. 63, 4; Kroll, p. 11, 27) and speak of the "domain of Carthage" instead of Roman "Africa" (Engelbrecht, p. 61, 6; Kroll, p. 7, 23 but in different context), a situation before 146 B.C. is assumed.

[14] Cf. above p. 283; p. 287.

[15] The text exists in several versions; the two best ones were edited by Maass, Anal. Erat., p. 141–149. The longer version is based on CCAG 8, 3, p. 61, Cod. 46, F. 9^v, the shorter one is recorded in CCAG 2, p. 1, Cod. 1, F. 221^v; 3, p. 10, Cod. 12, F. 188; 4, p. 23, Cod. 7, F. 88^v; 9, 2, p. 3, Cod. 38, F. 9. Abridged versions are CCAG 9, 2, p. 6, Cod. 39, F. 101 and CCAG 9, 2, p. 62, Cod. 65, F. 154; 11, 1, p. 6, Cod. 1, F. 126; p. 122, Cod. 14, F. 394^v.

[1] Even a hemispherical cupola is by no means an a priori concept; cf. below p. 577.

[2] Cf. above I E 2, 1 A.

influences. We know only that it was essentially completed in the time of Ptolemy[3] while it was far from completed in the time of Hipparchus. This holds not only for celestial coordinates but also for geography; as far as we know the first application of an orthogonal grid of geographical longitudes and latitudes, reckoned uniformly in degrees, is found in Ptolemy's "Geography."[4] As usual, the naive retrojection to Hipparchus of all concepts found in the work of Ptolemy has only obscured the history of hellenistic mathematical geography.

The geographical "fragments" of Hipparchus have been first assembled by H. Berger (1869) and again (1960) by D.R. Dicks. The first mentioned collection arranges the material according to topics whereas Dicks tries to follow the original sequence in Hipparchus' treatise against Eratosthenes. Both ways have their advantages and disadvantages, and both suffer from the elimination of the background of Strabo's narrative through their attempt to isolate the "authentic" fragments. Dicks prints his text without any apparatus, thus creating the impression of giving a secure text even if it is only based on a conjecture of some earlier edition.

A. Geographical Latitude

There are many ways of determining at a given locality the geographical latitude φ, e.g. directly by measuring the altitude of the pole or by gnomon observations (involving solar declinations), or by culminations of fixed stars, etc. It would be a quite unhistorical approach, however, to assume that these different procedures must always lead to the same value of the parameter φ. The mere fact that the length M of the longest daylight appears to be the most popular description of latitudinal positions (the whole concept of "climata" is based on it) suffices to show that one was satisfied, by and large, with the least accurate but practically most important parameter, not surprising at a time when one was not yet able to transform M into φ or vice versa mathematically correctly.

For the author of a "Geography" it must have been much easier to establish for certain regions reasonable estimates for M than to obtain some records of accurate measurements of angular quantities, even if such observations occasionally may have existed. It is therefore easy to understand that early geography consisted either mainly of descriptive material or concentrated completely on the strictly mathematical problem of providing the basic parameters, this latter approach being characteristic for Hipparchus, as Strabo tells us time and again.[1]

It is an obvious mathematical consequence of the sphericity of the earth that all horizons of equal "inclination" toward the earth's axis, i.e. all localities on the same "parallel" to the equator, will experience the same "phenomena" of the fixed stars (rising, settings, etc.) as well as the same length of daylight or solar altitude.

A systematization of this insight, consisting in a definition of distinct "climata" (i.e. inclinations) by their longest daylight, took place during the hellenistic

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[3] That there was still room for systematic improvements in the time of Ptolemy is shown by his introduction of new coordinates in the "Analemma" (cf. below V B 2, 5).

[4] Cf. below p. 934; also above p. 280.

[1] Mainly in Book II, 1 of his "Geography." Cf. also the sharp division of topics in Ptolemy's Geography: Book I mathematical theory, lists of localities and their coordinates in the remaining books.

period. Modern scholars proposed Eratosthenes, Hipparchus,[2] and of course Posidonius,[3] as inventors of the system of climata, using intricate arguments which carry little conviction.

The features which one tries to explain are indeed of a great variety. The terms "clima"[4] and "parallel" only gradually obtain (at least with some authors) an exact meaning in relation to a definite sequence of values M for the longest daylight. Unnecessary difficulties were created by the idea that only the sequence that progresses with 1/2-hour intervals represents "geographical" climata, in contrast to "astrological" climata. In fact the latter are nothing but a hellenistic generalization of Babylonian arithmetical methods for the determination of the length of daylight.[5] Finally there is the question why these climata (of any variety) are numbering seven, e.g. for the half-hour pattern from $M = 13^h$ to $M = 16^h$.

Our sources show all these features in a variety of combinations, a fact which makes it very unlikely to detect one definite author as the creator of "the" theory of climata. It seems certain that Hipparchus knew about climata; we hear from Strabo[6] that he had stressed the need for the investigation of climata in order to determine, e.g., whether Alexandria is to the north or to the south of Babylon (a question of obvious concern to Hipparchus). Though this statement is rather awkward in the form it appears in Strabo it probably means that one should design methods capable of establishing accurately on what parallel a given place is located.

Some results of Hipparchus in the arrangement of latitudinal data can be gathered from some longer summary in Strabo.[7] The most conspicuous feature of Strabo's report is the great number of distances between parallels, reckoned in stades; Fig. 291 gives a graphical representation of these data, drawn to scale (in units of 100^{st}). Two of these intervals are taken from Eratosthenes and deviate a little from the Hipparchian intervals.[8]

We know that Hipparchus followed Eratosthenes in assuming 700^{st} for 1° of a terrestrial meridian.[9] He therefore could have changed these distances between parallels to latitudes; but since the majority of the intervals are not divisible by 7 it is clear that the distances in stadia were not derived from geographical latitudes measured in degrees.[10] This is, of course, not surprising after we have shown[11] that the Hipparchian intervals, reckoned from Alexandria, form an arithmetical pattern for the distances counted in hundreds of stadia, a relationship which does not include divisibility by seven.

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[2] Honigmann, SK (widely accepted); Dicks [1955] opposing Honigmann.

[3] Reinhardt (cf., e.g., Honigmann, SK, p. 8f.).

[4] And derivations like ἔγκλιμα, etc. (listed, e.g., in Honigmann, SK, p. 5ff.; also Dicks [1955], p. 249f.).

[5] Cf., e.g. below IV D 1, 3.

[6] Strabo, Geogr. I 1, 12 (Loeb I, p. 23; Budé I, 1, p. 74); cf. also II 5, 34 (Loeb I, p. 503; Budé I, 2, p. 117).

[7] Geogr. II 5, 34–42 (Loeb I, p. 507–517; Budé I, 2, p. 118–123).

[8] The boundaries are Meroe (13^h) and Borysthenes (16^h). I think Honigmann is right (SK, p. 13) when he takes this as indicating that Eratosthenes was familiar with the seven half-hour zones.

[9] Cf. above p. 305, n. 27.

[10] It is, of course, absurd to give latitudes to seconds (rounded!), as, e.g., in the Loeb translation.

[11] Cf. above p. 305.

Of the three parallels south of Alexandria Syene is obviously determined by the condition fundamental for Eratosthenes' estimate of the earth's circumference: $\varphi = \varepsilon = 24°$, thus $24 \cdot 700 = 168000^{st}$ from the equator.[12] Meroe is put at the traditional 10000^{st} south of Alexandria.[13]

To the north of Alexandria the intermediate parallels are obviously inserted under the viewpoint of convenient degree intervals: Rhodes 5;12° from Alexandria, Byzantium 7° north of Rhodes, Mid-Pontus 2° more, all parallels of special interest to Hipparchus. Only the parallel of Phoenicia, 1° north of Carthage, seems to be an intruder, unrelated to the Hipparchian structure. Also Carthage itself has no obvious place in Hipparchus' geography and the equinoctial noon shadow 11:7 is crudely inaccurate.[14]

The parallels enumerated by Hipparchus include all of the seven which are known as the "seven climata." That the Hipparchian list exceeds these main parallels cannot be used as an argument against the existence of a special emphasis at his time, or earlier, on the canonical number seven. Ptolemy, who also explicitly mentions the "seven climata"[15] and who arranges important tables accordingly[16] exceeds this number of parallels in his trigonometric discussions[17] or reduces it to five in his "Phaseis" where the extremal regions with $M = 13^h$ and $M = 16^h$ are of little interest for weather predictions.[18]

Of purely astronomical elements Strabo's excerpts contain very little. A consistent set of data is only given for the solar altitudes at the solstices at the four northernmost parallels ($M = 16^h$ to 19^h). As shown before[19] these solar altitudes are again determined by an arithmetical scheme that solves effectively a problem of spherical astronomy.

Fixed star positions are only mentioned with five parallels, beginning with the first one through the Cinnamon-producing Country.[20] At this parallel the whole constellation of the Little Bear is said to be always visible since the southernmost star of this constellation, the "bright star at the Tip of the Tail" (α Umi) is located on the arctic circle.[21] The emphasis on this specific position is of interest to us because the same star is associated with a Latin fragment which we related to the Hipparchian catalogue of constellations.[22] There the polar distance of α Umi is given as 12 2/5°, also attested as a Hipparchian value by Marinus and by Ptolemy. Hence we have for the parallel through the Cinnamon Country a latitude of about 12;24° which is not part of the canonical sequence of climata.[23] Probably

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[12] Cf. below p. 653.

[13] Cf., e.g., the estimate in Strabo II 5, 7 or XVII 3, 1 (Loeb I, p. 439; VIII, p. 157). For the Cinnamon-producing country cf. below p. 335.

[14] Cf. below p. 746, n. 3.

[15] Alm. VI, 11 (Heiberg I, p. 538/539). Introduction to the Handy Tables (Opera II, ed. Heiberg, p. 174,17).

[16] Angles between meridian and ecliptic (Alm. II, 13; cf. above p. 50); Analemma (below p. 853 and p. 854). Similarly in the Handy Tables: oblique ascensions and parallaxes (below p. 978).

[17] Above p. 43f. and Table 2.

[18] Ptolemy, Opera II ed. Heiberg, p. 4, 3–20; below p. 928.

[19] Above p. 304.

[20] Cf. p. 1313, Fig. 291.

[21] Strabo II 5, 35 (Loeb I, p. 507; Budé I, 2, p. 119).

[22] Above p. 290.

[23] Alm. II, 6 gives $M = 12;45^h$ for $\varphi = 12;30°$. The 8800^{st} from the equator would correspond to $\varphi = 12;34°$.

it was the astronomical interest which assigned this parallel a place in the list of important latitudes.

Another of the northernmost constellations is associated with Syene ($\varphi = \varepsilon$), but in a less specific fashion. The major part of the Great Bear is there always visible with the exception of one of the stars of the Rectangle (γ Uma with $\bar{\delta} \approx 25;7°$), the star in the Tip of the Tail (η Uma; $\bar{\delta} \approx 29;20°$), and the stars in the Legs. Finally Arcturus (α Boo) is said to reach almost the zenith of Alexandria. Since Hipparchus placed this star at 59° from the north pole[24] the corresponding latitude of Alexandria would be a little more than 31° which agrees with the distance of 21800st from the equator (cf. Fig. 291, p. 1313).

No more astronomical data are mentioned by Strabo until $M = 15\,1/2^h$ and $M = 16^h$. The first mentioned parallel is said to be equidistant from pole and equator (thus $\varphi \approx 45°$). There the star "in the Neck of Cassiopeia" is on the arctic circle while the "Right Elbow of Perseus" is slightly to the north of it. At 16^h the whole of Cassiopeia lies within the artic circle. The star "in the Neck" is taken[25] to be α Cas because its distance from the pole is $\bar{\delta} \approx 45;12°$. In the Commentary to Aratus this star is not mentioned and in Ptolemy's catalogue it is placed "in the Breast" of Cassiopeia. The star in the Elbow of Perseus is mentioned in the same way as in Strabo by Ptolemy (η Per) but Hipparchus in the Aratus commentary places it in the "Right Hand."[26] All this is a good example of the ambiguity or fluidity of the ancient nomenclature for constellations.

There remain only two more astronomical data in Strabo which concern Hipparchus' geographical latitudes. This time they are given in the form of gnomon shadows but again in relation to the above-mentioned limits Alexandria and Byzantium.[27] For Alexandria the ratio of the gnomon to the equinoctial noon shadow is given[28] as

$$g : s_0 = 5:7.$$

Since this ratio is obviously absurd (it would give $\varphi = 54;28°$) the text is customarily changed to 5:3 (hence $\varphi = 30;58°$). I think, however, that Honigmann's suggestion[29] is much more plausible to assume a mistaken rendering of a passage as we find it in Hypsicles who says that he determined the ratio of the extremal lengths of daylight in Alexandria to be $m : M = 5:7$, using the solstitial noon shadows of the gnomon. Also Strabo refers in the case of Byzantium[30] to a solstitial shadow, saying that

$$g : s_1 = 120 : 41\,4/5$$

holds for the summer solstice (which gives $\varphi - \varepsilon = 19;12$ thus $\varphi = 43;12°$).

It is amazing to see how many huge and totally useless projects were attributed to Hipparchus. Obviously in his spare time, after having computed some 2500 eclipses and filled 12 Books with plane trigonometry, he felt the urge to determine

[24] Aratus Comm., p. 82, 24f. Actually $\delta \approx 31;17°$ in -125.
[25] E.g. Manitius in Ar. Comm., p. 301, n. 28. Cf., however, Schjellerup [1881], p. 30 about η Cas.
[26] Manitius, p. 120, 18.
[27] Cf. above p. 335.
[28] Strabo II 5, 38 (Loeb I, p. 511, Budé I, 2, p. 120).
[29] Mich. Pap. III, p. 316.
[30] Strabo II 5, 41 (Loeb I, p. 514, Budé I, 2, p. 122).

for each single degree of a meridian quadrant an assortment of astronomical data – so at least one elaborates admiringly Strabo's introductory remarks[31] to his summary of which we made use in the preceding pages. Fortunately one need not to rely on common sense to see that such a catalogue (prepared without the help of spherical trigonometry) never existed. Not only do we have the explicit statement of Ptolemy[32] that Hipparchus gave only for comparatively few localities the altitude of the pole, but Hipparchus himself explains[33] that he had given the derivations (or proofs: *ἀποδείξεις*) which make it possible to establish (*παρακολουθεῖν*) for practically any place in the oikoumene the simultaneous risings and settings of the constellations and the zodiacal signs. What he claims to have done is a great achievement indeed: to develop methods, valid for parameters which concern the oikoumene (obviously not the polar regions where such rules usually fail) to solve problems in general which were fundamental for the critical discussion of "phenomena" as presented by Aratus and other astronomers for one specific latitude ($M = 14\ 1/2^{h}$). Unfortunately we do not have the underlying mathematical discussion but how his problems looked and in what form he answered them we can very well evaluate from his preserved work without postulating the existence of tabulations of a totally unattested type.

B. Longitudes

There is no doubt that Hipparchus underlined in his writings the necessity of using for the determination of geographical longitudes the differences in local time, made evident through the observations of lunar eclipses. That Strabo speaks in this context about lunar and solar eclipses is perhaps only a slip of the pen.[1] I do not know, however, of any compelling reason to assume that the geographical importance of eclipse observations was unknown before Hipparchus.

We do not know how Hipparchus counted longitudes. A remark in Strabo seems to indicate that he followed Eratosthenes in the strictly sexagesimal division of the equator.[2] The fact that his reckoning of latitudes appears to have been based on stades, while longitudes are counted in sixtieths of the equator, seems not to favor the assumption that Hipparchus constructed a terrestrial map. Of course he could have converted all units uniformly to the same norm, e.g. degrees, but one would expect to find some traces of such an adjustment. At present we see no such steps made before Marinus and Ptolemy.[3]

We know that Hipparchus in his list of latitudes accepted with Eratosthenes the line Meroe-Alexandria-Borysthenes as basic meridian.[4] It seems plausible that he counted longitudes to the east and to the west from this zero-meridian but explicit evidence for any such norm is lacking.

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[31] Strabo II 5, 34 (Loeb I, p. 503, Budé I, 2, p. 117).
[32] Ptolemy, Geogr. I 4, 2 (Nobbe, p. 11; Mžik, p. 21).
[33] Aratus Comm. II 4, 3 (Manitius, p. 184/185). The above quoted passage in Strabo (above note 31) is only a clumsy paraphrase of Hipparchus' words to which Strabo added of his own (*λέγω δὲ*) "from the equator to the north pole."
[1] Strabo, Geogr. I 1, 12 (Loeb I, p. 25; Budé I, 1, p. 74); cf. also below p. 667.
[2] Strabo, Geogr. II, 5, 7 (Loeb I, p. 439; Budé I, 2, p. 86); cf. also below p. 590.
[3] Cf. below V B 4, 1.
[4] Strabo, Geogr. I 4, 1 (Loeb I, p. 233; Budé I, 1, p. 167); cf. also below p. 652.

How little we know about Hipparchus' geography is evident from the fact that a remark by Pliny can cause more difficulties than intelligible information. Pliny tells us (right after the story about the 600 years of eclipse predictions) that Hipparchus' work included "the months of the nations, and days and hours, as well as the place of localities and the appearance of peoples."[5] This seems to mean that he had discussed (according to Pliny) also local calendars (useful information when one wishes to relate local observations to, e.g., Alexandria and the Egyptian calendar), that he gave information about the length of daylight (needed to reduce local seasonal hours to equinoctial hours) and that he added general descriptions about places and people. Since the latter remark contradicts the axiom that Hipparchus would not condescend to narrative geography one is willing to postulate a new terminology according to which "*situs* denotes latitude and *visus* longitude"[6] or to translate what is not in the text "les aspects du ciel pour les différents peuples."[7] I prefer to admit my ignorance.

4. Fragments

It has been assumed[1] from a passage in the Almagest[2] that Hipparchus had published a list of his writings but this interpretation is by no means certain. Both Halma and Manitius[3] see in the words in question not more than a direct quotation by Hipparchus from a work of his own. At any rate, nowhere else do we have a trace of an autobiographical document.

The numerous references to Hipparchian weather predictions in Ptolemy's "Phaseis"[4] suggest a special treatise containing a parapegma, but Ptolemy remains our only source; not even the title is known.

The popularity of Aratus' poem has not only preserved Hipparchus' criticism but it also caused to assume his authorship for all kinds of references to constellations, usually of very little interest.[5]

A few quotations in ancient literature lead us outside of the field of astronomy. A passage from a treatise on gravitational forces is cited by Simplicius in his Commentary to Aristotle's De caelo.[6] Another small fragment concerns the theory of vision.[7] Finally we have from Plutarch statements about combinatorial logic which have, however, so far eluded a satisfactory explanation.[8]

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[5] NH II, 53; Jan-Mayhoff I, p. 143, 14f. *situs locorum et visus populorum.*

[6] Loeb I, p. 202, note b.

[7] Budé II, p. 24. Honigmann, SK, p. 72/73, note 3 obscures the situation with learned irrelevancies.

[1] Rehm in RE 8, 2, col. 1666, 46 and 1671, 25, following Heiberg in Ptol. Opera II, Index, p. 276 s.v. Ἀναγραφή.

[2] Alm. III, 1 Heiberg, p. 207, 18.

[3] Halma I, p. 164, Manitius I, p. 145, 23.

[4] Cf. below V B 8, 1 B.

[5] Cf. Maass, Comm. Ar. rel., p. 330 and Anal. Erat., p. 45–49; p. 139. Cf. also Gudeman in RE 3A, 2, col. 1879f.

[6] Comm. in Arist., Vol. VII, p. 264, 25–266, 29. Discussed, e.g., by Duhem, SM I, p. 386, p. 394.

[7] Diels, Dox., p. 404 (also Diels VS$^{(5)}$ I, p. 226, 25 or Galen, Opera XIX, p. 307 ed. Kühn).

[8] Plutarch, Moralia 732 F (Loeb IX, p. 196/7) and 1047 C, D. Cf. Biermann-Mau, J. of Symbolic Logic 23 (1958), p. 129–132; also Rome, Annales de la Soc. Sci. de Bruxelles, Sér. A 50 (1930), Mém., p. 101.

In the Greek medical literature Galen is occasionally related to Hipparchus, no doubt correctly because of his interest in the influence the seasonal variations have on the human body.[9] This relationship seems to have resulted, however, in ascribing to Galen a work "On the length of the year" which is actually a treatise by Hipparchus, cited by exactly the same title in the Almagest.[10] The Arabic medical literature mentions a "Book on the motion of the sphere"[11] which may correspond to some unknown treatises on spherical astronomy.

The fame of Hipparchus remains very much alive during the Middle Ages though the transition through Arabic distorted his name to Abrachis, or similar. As far as I know, however, the oriental tradition has not brought anything to light which we do not know from the still extant Greek sources. Steinschneider knows only of a work[12] on the "Secrets of the stars." The same, obviously apocryphal, work is known to Bar Hebraeus (≈ 1250) when he says[13] "At the time of Nebuchadnezzar, Hipparchus, the philosopher and mathematician, was also famous.... Only one book is extant today, the "Mysteries of the Luminaries," whereby is made known the renewal of the various kingdoms of the world."

In late antiquity a correspondence between Hipparchus and the Pythagorean Lysis (who lived some three centuries earlier) was manufactured,[14] taken as authentic by Copernicus.[15] Finally in the 19th century Hipparchus' astronomy was practically identified with Ptolemy's, some three centuries later. This myth still dominates the literature.

§ 7. Hipparchus' Astronomy. Summary

The commonly held assumption that we know comparatively much about Hipparchus, simply because we have the Almagest, has resulted in neglecting

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[9] Cf. also above p. 293.

[10] In one of his commentaries to Hippocrates (edited in the Corp. Med. Gr. V 9, 2, p. 333, 12–334, 14 = ed. Kühn XVII, 2, p. 240), commenting on a remark by Hippocrates that neither the year nor the lunar month amounts to an integer number of days, Galen says that in particular the question of the length of the interval between consecutive conjunctions of sun and moon requires long discussions and that "Hipparchus wrote a whole book" about it, "such as our work On the length of the year" (*Περὶ τοῦ ἐνιαυσίου χρόνου σύγγραμμα ἡμέτερον*). The same title, however, is quoted by Hipparchus himself for one of his writings (Alm. III, 1 Heiberg, p. 207, 20; cf. above p. 292, n. 3) and Galen in De crisibus III (ed. Kühn IX, p. 907, 14–16) again mentions only Hipparchus as having written "one whole book" on the length of the lunar month. All this casts doubts on Galen's authorship of a book on the length of the year and suggests a corruption of the text in the commentary on Hippocrates. Unfortunately the problem has again been obscured by a conjecture of Bergsträsser; cf. the subsequent note.

[11] Pseudo-Galen, Comm. Hippocr. De septimanis, ed. Bergsträsser, Corp. Med. Gr. XI 2, 1, p. 35 and p. 83. In the first mentioned passage (11ʳa, b) Hippocrates in quoted as claiming the authorship of a work called "Memoir on the motion (ḥaraka) of the sphere", tentatively (and surely incorrectly) identified by Bergsträsser with *Περὶ τοῦ ἐνιαυσίου χρόνου* (cf. the preceding note). Subsequently (p. 82/83: 26ʳc) a similar title "Book on the motion (ḥaraka) of the sphere" is ascribed to Hipparchus, taken by Bergsträsser (p. 203) to refer to *Ἡ τῶν συνανατολῶν πραγματεία* (cf. above p. 301, n. 1) which is at least possible.

[12] Arab. Übers. aus dem Griech., p. 226 (from ZDMG 50, 1896, p. 350).

[13] Chronogr. IV, trsl. Budge, p. 29.

[14] Cf., e.g., Diels VS[(5)] I, p. 421, No. 5.

[15] Cf. his preface to De revolutionibus (Opera II, p. 3 and p. 30 note, for Copernicus' Latin translation which he deleted from the printed edition).

even that little real evidence which with some degree of confidence can be attributed to Hipparchus himself. This whole attitude should have changed drastically after H. Vogt's brilliant study (1925) on Hipparchus' "Catalogue of Stars," where he showed on the basis of the data preserved in Hipparchus' Commentary to Aratus that Ptolemy's catalogue could not have been derived from Hipparchus' stellar positions. Yet Vogt's paper remained practically unknown and its methodological lesson unheeded. It was therefore necessary to begin from scratch the investigation of all sections of Hipparchian astronomy without conveniently supplementing the gaps in our sources from Ptolemy's writings. The complexity of this undertaking may well have seriously taxed the patience of the reader. I therefore think it may be useful to give a short summary of what I consider the essential results arrived at in the preceding chapters.

To begin again with the "Catalogue of Stars": after Vogt had shown that it cannot be reconstructed from the Almagest[1] one can make a step in the positive direction and suggest that the way in which Hipparchus describes stellar positions in his Commentary to Aratus represents the type of his own catalogue, i.e. an arrangement according to constellations and their boundaries, based on a variety of coordinates, among which the "polar latitudes" and "longitudes" are the most important ones from a historical viewpoint,[2] because they suggest connections with stereographic projection[3] and because it is this form of determination that reached India.[4] Since the Aratus Commentary is mainly concerned with simultaneous rising and settings it ignores the always visible northernmost constellations, but some Latin fragments seem to fill the gap.[5] No doubt Hipparchus was striving for high observational accuracy for whatever form of determination of celestial positions seemed in each case convenient. But a certain primitivity of the system as a whole is unmistakable[6]; in particular ecliptic coordinates are almost totally absent, excepting for Spica and Regulus because of the importance of these stars for precession.[7] Also the later classification in six stellar magnitudes seems not yet in existence.[8] The question whether the story of the Nova Scorpii is true or not[9] should be left open.

Hipparchus' most important discovery is, of course, this distinction between sidereal and tropical year.[10] He fully realized that the observations at his disposal did not suffice for a definitive determination of the constant of precession. I even think one should keep the possibility in mind that he is connected with, if not responsible for, the theory of trepidation of the vernal point with an 8° amplitude[11] since he was obviously not convinced of the constancy of the tropical year.[12]

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[1] Above I E 2, 1 B.
[2] Above p. 283.
[3] V B 3, 7 B.
[4] I E 3, 1, p. 299; VI B 1, 6.
[5] I E 2, 1 C, p. 287; I E 2, 1 C 1.
[6] E.g. the use of "zodiacal signs" for arcs in any direction (cf. I E 2, 1 A, p. 278).
[7] I E 2, 1 B, p. 283, n. 13.
[8] I E 2, 1 D; I E 6, 1, p. 330.
[9] I E 2, 1 C, p. 284f.
[10] I E 2, 2 A.
[11] I E 2, 2 C, p. 298.
[12] I E 2, 2 A, p. 294; I E 2, 2 C, p. 298.

Hipparchus' interest in the cardinal points of the solar year is, of course, also motivated by his solar model, the parameters of which were still considered valid by Ptolemy.[13] It is not certain, however, that he preferred the eccenter version for the description of the solar anomaly; in fact the discussion of a set of four lunar eclipses seems to indicate a ready shift between eccenter and epicycle models[14]; epicyclic planetary models influenced Indian astronomy.[15] Also the constancy of the radius of the lunar epicycle (or of the corresponding eccentricity) seems to have been subject to doubt. That Hipparchus seems not to have normed the radius of the deferent[16] (later $R=60$) is another indication of a rather early stage of the theory.

We know from Ptolemy that Hipparchus was well aware of the insufficiency of the simple epicyclic lunar theory but that he did not succeed in disentangling the observational discrepancies.[17] Fortunately these difficulties did not concern the syzygies, so that Hipparchus was able to develop a very advanced theory for the intervals between eclipses, lunar as well as solar. The latter involves considerations about parallax, i.e. geographical parameters considering both hemispheres.[18]

Hipparchus adopted data in his lunar theory from the Babylonian "System B" for the parameters of mean motions, a fact which was discovered by Kugler (about 1900) and placed Greek astronomy into an entirely new historical perspective.[19] This connection of Hipparchus with the basic data of the Babylonian lunar theory was confirmed by his borrowings also of planetary periods of Mesopotamian origin.[20] Nevertheless this dependence is not without its problems, one of which is the assyriological difficulty of transmitting more than simple numerical parameters from Babylonian texts to Greek astronomers. I think one can say that the Hipparchian cinematic theory of sun, moon, and planets cannot have profited much from Babylonian methods. In purely mathematical respects, however, early Greek astronomy is undoubtedly influenced by the Babylonian arithmetical methods, i.e. the use of linear zigzag functions for the description of periodic phenomena, and arithmetical sequences of different order for interpolation problems, much in the same way as the sine function was used later in India for trigonometric interpolation.[21] That also the trigonometry of Hipparchus was influenced in this general fashion by Babylonian procedures cannot be doubted.

This influence is best visible where Greek mathematics had not yet developed the proper tools to solve important astronomical questions. One such problem is the variation of the length of daylight as function of the solar longitude. Babylonian astronomy had long been in the possession of quite satisfactory

[13] I E 4.
[14] I E 5, 1 C, p. 317.
[15] Cf. below IV C 3, 8.
[16] I E 5, 1 C, p. 315.
[17] I E 5, 1 C, p. 315f.
[18] I E 5, 2 B and I B 6, 4, p. 129.
[19] I E 5, 1 A.
[20] I E 6, 1, p. 329. Also the use of Babylonian units ("cubit" of 2°) points in the same direction (cf. I E 2, 1 A, p. 279; I E 3, 2, p. 304).
[21] I E 3, 2, p. 306.

approximate solutions[22] which appear in the period of Hipparchus in a form adapted to Greek geographical knowledge.[23] In other words we have here a case where arithmetical procedures were used to give numerical answers to problems which would require spherical trigonometry for their accurate solution. Precisely this is also the way followed by Hipparchus when he had to connect terrestrial distances on a meridian with the change of longest daylight[24] and again in the determination of solstitial solar altitudes in northern regions.[25] This arithmetical approach to spherical astronomy does not exclude geometrical methods, known as "analemma"[26] which permit one to obtain mathematically accurate solutions that require only plane trigonometry. We have no explicit proof for the use of such methods by Hipparchus but they are attested in the first century B.C. (Vitruvius)[27] and their appearance in India[28] can perhaps be taken as part of Greek methodology based on Hipparchian works.

To make a special mathematical discipline out of plane "trigonometry" is, of course, nothing but a misleading anachronism based on (antiquated) modern teaching methods. In fact, to solve right triangles, or to break up general triangles into two right ones, is a technique well know since Old-Babylonian times. What constitutes real progress, however, is the decision not to solve individually every problem as it arises but to tabulate the solutions of right triangles once and for all as function of one of its angles. This idea may well be Hipparchus' though familiarity with Babylonian numerical tables could have played a role. In the construction of such tables a proper choice of the interval must be made and a uniform method of computation should be developed. Concerning the latter point we are left almost completely in the dark, excepting that right triangles were embedded in a circle, again an Old-Babylonian mathematical insight. The units, however, are most likely arcs in various ways and contexts attested in early Greek astronomy: 30°, 15°, 7;30° connected with the concept of "steps."[29] Accepting these units leads to a table of chords with steps of 7;30° for the argument. As basic radius Hipparchus chose 3438 minutes, a quantity which corresponds to a circumference of 360 degrees. The same norm is found in the Indian table of sines which has 3;45° as lowest argument, corresponding to the chord 7;30° in the Greek table.[30] For intermediate angles linear interpolation leads to sufficiently accurate approximations for chords and sines.[31]

We know practically nothing about Hipparchus' planetary theory excepting what we can conclude from the few introductory remarks by Ptolemy in the Almagest,[32] i.e. that he used epicyclic models without, however, reaching sufficient agreement with observational data.

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[22] IV D 1, 2.
[23] IV D 1, 2 A.
[24] I E 3, 2, p. 305.
[25] I E 3, 2, p. 304.
[26] I E 3, 2, p. 301 f.
[27] V B 2, 3.
[28] Cf. p. 302, n. 11.
[29] I E 3, 1.
[30] Cf. p. 299 f. and p. 1132, Table 8.
[31] V A 2, 1 D 1, p. 819, Table 7.
[32] I E 6, 1, p. 330, n. 4.

Beyond the cinematic description of planetary motion he also tried to obtain information about actual distances and sizes. We know of a comparison of the apparent diameters of planets, of the sun, and of fixed stars.[33] The equality of the apparent diameters of sun and moon, combined with the apparent diameter of the earth's shadow (obtainable from lunar eclipses) led Hipparchus to a geometrical arrangement which allows one to determine even absolute distances (reckoned in earth radii). This arrangement is well known from Almagest V, 11 and was still used for the determination of the sun's distance by Copernicus.[34] Ptolemy's result of a solar distance of about $1210\, r_e$ was for many centuries accepted as essentially correct[35]; modern scholars, however, gave Hipparchus credit for a better result by accepting the emendation $2490\, r_e$ of a solar distance $490\, r_e$ ascribed by Pappus to Hipparchus.[36] Again it took about 70 years before Pappus' report on Hipparchus' methods and results was cleansed from the accretions of modern "histoire moralisant" and the whole Hipparchian procedure in its rather unexpected form restored to its proper historical position.[37]

Hipparchus' role in geography is rather minor. He followed in basic assumptions Eratosthenes, obviously with little enthusiasm and faute de mieux.[38] His insistence on mathematical accuracy in geographical problems had, of course, little effect because it is more pedantic than useful in practice. One can compare with Hipparchus' pious hopes the methodical work and computations of al-Bīrūnī[39] in order to see how ancient geography could have looked if its practical problems had been taken seriously. The recommendation of lunar eclipses is certainly nothing that could have impressed astronomers of the hellenistic period as a great discovery. Its praise in modern times as a major contribution by Hipparchus is out of all proportion.

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[33] I E 6, 1, p. 330; also I C 8, 5, p. 261.

[34] I B 5, 4 A, p. 109.

[35] Copernicus found $1179\, r_e$ (De revol. IV, 19).

[36] I E 5, 4 A.

[37] Swerdlow [1969], against Hultsch [1900].

[38] Acceptance of the meridian Alexandria-Rhodes (cf. below p. 939) or the equivalence of 700 stades and 1° on the meridian (I E 6, 3 A).

[39] Bīrūnī, Taḥdīd (trsl. Ali), Chaps. V to XXII and Kennedy's commentary. In contrast the nonsense about the "hellenische Geist," approvingly cited by Kubitschek, RE 10, 2, col. 2058, 43–49.

Book II

Babylonian Astronomy

Quicunque autem astronomicae peritiam disciplinae affectat, hunc tractatum tota mentis intentione amplectatur.

Ezich Elkaurezmi per Athelardum Bathoniensem ex arabico sumptus (ed. Suter, p. 17/18)

Nota quod supra multa continentur quod modicum valebunt post mille annos persicos.

Entry at the end of a register of the English Nation at the University of Paris

Introduction

Delambre's "Histoire de l'astronomie moderne," published in 1821, begins as follows: Les recherches les plus exactes et les plus scrupuleuses n'ont pu jusqu'ici nous faire découvrir d'autre Astronomie que celle des Grecs. Partout nous retrouvons les idées d'Hipparque et de Ptolémée; leur Astronomie est celle des Arabes, des Persans, des Tartares, des Indiens, des Chinois, et celle des Européens jusqu'à Copernic.

Eighty years later Kugler's "Babylonische Mondrechnung" brought irrefutable evidence that Hipparchus had obtained the fundamental parameters of the lunar theory from slightly older or contemporary Babylonian sources; these discoveries revealed the existence, during the hellenistic age in Mesopotamia, of a highly sophisticated mathematical astronomy, very comparable to any one of the astronomical systems enumerated by Delambre. Continued research could only confirm and deepen Kugler's results. We now know that the Babylonian "arithmetical methods" profoundly influenced ancient astronomy and its mediaeval descendants, to mention only the sexagesimal place value system (including the use of a symbol for zero), the "tithis" of Indian astronomy, or the linear schemes in Greek astrology or geography.

Yet, Delambre's statement retains a good deal of validity and we can now understand much better why. It is the overshadowing influence of the Almagest on late ancient and Islamic astronomy that makes Delambre's dictum factually correct. Only since the cuneiform sources have become available are we in a position to recognize the common origin of methods which do not belong to the framework of the Almagest but are part of an older Mesopotamian tradition, visible only in isolated fragments.

It is not only for this reason that the student of Babylonian mathematical astronomy faces a very peculiar situation. His sources are authentic contemporary documents,[1] the majority exactly dated (or datable) and available in an abundance which, in spite of all gaps, has no parallel before a thousand years later. On the other hand these sources are very narrowly limited in several respects. They practically all come from two, accidentally found (i.e. plundered) archives, one in Babylon (of otherwise unknown locality), and one from a famous temple (the "Rēsh" sanctuary) in Uruk. With few exceptions they belong to the last three centuries B.C.; they are all, again with a few exceptions, of the same type: ephemerides for the moon and for the planets and some "procedure texts" which give the rules showing how to compute the ephemerides. Hence we are today able (optimistically speaking) to compute with Babylonian methods say a planetary position or a lunar eclipse, much in the same way as we can use the

[1] This is to be understood in a general sense. There exist texts which are copies of earlier tablets and probably all texts we do have are final copies of rough drafts. Nevertheless there will hardly ever arise problems as with mediaeval manuscript tradition.

"Handy Tables". But there is an essential difference. Thanks to the Almagest we understand exactly the reasoning which underlies the methods of the Handy Tables. For the cuneiform ephemerides we can penetrate the astronomical significance of the individual steps, as one may expect with any sufficiently complex mathematical structure. But we have practically no concept of the arguments, mathematical as well as astronomical, which guided the inventors of these procedures. This historical problem is still more involved because we have, beside the mathematical-astronomical material, equally extensive records in which many predicted data are embedded without these predictions being based on the contemporary mathematical methods. Hence we are very far from any "history" of Babylonian astronomy and must be satisfied to accept it as a completed system of admirable elegance and efficiency but without really understanding its development. In short we probably know as much and as little about Babylonian astronomy as a Greek astronomer of the hellenistic age knew.

For the modern historian of "ideas" Babylonian astronomy is no pleasant hunting ground since one cannot grasp the underlying concepts without a full command of the numerical details which are the backbone of the whole system. And for the purely historical question how such details could reach astronomers in Alexandria or Ujjain he will find no answer since our sources fail us here completely. Hence it is a rather self-contained problem of great technical complexity that will occupy us in the following chapters. But it will reveal to us the working of a theoretical astronomy which operates without any model of a spherical universe, without circular motions and all the other concepts which seemed "a priori" necessary for the investigation of celestial phenomena. The influence of the available mathematical tools on the development of a scientific theory is perhaps far greater than we customarily admit.

§ 1. The Decipherment of the Astronomical Texts

All serious studies of Babylonian astronomy are founded on the work of three Jesuit fathers, Johann Nepomuk Strassmaier (1846–1920), Joseph Epping (1835–1894), and Franz Xaver Kugler (1862–1929).[1] Strassmaier was an orientalist and one of the leading pioneers in assyriological studies. From 1878 to 1881 and again from 1884 to 1897, when he fell seriously ill, he worked in the British Museum, relentlessly striving to bring order into the colossal amount of texts and to make them generally available by publishing copies. During this work which was to a great extent centered on Neo-Babylonian material he ran into astronomical texts, recognizable as such by the large amount of numerical material combined with names of months and often dated in the Seleucid or Arsacid era but otherwise totally incomprehensible to Strassmaier. He implored the help of Epping who had been professor of mathematics and astronomy in Maria Laach,[2] then (from 1872–1876) in Quito, Ecuador, and finally in Holland. Epping

[1] For biographies cf. Deimel [1920], Baumgartner [1894], and Schaumberger in Kugler-Schaumberger [1933], p. 97–100, respectively.

[2] West of the Rhine, near Koblenz.

reluctantly agreed to embark on these investigations, all new to him. But one year later he published in the "Stimmen aus Maria Laach," a periodical concerned mostly with catholic theology, a short paper of 11 pages[3] which is a masterpiece of a systematic analysis of numerical data of unknown significance. He succeeded in understanding the concluding columns of a lunar ephemeris[4] and from an observational text[5] he identified correctly the ideograms for the planets and thus was able to give the exact julian equivalents of dates in the Seleucid era (he found S.E. 189 Nisan $1 = -122$ March 25), in itself a result of great interest for the chronology of hellenistic history.

Epping fully realized the significance of his discoveries. The two columns from a lunar ephemeris which he had deciphered, he said, "give us more information about Babylonian science than all the notices from classical antiquity combined"[6] – a fact which cannot be emphasized too often. And he foresaw clearly that the new material would become of great importance for ancient chronology, for Assyriology in general, and even for modern astronomy.[7]

With the basic principles of Babylonian methods understood, Epping was now able to penetrate deeper and deeper into the texts furnished him by Strassmaier.[8] The results were finally presented in a small but brilliant book "Astronomisches aus Babylon" (1889) of which Kugler's "Babylonische Mondrechnung" (1900) and "Sternkunde" (1907 to 1924, with Schaumberger's supplementary volume of 1935) are the direct continuation. With these works the pioneering period in the decipherment of Babylonian astronomy can be considered to be closed.

It is not my intention to describe here the subsequent history of the study of Babylonian astronomy. Not only would this require a great deal of space and explanations of assyriological details but also a description of the absurd so-called "Bibel-Babel" controversy as well as the equally absurd "Panbabylonistic" doctrines. Miraculously the first world war made these disputes vanish from the scene as if they had never existed.

In the meantime progress was made in the analysis of Kugler's texts, and in particular, through the discovery of the Dutch astronomer Pannekoek that the planetary ephemerides were based on mean synodic months as unit of time, divided in 30 parts[9] – called "*tithi*" in Indian astronomy.

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[3] Epping-Strassmaier [1881]; cf. also Epping [1890].

[4] Now part of ACT No. 122 (rev. X/XI lines 2 to 14 from SH 81-7-6, 277 in Fig. 1, p. 1315). Cf. also Pl. IV.

[5] Sp. 129 (=BM 34033); later published in Epping AB, pl. 1-3.

[6] Epping-Strassmaier [1881], p. 285. The same could be said, 50 years later, about Babylonian mathematics.

[7] Epping-Strassmaier [1881], p. 291 f.

[8] Pl. II shows a page from Strassmaier's notebooks, Pl. III gives one of his copies redrawn for the use of Epping and with later notes by Kugler; cf. now ACT No. 5 (Vol. I, p. 90 and photo Vol. III, pl. 255). The copies in the notebooks contain remarks in an oldfashioned shorthand ("Gabelsberger" which I had fortunately learnt in school) explaining the condition of the tablet and occasionally other remarks, e.g. at Sp. II, 604 (Pl. III left) "ein dickes Fragment einer ganz dicken Tafel, so: ..." and "für P. Epping kopiert 1/3 93." Sp. II, 604 is now joined with Sp. II, 453 in ACT, No. 70; cf. Pinches' copy in Pinches-Sachs, LBAT, No. 51, p. [13].

[9] Pannekoek [1916], p. 689, rediscovered by van der Waerden [1941], p. 28 (note 10); neither one realized the relation to Indian astronomy. Cf. Neugebauer [1947], p. 146. Cf. below p. 358.

Of major importance, however, was the appearance of new texts. The Strassmaier material from the British Museum came, as we know now (mainly through the work of A. Sachs in the British Museum in 1953/54) from Babylon. German excavations in Uruk (1912/13), however, apparently had reached an archive and a great number of well-preserved tablets (astronomical and others) came through antiquity dealers to the museums in Paris and Berlin. The French acquisitions were published in excellent copies by Thureau-Dangin (1922)[10] whereas several astronomical texts from Berlin[11] were transcribed and discussed in a book by Schnabel on Berossus (1923),[12] unfortunately with countless errors, incomplete, and marred by a chaos of hypotheses and misinterpretations.

Except for the tablets from Uruk, sold by dealers, the majority of texts are in a deplorable state of preservation, broken in many pieces, usually without any indication of a connection between scattered fragments. As an example see Pl. IV, showing a lunar ephemeris,[13] now almost completely restored from nine fragments (cf. Fig. 1, p. 1315), two of which played a crucial role in the decipherment by Epping.[14] The process of rebuilding this text took some 75 years, beginning with Strassmaier's first copies in 1879 and through successive joins made by him, Schaumberger, and myself. Another example for the condition of our source material is shown in Pl. V which is the reproduction of a photograph made by the German expedition in Uruk – now the only extant record for these fragments. Several of these splinters from tablets belong to ephemerides and could be dated, disclosing that they were fragments from larger tablets which had come into the hands of dealers. Hence also the provenance of the larger tablets is now accurately known.

To establish connections between fragments ("*joins*" in the professional jargon) is by no means only a matter of good luck or of paying close attention to shape, writing, color, etc. as in a puzzle. Ephemerides are based on columns of numbers which are computed following definite arithmetical rules (e.g. difference sequences). Consequently it is possible to develop purely mathematical criteria which must be satisfied if the numbers in a certain column of one fragment are to be the numerical continuation of the number in another fragment. If such a condition is violated it is clear that the two fragments are independent. If, however, an arithmetical connection is possible one can establish the number of steps (i.e. lines) required to reach one section from the other.[15] Hence one knows exactly the relative position of the two fragments and the question of an actual join can be tested. In this way it was possible to repair many of the fragmentary pieces and to bring secure chronological order into a large group of texts. On this method is based the edition of all previously known mathematical astronomical texts of the Seleucid period[16] which I published in 1955 as "Astronomical Cuneiform Texts" (henceforth quoted as ACT). The material included

[10] Thureau-Dangin, TU.
[11] Signature VAT, i.e. "Vorderasiatische Tontafel".
[12] Schnabel, Ber.
[13] ACT, No. 122.
[14] Epping [1890], "Tafel A" and Epping AB, "Tablet A" and "Tablet C" without realizing the join; Kugler BMR, p. 12/13.
[15] Cf. Neugebauer [1936], [1937, 1 to 3].
[16] "*Seleucid*" should always mean Seleucid *and* Parthian (if necessary even Roman).

in this edition does not only comprise the previously known Strassmaier-Kugler texts and the published texts from Uruk but many hitherto unknown tablets. In part this was made possible only through the unpublished notebooks of Strassmaier[17] which were put at my disposal by the Istituto Pontificio Biblico in Rome on the recommendation of P. A. Deimel. These notebooks provided us with the accession numbers of texts which were otherwise kept inaccessible in the British Museum. Finally the obstructive rules for the use of the collection of cuneiform texts were slowly relaxed and A. Sachs was allowed (in 1953/54) to conduct a systematic search for all types of astronomical texts. He was even given access to notebooks filled with excellent copies made during the years 1895 to 1900 by T. G. Pinches and since then kept strictly secret in the British Museum.[18] Only through his patient efforts and thanks to the labors of Strassmaier and Pinches do we now have a collection of material which seems to be fairly representative for the astronomy of Seleucid-Parthian Mesopotamia.

§ 2. The Sources

The following discussion of the "mathematical-astronomical" cuneiform texts is based on about 300 tablets, each one usually rebuilt from several fragments. They comprise ephemerides in the modern sense of the word, i.e. day-by-day positions, or in an extended meaning of the term, month-by-month positions, or other regular sequences of events. More than half of our material concerns the moon (and, of course, the sun because of the syzygies and eclipses), the rest, the five planets. To these "*ephemerides*" belong also certain "*auxiliary texts*" which tabulate numerical sequences needed in the computation of the ephemerides. Finally we have the "*procedure texts*" which give the rules for computing ephemerides, for us often very difficult to understand. For brevity, I shall refer to these classes of texts as "*ACT-texts*" even if they are not included in the ACT publication (i.e. identified after 1955).

Beside this group we have a huge mass (some 1400 tablets) of astronomical texts which are not concerned with the computation of ephemerides. Many of these texts have been studied already by Epping and Kugler but it was not until 1950 that a precise classification of this diverse material was achieved by A. Sachs.[1] The major types, in the terminology of Sachs, are the "*Goal Year Texts*" (designed for the prediction of lunar and planetary phenomena based on certain fundamental periods), the "*Almanacs*" (yearly lists of lunar and planetary phenomena, solstices and equinoxes, etc.), the "*Diaries*" (which concern astronomical and historical events in chronological order, month by month in a given year), and "*Excerpts*" from the above-mentioned groups. In order to refer to this large group of non-ACT astronomical texts I shall use the abbreviation *GADEx*. Outside the ACT and the GADEx texts one still has several documents which concern specific planets, eclipses, etc., but without constituting a uniform class of

[17] Cf. Pl. II, p. 1449.
[18] Now published in Pinches-Sachs LBAT (1955).
[1] Sachs [1948] and Pinches-Sachs LBAT.

sources. All these texts will not concern us in any detail before the third section[2] when we shall try to obtain some information about the historical development of Babylonian astronomy. While the ACT-texts form a very uniform group which belongs to the time from about -300 to about $+50$ the non-ACT-texts reach back to the sixth century B.C., though their overwhelming majority again belongs to the last three centuries B.C. with the exception of eclipse texts which are more uniformly distributed over the whole time beginning in the middle of the 8th century B.C., in perfect agreement with Ptolemy's statement that the ancient observations were almost completely (ἐπίπαν) preserved from the time of Nabonassar on.[3]

Today the texts discussed here are scattered over many museums the world over. Almost all texts from Babylon are in the British Museum, and a few are in Berlin or in the Metropolitan Museum, New York. The material from Uruk is now in Istanbul, Berlin, Paris (Louvre), Chicago (Oriental Institute), and in Baghdad (inaccessible in the Iraq Museum). None of these museums possess a catalogue of their tablets which may number many thousands, if not hundreds of thousands (as in the British Museum). Only a small fraction of such collections has ever been published and every excavation produces new texts, often in vast quantities, which will be buried in the cellars of museums under much more adverse conditions than in the ruins.

Only in a few exceptional cases do we have a record of the place of discovery of our texts. As mentioned before at least the texts from Uruk are relatively accurately localized.[4] Otherwise one has to rely on secondary criteria which are peculiar to a certain scribal school, as, e.g., in the Uruk texts the use of a "zero" sign[5] before units which follow a ten in a higher order (as in 20,.5 as distinct from 25). Of course the safest criterium is a "colophon" at the end of a tablet, giving date, locality, and name of the scribe.[6] But it was just the misreading of such a colophon on a Babylon tablet[7] that caused the often repeated assertion that a group of our texts belonged to an archive in Sippar,[8] thus seemingly supporting a passage in Pliny where he speaks about three astronomical schools in Mesopotamia: Babylon, Hipparenum = Sippar (?), and Uruk.[9] We have no reason to doubt this but certainly at present no text from Sippar has reached us.

The texts from Babylon, now in the British Museum, were all purchased in the 19th century from native diggers of the Baghdad area. Before such texts even got an accession number years might elapse as one can establish through Strassmaier's notebooks.[10] These accession numbers were arranged according to "collections," that is according to purchasers or shipments that reached London. Some texts are marked, e.g., as "Rm" which means acquired through Hormuzd

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[2] Below II C 3.
[3] Alm. III, (Heiberg, p. 254, 10–13). Cf. also above p. 74; p. 118.
[4] Cf. above p. 350.
[5] Originally a separation sign, here transcribed by a period. Cf. Neugebauer [1941] and ACT, p. 4, p. 511; Aaboe-Sachs [1966], p. 3; Neugebauer-Sachs [1967], p. 210/12.
[6] Cf. for colophons ACT I, p. 11 to 24.
[7] On the lower edge of ACT, No. 122, shown Pl. IV; cf. also the copy ACT III, Pl. 221.
[8] The error was detected by A. Sachs; cf. ACT I, p. 5, note 14 and Sachs [1948], p. 272, note 3.
[9] Cf. ACT I, p. 5.
[10] Cf., e.g., Epping-Strassmaier [1881], p. 283, note 1.

Rassam who was for many years British consul in Mosul (he also conducted digs in the ruins of that area during the years 1877 to 1883). S† indicates George Smith whom the Daily Telegraph (hence also "DT"-texts) sent out to find more fragments of the creation epic and the story of the Deluge; he died in Aleppo from exhaustion on the way back from his third expedition (1876). Other tablets got their signatures in commemoration of dealers, so SH from Shemtob and Sp from Spartali. Finally a four-parametric number like 81-6-25,246 conveys the important information that this tablet was the 246th that got its number on the 25th of June, 1881. Future archaeologists may determine from omitted dates on cuneiform tablets in the British Museum the Sundays of the 19th century; for the present the only useful element in this elaborate system is a possibility to reconstruct (by long experience and careful statistics) a certain indication of the origin of a text. Even this modest chance has been eliminated in recent years by renumbering the majority of texts by five-digit numbers (BM) which refer to drawers and boxes in which the texts are kept.[11]

§ 3. Calendaric Concepts

The concept "month" obviously takes its origin from the phases of the moon. It is therefore not surprising to find months reckoned, e.g., from first visibility to first visibility (as in Mesopotamia), or from last visibility to last visibility (as in Egypt[1]). In a progressing civilization, however, such a naive definition of one of the most important calendaric units becomes increasingly inconvenient since it requires continued observation of a phenomenon which varies in a complicated fashion, unfit for a prediction by simply counting days. For this reason schematic arrangements were developed as substitutes for the "true" lunar months.

The simplest procedure consists in the introduction of "months" of constant length, thus abolishing completely any consideration of the actual moon. Such are the months of the Egyptian civil calendar, all 30 days long.[2] Or, remaining nearer to reality, one can operate with alternating "full" and "hollow" months (of 30 and 29 days, respectively) with some periodic corrections. This is the method adopted by the schematic Islamic calendar in which 11 months in 30 years are changed from hollow to full.[3] But a radically different approach is found in the late Babylonian period where one made an attempt to detect the causes for the variability of the length of the synodic months and to predict as accurately as possible the days of first visibility. This is the origin of the lunar theory known to us from cuneiform texts, to be discussed at length in section II B.

Another obviously related problem concerns the correlation between months and years, whatever these terms may mean. In Babylonia the occasional addition of a 13th month kept the twelve ordinary lunar months in a fairly stable relation

[11] Exceptions are the Kuyundjik (= Nineveh) Collection (K), DT, and Rm. Concordances for the astronomical texts are given in Pinches-Sachs, LBAT, p. XXXIX ff.

[1] Also full moons can be used, as in India.

[2] Cf. below III 1 and VI A 2, 1.

[3] The resulting "calendaric mean synodic month" is $29;31,50^{d}$ long; cf. below p. 548.

to the agricultural year, characterized by the date harvest in the fall and by the barley harvest in the spring. Only in the latest phase of Mesopotamian history was the relation between months and years definitively regulated by the adoption of a 19-year cycle.

A different and more radical solution consists in ignoring the seasonal (or "solar") year altogether by calling "year" every sequence of 12 consecutive months (technically known as "lunar year') which then rotates through all seasons, accumulating in about 33 years a deficit of about one solar year. Such are the "years" of the Muslim calendar and probably also of the Assyrian calendar at some time in the second millenium B.C.[4]

Most convenient of all arrangements is, of course, the Egyptian "year" of 12 "months" and 5 epagomenal days – much easier to handle than its "julian" modification with intercalations or even with months of unequal lengths.

But beside the simple civil calendar vestiges of the primitive true lunar calendar are preserved in the calendar of religious festivals in Egypt.[5] Similarly the Christian ecclesiastical calendar maintains to the present day in its Easter computus the basic principle of the Babylonian cyclic lunar calendar.

1. The 19-Year Cycle

The history of the 19-year cycle is still much in the dark; it is only certain that it is a comparatively late invention. Nabonid (−555 to −538) still decreed individual intercalations[1] and the same situation exists under Cyrus and Cambyses, i.e. at least until −525.

We have no evidence of tentative cyclic intercalations between this period and the appearance of the 19-year cycle. Attempts to demonstrate a short-lived existence of an octaeteris[2] (as in Greece[3]) were not substantiated by increasing source material.

The only way open to ascertain the method of intercalation consists in collecting dated documents which mention explicitly an intercalary month – either, in autumn, a second Ulul (VI_2) or, in spring, a second Adar (XII_2) – and occasionally a longer sequence of months that establishes a year as ordinary.[4] In this way we know that a regular 19-year cycle of intercalations existed since −380/79 (reign of Artaxerxes II). Four years earlier, instead of the expected three years earlier, i.e. in −384/3, one finds an intercalary XII_2. Except for this

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[4] Weidner [1935], p. 28/29 has shown that in the middle of the 12th cent. B.C. the same Assyrian month could coincide with seven different Babylonian months. This implies a rapidly shifting Assyrian calendar with respect to the essentially stable Babylonian calendar.

[5] Cf. Parker, Calendars.

[1] Parker-Dubberstein, BC, p. 1ff.

[2] Kugler, Sternk. II, p. 422ff. and van der Waerden AA, p. 112 on the basis of four VI_2 intercalations in similar positions within the 25 years from B.C. 527 to 503. Nobody takes it as evidence, however, for a cycle when four VI_2 agree in the 16 years from B.C. 614 to 599.

[3] Cf. below p. 620.

[4] Such are the sources of the tabulation in Parker-Dubberstein, BC, p. 6, on which the following discussion is based ($U = VI_2$, $A = XII_2$).

isolated (but well attested) violation of the rule the ordinary procedure can be followed back to $-424/3$ (Artaxerxes I). One could go even to $-497/6$[5] (Darius I) if one were to allow a year to intercalate a VI_2 instead of a XII_2 required by the final pattern.[6] Before -497 the intercalations seem definitely irregular. Hence the invention of the 19-year cycle, or at least its first practical use, seems to belong to the century between -500 and -400. Meton is said to have inaugurated the same cycle in Athens in -431.[7]

If we now discuss some arithmetical features of the 19-year cycle it must be kept in mind that from a purely formal viewpoint no astronomical concepts are involved. In particular the question of the accuracy of the cycle remains completely outside the present investigation.[8] At the moment the 19-year cycle is only defined as a relation between one unit, called "year" (y), and another unit, called "month" (m), such that always

$$19^{y} = 12 \cdot 19 + 7^{m} = 3{,}55^{m}. \tag{1}$$

It follows from this definition that

$$1^{y} = \frac{3{,}55}{19} \approx 12;22{,}6{,}19^{m}. \tag{2}$$

This approximation is very good since

$$12;22{,}6{,}19 \cdot 19 = 3{,}55;0{,}0{,}1.$$

If we accept for a moment that $1^{m} \approx 30^{d}$ and $1^{d} = 24^{h}$ we see that

$$12;22{,}6{,}19 \cdot 19 \approx 3{,}55^{m} + 0;0{,}0{,}30^{d} = 3{,}55^{m} + 0;0{,}12^{h}. \tag{2a}$$

In other words, if we define with (2) that

$$1^{y} = 12;22{,}6{,}19^{m} \tag{2b}$$

we deviate from the definition (1) only by about 12 seconds each cycle.

We can go further and replace (2) by the round number

$$1^{y} = 12;22{,}6{,}20^{m} \tag{3}$$

which means the same as replacing (1) by

$$19^{y} = 3{,}55^{m} + 0;0{,}0{,}20^{m} \approx 3{,}55^{m} + 0;4^{h}. \tag{3a}$$

Hence we commit an error of $0;0{,}0{,}20^{m} \approx 4$ minutes in 19 years if we consider (3) as the equivalent of the definition (1) of the 19-year cycle. If we again estimate that $1^{d} \approx 0;2^{m}$ we see that it takes 6,0 cycles or $360 \cdot 19 = 6840^{y}$ until (3) has accumulated an error of 1 day in comparison with (1).

For the arrangement of a civil calendar with 7 intercalary months in 19 years, according to (1), an error of 1 day in about 7000 years is of no significance. We

[5] Parker-Dubberstein, BC, p. 6, cycle 14 year III. One should realize, however, that references to early intercalation may be untrustworthy. Aaboe-Sachs [1969], p. 21, n. 16 observed that two texts, both concerned with eclipses, assign the year Xerxes 18 ($-467/6$) a VI_2 and a XII_2, respectively.

[6] The years in question are $-445/4$ and $-426/5$; Parker-Dubberstein BC, p. 6, year XVII in cycles 16 and 17. Cf. also below p. 364.

[7] Diodorus XII, 36; cf. below p. 622.

[8] This problem will be of interest, of course, in a later context (cf. below p. 542).

Table 1

Cycle	−2		−1		0		1		Cycle
Year	S.E.	jul.	S.E.	jul.	S.E.	jul.	S.E.	jul.	Year
1	−38	−349/8	−19	−330/29	0	−311/10	19	−292/1	1
2	−37*	−348/7	−18*	−329/8	1*	−310/9	20*	−291/90	2
3	−36	−347/6	−17	−328/7	2	−309/8	21	−290/89	3
4	−35	−346/5	−16	−327/6	3	−308/7	22	−289/8	4
5	−34*	−345/4	−15*	−326/5	4*	−307/6	23*	−288/7	5
6	−33	−344/3	−14	−325/4	5	−306/5	24	−287/6	6
7	−32	−343/2	−13	−324/3	6	−305/4	25	−286/5	7
8	−31*	−342/1	−12*	−323/2	7*	−304/3	26*	−285/4	8
9	−30	−341/40	−11	−322/1	8	−303/2	27	−284/3	9
10	−29*	−340/39	−10*	−321/20	9*	−302/1	28*	−283/2	10
11	−28	−339/8	−9	−320/19	10	−301/300	29	−282/1	11
12	−27	−338/7	−8	−319/8	11	−300/299	30	−281/80	12
13	−26*	−337/6	−7*	−318/7	12*	−299/8	31*	−280/79	13
14	−25	−336/5	−6	−317/6	13	−298/7	32	−279/8	14
15	−24	−335/4	−5	−316/5	14	−297/6	33	−278/7	15
16	−23*	−334/3	−4*	−315/4	15*	−296/5	34*	−277/6	16
17	−22	−333/2	−3	−314/3	16	−295/4	35	−276/5	17
18	−21	−332/1	−2	−313/2	17	−294/3	36	−275/4	18
19	−20**	−331/30	−1**	−312/1	18**	−293/2	37**	−274/3	19

therefore may consider (1) and (3) as equivalent definitions of the 19-year cycle. In the following we will make use of the form (3) or (1) as convenience dictates.

The distribution of the seven intercalary months in each cycle is, of course, only a matter of convention. It is natural to space the intercalations roughly at even distance, i.e. about 3 years apart. By long tradition, probably motivated by agricultural considerations, intercalations were only made in spring (XII_2) and autumn (VI_2). The 19-year cycle, as it is known from −380 on,[9] makes the following years of the Seleucid era intercalary:[10]

$$\text{S.E. } 1^* \quad 4^* \quad 7^* \quad 9^* \quad 12^* \quad 15^* \quad 18^{**} \tag{4}$$

where

$$* = XII_2 \quad ** = VI_2. \tag{4a}$$

All other years are ordinary and the rule (4) applies to all years which are congruent modulo 19 to the sequence in (4). All other years are ordinary years.

If we arbitrarily define the years S.E. 0 to 18 as "cycle 0" and count S.E. 0 as year 1 of this cycle, then we can construct a table for as many cycles as we want (cf. Table 1). As a convenient notation we introduce[11]

$$\text{cycle } m, \text{ year } n = [m:n]. \tag{5}$$

Example: S.E. $4^* = -307/6 = [0:5]$ and $[2:14] = \text{S.E. } 51 = -260/59$.

[9] Cf. above p. 354.

[10] This pattern was first discovered by Kugler, Sternk. I, p. 212 (1907); cf. also Sternk. II, p. 425.

[11] If we denote Parker-Dubberstein BC, p. 6 cycle p, year q by $(p:q)$ then we have with (5)

$$(p:q) = [p-24:q+2] \quad \text{or} \quad [m:n] = (m+24:n-2).$$

We do not know why only S.E. 18 was made a **-year and not also S.E. 10. The latter norm would have resulted in a more regular distribution of intercalary months in groups of either 37 or 31 months length, instead of with one interval as short as 25 months (from S.E. 8 I to S.E. 9 XII_2). We shall see presently[12] that another question is raised by this peculiarity in the intercalation pattern.

2. Solstices and Equinoxes

Among the many fragments of tablets from Uruk now in the Arkeoloji Müzeleri in Istanbul two small pieces joined and gave us a text just large enough to reveal the existence of an arithmetical scheme by means of which for some three centuries dates were assigned to solstices and equinoxes.[1] The existence of such a schematic determination of the cardinal points in the solar year – henceforth simply called "the *Uruk scheme*" – is not only in itself of great historical interest but it also provided a powerful tool in the dating of fragmentary texts from the GADEx group since these texts normally contain dates of solstices and equinoxes.[2] The requirements of a definite arithmetical pattern usually reduce the possibilities for the date of a fragment to a few cases among which other astronomical events mentioned in such a text may lead to a unique solution, in spite of their frequency (e.g. lunar positions).

Arithmetical Structure of the Uruk Scheme

It suffices for our present purposes to transcribe the following lines (5 to 10) of the Uruk Fragment:

5.	2]8	IV	3,50,50 a	
6.	]29	III	14,54	
7.	]30	III	25,57,10	(A)
8.	]31	IV	7,.,20 kin-a	
9.	]32	III	18,3,30	
10.	]3[3]	III	29,6,40 a	

The ideograms "a" and "kin-a" are the standard signs for intercalary years, "a" indicating the intercalation of a XII_2, "kin-a" of a VI_2. The lines 5, 8, and 10 correspond exactly to the pattern of the years S.E. 15*, 18**, and 20*.[3] This suggests the interpretation of the fragmentary numbers in the first column as year numbers in the Seleucid era, followed by calendar months and days with fractions in the second and third column.

To test these interpretations we use first the **-year $x+31$ (in line 8) which must be $k \cdot 19$ years distant from S.E. 18**. Since year numbers may be written

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[12] Below p. 365.

[1] Neugebauer [1947, 1] and [1948]; now ACT No. 199 (and Pl. 136) from Istanbul U 107 and U 124.

[2] Cf. below p. 542.

[3] Cf. above p. 356, Table 1, years [0:16], [0:19], and [1:2], respectively.

either sexagesimally or decimally we have

$$x=\begin{cases} a\cdot 60 & 1\leqq a\leqq 6 \\ b\cdot 100 & 1\leqq b\leqq 3 \end{cases}$$

where the integers a and b are bounded by obvious historical reasons. By direct check of

$$x+31=18+k\cdot 19 \quad \text{or} \quad x+13=k\cdot 19$$

one finds that these equations have only the solution $a=2$, $k=7$. Hence the years in (A) run from S.E. 2,28 to 2,33 and, in particular, 2,31 is the **-year [7:19].

The second column poses no problem since III and IV represent the usual month names of the Babylonian calendar.

The subsequent numbers form a sequence of constant difference $\delta=11{,}3{,}10$, reckoned modulo 30,0,0. A modul of 30 days in a lunar calendar would rapidly accumulate an intolerable error. Hence "30" must represent a mean lunar month, whatever its length may be in ordinary days. This fact leads to an important new concept. We introduce schematic months of constant length (hence called "mean lunar months") which we divide into 30 parts, also of equal length. We call such a part (with a Sanskrit term) "*tithi*" and denote it by τ. Hence we define

$$1^{\mathrm{m}}=30^{\tau}. \tag{1}$$

Consequently we write now line 5 as

$$\text{S.E. } 2{,}28^{*} \quad \text{IV} \quad 3;50{,}50^{\tau}$$

and the difference from line to line as

$$\Delta=12^{\mathrm{m}}+11;3{,}10^{\tau}. \tag{2}$$

Adding this Δ to line 5 and taking into consideration that the year 2,28 contains 13 months we obtain for line 6

$$\text{S.E. } 2{,}29 \quad \text{III} \quad 14;54{,}0^{\tau}$$

in agreement with the text. Similarly line 7 will be S.E. 2,30 III $25;57{,}10^{\tau}$ and line 8 moves into month IV according to (1):

$$\text{S.E. } 2{,}31 \quad \text{IV} \quad 7;0{,}20^{\tau}.$$

The insertion of a month VI_2 in S.E. 2,31 produces in the next line again a month III, and so forth.

The definition (1) of the tithis can also be written as

$$1^{\tau}=0;2^{\mathrm{m}} \tag{3}$$

and hence the constant difference Δ becomes

$$\Delta=12;22{,}6{,}20^{\mathrm{m}}. \tag{4}$$

This is exactly the same number which we have found in the preceding section (p. 355) as being characteristic for the 19-year cycle

$$1^{\mathrm{y}}=12;22{,}6{,}20^{\mathrm{m}}. \tag{5}$$

Hence we can say: also the "Uruk scheme" (A) can be used as a definition of the 19-year cycle, including the rule (4), p. 356 for the arrangement of the intercalary years.

As we have seen earlier (p. 355) the relation (5), i.e. the Uruk scheme, is not exactly the same as the original definition (1), p. 355 of the 19-year cycle since it follows from (5) that[4]

$$19^y = 3{,}55^m + 0;0{,}10^\tau. \qquad (6)$$

Our fragment (A) belongs, in the terminology of p. 356, to cycle 7. Hence we have because of (6) the dates

[7:19]	S.E.	151**	IV	7;0,20$^\tau$
[6:19]		132**	IV	7;0,10
[5:19]		113**	IV	7;0,0.

Since each cycle modifies the tithis only by $0;0{,}10^\tau$ it is clear that [5:19] is the only instance in the whole scheme where we find an integer number of tithis in the date. Because [5:19] = [6:0] we can now compute all dates for cycle 6, year 0 (= S.E. 113**) to year 19 (= S.E. 132**):

S.E.	113**	IV	7; 0, 0$^\tau$	S.E.	123*	III	27;31,40$^\tau$
	114	III	18; 3,10		124	III	8;34,50
	115*	III	29; 6,20		125	III	19;38, 0
	116	III	10; 9,30		126*	III	30;41,10
	117	III	21;12,40		127	III	11;44,20
	118*	IV	2;15,50		128	III	22;47,30
	119	III	13;19, 0		129*	IV	3;50,40
	120	III	24;22,10		130	III	14;53,50
	121*	IV	5;25,20		131	III	25;57, 0
	122	III	16;28,30		132**	IV	7; 0,10.

(B)

All cycles can be derived from (B) by adding $0;0{,}10^\tau$ in each line for each subsequent cycle.

Purpose of the Uruk Scheme

We still have to answer the question: what is the astronomical significance of the dates in (B)? Since they remain almost always in month III and since month I is known to be near the vernal equinox it is clear that (B) must be near the summer solstice. This can be easily checked since one knows, e.g., that the lunar month S.E. 113 IV begins about −198 June 22[5]; hence S.E. 113 IV 7 ≈ −198 June 28. On that day the sun had a longitude of ≈91°,[6] hence our first date in (B) is indeed very near the summer solstice. If the length of the "year," used in (5), is reasonably accurate then it follows from

$$\Delta = 12^m + 11;3{,}10^\tau = 1^y \qquad (7)$$

[4] Cf. p. 355 (3a).

[5] Using Parker-Dubberstein BC, p. 40 (based on modern computations of first visibility of the moon).

[6] Tuckerman, Tables I. Our discussion is only intended to produce an estimate for the significance of our dates but not an evaluation of their accuracy.

that all dates in the Uruk scheme represent summer solstices. Finally the dates of solstices in GADEx texts explicitly confirm the dates in (B) as dates of solstices.[7] The term

$$e = 11;3,10^{\tau} \tag{8}$$

is called the "epact" of the 19-year cycle.

It remains to be noted that the concept of "tithi" as unit of time is by no means restricted to the computation of solstices. It will be encountered again in the planetary theory and in the lunar theory itself.[8] The same units play an important role in Indian astronomy to the present day.[9] It can be hardly doubted that we have a case here of far reaching influence of Babylonian astronomy on the astronomy of the Middle Ages.

The practical advantage of introducing such a unit is evident from its use discussed so far. Since calendaric lunar months are either 29 or 30 days long a tithi will be nearly the same as an ordinary day. Hence, if we ignore fractions and call the integer parts of the tithis simply "days" within a lunar month we will always keep near to the corresponding calendar dates, but avoid the determination for each month of its actual length. In other words the tithis help to eliminate the painful consequences of a strictly lunar calendar.

Abbreviated Scheme

GADEx texts as well as some special astronomical texts[1] contain dates of solstices and equinoxes. The summer solstice dates always agree numerically with the Uruk scheme in so far as all fractions of tithis are simply ignored (e.g. even $0;59^{\tau}$) while the integer parts are taken as days of the lunar calendar. Hence in practice the scheme (B) of p. 359 will result in the following calendar dates of summer solstices:

S.E. 113**	IV	7		etc.		
114	III	18	130	III	14	
115*	III	29	131	III	25	(1)
116	III	10	132**	IV	7.	
etc.						

Since corresponding years in consecutive cycles differ only by $0;0,10^{\tau}$ in historical times this difference will not affect the day numbers given in (1) with the exception only of the **-years before and after S.E. 113**. From this year on the date in all **-years will be IV 7 (originating from an unabbreviated $7;0,0^{\tau}$ $7;0,10^{\tau}$ etc.) while before it (ending with S.E. 94**) all dates should be only IV 6 (originating from ..., $6;59,40^{\tau}$ $6;59,50^{\tau}$). But all non **-years will show the same solstice date in each cycle, obtainable from the day number in cycle-year 0 by adding 11 from year to year. Table 2 displays this simple pattern with its only apparent singularity at S.E. 113**.

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[7] For the accurate meaning of this statement cf. the next section (p. 360f.).
[8] Below p. 395ff., p. 500, etc.
[9] Cf., e.g., Sûryasiddhânta I, 13.
[1] Of non-ACT type, e.g. a text for Mercury (cf. Neugebauer [1948, 1], p. 212f.).

Table 2

Cycle	3	4	5	6	7	Summer Solstice
Y. 0		S.E.		113**	132**	IV 7^r
1	57	76	95	114	133	III 18
2	58*	77*	96*	115*	134*	III 29
3	59	78	97	116	135	III 10
4	60	79	98	117	136	III 21
5	61*	80*	99*	118*	137*	IV 2
6	62	81	100	119	138	III 13
7	63	82	101	120	139	III 24
8	64*	83*	102*	121*	140*	IV 5
9	65	84	103	122	141	III 16
10	66*	85*	104*	123*	142*	III 27
11	67	86	105	124	143	III 8
12	68	87	106	125	144	III 19
13	69*	88*	107*	126*	145*	III 30
14	70	89	108	127	146	III 11
15	71	90	109	128	147	III 22
16	72*	91*	110*	129*	148*	IV 3
17	73	92	111	130	149	III 14
18	74	93	112	131	150	III 25
19	75**	94**				IV 6

The above-mentioned texts not only contain summer solstice dates but also winter solstices and equinoxes. The pattern followed by these dates is very simple indeed: they are evenly spaced[2] with distances of $3^m 3^r$ beginning at the summer solstice. Table 3 gives the resulting dates,[3] directly derived from Table 2. In each column the difference of the day numbers is 11 (mod. 30)[4] and thus the return from a vernal equinox to the next summer solstice is shortened to $3^m 2^r$.

Again, Table 3 can be considered as a form of definition of the 19-year cycle. It is the direct consequence of the definitions (1) or (3), p. 355f. to which has been added the intercalation pattern (4), p. 356 and the placing of the cardinal points of each year at equidistant intervals.

Long before the schematic character of solstice- and equinox-dates had become known through the discovery of the Uruk scheme Kugler had attempted[5] to compare these dates with modern determinations of the cardinal points. He found the best agreement with modern results for the autumn equinoxes but considering precession he decided that the vernal equinoxes had provided the basis for the other three dates in each year. We know now that neither one of

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[2] This was already suggested by Epping, AB, p. 151.

[3] The three cases where the vernal equinox falls into a month I require, of course, a year number one higher than listed.

[4] This pattern is confirmed by the large number of texts which give such dates. It is easy to show that the simple rule of always adding 11 from line to line in each column of Table 3 implies that these columns are derived from the truncated scheme (1), p. 360 by adding $3^m 3^r$ from column to column and not $3^m 2;45,47,30^r$ which would be the difference for exactly equidistant cardinal points (cf. Neugebauer [1948, 1], p. 214ff.).

[5] Kugler, Sternk. II, p. 606f.

Table 3

Cycle	4	5	6	7	Summer Solstice	Autumn Equinox	Winter Solstice	Vernal Equinox
0	S.E.		113**	132**	IV 7	VI$_2$ 10	IX 13	XII 16
1	76	95	114	133	III 18	VI 21	IX 24	XII 27
2	77*	96*	115*	134*	III 29	VII 2	X 5	XII$_2$ 8
3	78	97	116	135	III 13	VI 13	IX 16	XII 19
4	79	98	117	136	III 21	VI 24	IX 27	XII 30
5	80*	99*	118*	137*	IV 2	VII 5	X 8	XII$_2$ 11
6	81	100	119	138	III 13	VI 16	IX 19	XII 22
7	82	101	120	139	III 24	VI 27	IX 30	I 3
8	83*	102*	121*	140*	IV 5	VII 8	X 11	XII$_2$ 14
9	84	103	122	141	III 16	VI 19	IX 22	XII 25
10	85*	104*	123*	142*	III 27	VI 30	X 3	XII$_2$ 6
11	86	105	124	143	III 8	VI 11	IX 14	XII 17
12	87	106	125	144	III 19	VI 22	IX 25	XII 28
13	88*	107*	126*	145*	III 30	VII 3	X 6	XII$_2$ 9
14	89	108	127	146	III 11	VI 14	IX 17	XII 20
15	90	109	128	147	III 22	VI 25	IX 28	I 1
16	91*	110*	129*	148*	IV 3	VII 6	X 9	XII$_2$ 12
17	92	111	130	149	III 14	VI 17	IX 20	XII 23
18	93	112	131	150	III 25	VI 28	X 1	I 4
19	94**				IV 6	VI$_2$ 9	IX 12	XII 15

the equinoxes represents a primary date and that all dates are unfit for comparison with accurate astronomical computations since they are the result of truncation of numbers which represent tithis and not actual solar days. This is a good example of the pitfalls of a comparison of ancient data with modern astronomy as long as the origin of the ancient data is not fully known.

While the investigation of the Uruk scheme leaves no doubt about its equivalence with the 19-year cycle the historical question of its time of origin remains almost completely in the dark. It seems plausible to consider the strict Uruk scheme as of some later origin than the 19-year cycle. But we know so little about the chronology of the cycle itself[6] that it is impossible to derive from it conclusions about the invention of the detailed solstice-equinox pattern. The direct textual evidence proves the actual use of the scheme during most of the Seleucid period.[7] An earlier application has been found[8] on the reverse of a tablet which concerns the characteristic phenomena of Mars.[9] At the moment it suffices to give the following excerpt for solstice dates:

(year)	8**	IV 7
	9	III 18
	10*	III 29
	11	III 10
	12	III 21.

[6] Cf. above p. 354f.

[7] Cf. the tabulation in Neugebauer [1948, 1], p. 218.

[8] Aaboe-Sachs [1966], p. 11f.

[9] Cf. below II A 6, 1 B (p. 424).

A year 8** excludes, of course, the Seleucid or Arsacid era. Hence we must deal with regnal years from the preceding period and it is only Artaxerxes III (who ruled 21 years) whose 8th year is a **-year, corresponding to the year [−3:19] =S.E. −39=−350/49. Unfortunately this does not establish the existence of the Uruk scheme around −350 because at that time the summer solstice in a **-year should have the date IV 6 and not IV 7 which is the proper date only after S.E. 113 (cf. above p. 360 and Table 3, p. 362). Hence we cannot exclude the possibility that the text was either written after S.E. 113 by a scribe who used the **-date of his own period to check solstices of an earlier time, or that a non-Uruk pattern existed in the fourth century B.C.

3. Sirius Dates

What we have established so far is the fact that the dates given for solstices and equinoxes in the astronomical texts of the Seleucid period are not at all observed but an arithmetical consequence of a pattern based on the 19-year cycle of intercalations. All solstice and equinox dates in Table 3, p. 362 are known if a single one of them is given. It was nevertheless surprising to learn that a similar pattern also included phenomena which one never doubted to be individually observed, the heliacal risings and settings of Sirius.

The "Almanacs" and "Diaries" mention regularly three characteristic phenomena for Sirius, schematically illustrated in Fig. 2, p. 1315: the heliacal rising Γ (in July), the heliacal setting Ω (in May), and in between what we conventionally call "opposition" Θ instead of using the correct but clumsy term "apparent acronychal rising," i.e. the last rising of the star visible after sunset.[1] Investigating these data in the GADEx material A. Sachs found[2] that they followed a fixed scheme directly related to the 19-year cycle in the same fashion as the solstices and equinoxes (cf. Table 4[3]). If S.S. denotes the date of the summer solstice, as taken from Table 2 or 3, we see that

$$\begin{aligned} \Gamma &= \text{S.S.} + 21^{\tau} \\ \Omega &= \text{S.S.} - 44^{\tau} \\ \Theta &= \text{S.S.} + 6^{m} + 11^{\tau}, \end{aligned} \tag{1a}$$

or, using the dates of the winter solstices and vernal equinoxes from Table 3

$$\begin{aligned} \Gamma &= \text{S.S.} \ + 21^{\tau} \\ \Omega &= \text{V.E.} \ + 48^{\tau} \\ \Theta &= \text{W.S.} + 5^{\tau} \end{aligned} \tag{1b}$$

Fig. 3 illustrates these relations.

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[1] Our present Θ corresponds to Θ_1, p. 1091.

[2] Sachs [1952, 1].

[3] I have arbitrarily given the year numbers of cycle 7. The dates shown remain the same for any other cycle.

Table 4

Cycle 7		Summer Solstice	Sirius		
	S.E.		Ω	Γ	Θ
0	132**	IV 7		IV 28	IX 18
1	133	III 18	II 4	IV 9	IX 29
2	134*	III 29	II 15	IV 20	X 10
3	135	III 10	I 26	IV 1	IX 21
4	136	III 21	II 7	IV 12	X 2
5	137*	IV 2	II 18	IV 23	X 13
6	138	III 13	I 29	IV 4	IX 24
7	139	III 24	II 10	IV 15	X 5
8	140*	IV 5	II 21	IV 26	X 16
9	141	III 16	II 2	IV 7	IX 27
10	142*	III 27	II 13	IV 18	X 8
11	143	III 8	I 24	III 29	IX 19
12	144	III 19	II 5	IV 10	IX 30
13	145*	III 30	II 16	IV 21	X 11
14	146	III 11	I 27	IV 2	IX 22
15	147	III 22	II 8	IV 13	X 3
16	148*	IV 3	II 19	IV 24	X 14
17	149	III 14	I 30	IV 5	IX 25
18	150	III 25	II 11	IV 16	X 6
19	151**		II 22		

As an example we can estimate the julian dates which must approximately result for the year S.E. 133/4:

Ω:	S.E. 133	II 4	≈ −178	May 15
Γ:		IV 9		July 18
Θ:		IX 29	−177	Jan. 2
Ω:		II 15		May 16.

Since the solstice- and equinox-dates in (1a) and (1b) are tithis it is clear that the same holds also for the differences which lead to the Sirius dates. But tithis may terminate at any time of a day and therefore are unrelated to the fact that Γ always belongs to the morning, Ω and Θ to the evening. This again underlines that the tithis are used as convenient arithmetical representations of ordinary calendar dates, regardless of the fact that an error of ±1 day may easily result.

Sachs also investigated the chronological range for which the pattern of Table 4 is valid. He found S.E. −11 = −322/1 = [−1:9] as earliest date for which the scheme is attested while definite contradictions appear in cycle −6 (specifically in −418/7 = [−6:8] and in −424/3 = [−6:2]). This is reminiscent of the facts that the 19-year cycle itself is in undisturbed validity only after −380 = [−4:8][4], and that the summer solstice of −350/49 = [−3:19] has a day

[4] Cf. above p. 354.

number one too high.[5] All this seems to indicate that the solstice and Sirius patterns were introduced near the very end of the Persian period.[6]

Tables 3 and 4 show that the heliacal rising of Sirius (Γ) is the only element which only once, in cycle year 11, falls outside the same calendar month (IV). Even this single exception could have been easily eliminated by replacing the intercalation in the cycle year 10* by an intercalation 11**. Then Γ in IV 18 of the year 10 would have been followed by IV 29 in the year 11**, and again by IV 10 in year 12 as before. This change in the intercalation rule would have had the added advantage of spacing the intercalatory months in the whole 19-year cycle as evenly as possible.[7] Why one nevertheless avoided a second **-year in a position analogous to cycle year 19 is a question we have no means to answer. This fact illustrates our ignorance concerning the origin and the guiding principles in the construction of the 19-year cycle.

4. Summary

We have found that during the whole Seleucid-Parthian period from which we have cuneiform astronomical texts the intercalations follow a strict 19-year cycle

$$19^y = 3{,}55^m \tag{1}$$

with only one **-year in each cycle ($\equiv$S.E. 18 mod. 19). The dates for solstices, equinoxes, and Sirius phenomena are based on the solstice dates which, in turn, are derived from the relation

$$1^y = 12;22,6,20^m = 12^m + 11;3,10^\tau \tag{2}$$

which can be considered the equivalent of the definition (1) of the 19-year cycle.

Consequently we see that the whole table of solstice-equinox-Sirius dates forms a matrix of $7 \cdot 19$ elements[1] in which all elements are known if any one of them is given.

It is obviously senseless to ask for the observational basis for any one date which then determines all the others. In other words it makes no sense to investigate the agreement of any sequence of solstice-, equinox-, or Sirius dates with accurately determined astronomical elements.

We shall see in the following that the Babylonian astronomers were not satisfied with a calendarically convenient determination of the relation between the solar year and lunar months as expressed by (1). Instead other relations, e.g.

$$1^y = 12;22,8^m$$

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[5] Cf. above p.

[6] We ignore here some errors in a text which gives correct dates for Ω and Θ but day numbers one too low for Γ during the years S.E. 62 = [3:6] to S.E. 69 = [3:13]. There is also some trouble with S.E. 189** = [9:19] where one finds II 22 for Ω instead of the expected II 23 (cf. Table 4, cycle years 0 and 19).

[7] Cf. above p. 357.

[1] Our Tables 3 and 4.

appear in use in mathematical astronomy where "year" and "month" must be exactly defined beyond the naive use in calendaric context. Also the dependence of equinoxes and solstices on the true solar motion is fully recognized in the lunar theory.[2]

The insight that the solstice-equinox-Sirius dates were based exclusively on the cycle (1) without any further observations shatters the traditional belief – inherited from late antiquity – in extensive Babylonian observational activities, stretching over a great length of time. One could very well imagine that someone looking at any archive extending over a century or two, could find that seven intercalations in 19 years seemed to satisfy practical needs. A simple division would then lead to (2) and one given date of a summer solstice would provide all future solstice-equinox-Sirius dates without ever making an observation again. One quiet evening's work could have settled the whole affair.

I do not suggest this as the true story of the origin of the 19-year cycle and of the Uruk scheme. But one must admit that the scheme itself leaves no room for successive empirical adjustments and one has to consider it as a definite possibility that the whole structure was erected in a comparatively short interval, perhaps shortly before −380.

It may be added that the experience with the calendaric patterns does not stand alone in Babylonian astronomical methodology. We will find that the same strictly determined matrix type of computed data is also fundamental for the planetary theory and similar tendencies are visible in many aspects of Babylonian astronomy. It is important to realize that it is only since the invention of the telescope that the observational component has become a dominating element in the whole field. Ancient astronomy tends far more to mathematical schematisation than one is willing to assume on the basis of our modern background. But this makes it also possible to obtain definitive results comparatively quickly which then remain valid for a long period to come.

§ 4. Length of Daylight

It is one of Kugler's many important discoveries that the Babylonian astronomers of the Seleucid period operated with the simple arithmetical ratio

$$M : m = 3 : 2 \tag{1}$$

for the length M of the longest daylight and the length m of the shortest day.[1] The absolute values of M and m are

$$M = 3;36^{\mathrm{H}}, \qquad m = 2;24^{\mathrm{H}} \tag{2}$$

expressed in units which we call "*large hours*" H. Since the mean value is $1/2(M+m) = 3^{\mathrm{H}}$ we see that

$$1^{\mathrm{d}} = 6^{\mathrm{H}} \quad \text{or} \quad 1^{\mathrm{H}} = 4^{\mathrm{h}}. \tag{3}$$

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[2] Cf. below p. 368.

[1] Kugler, Mondr., p. 74ff.

The term "large hour" is introduced here only for convenience's sake without any equivalent Babylonian term. The texts divide the day into 360 units called uš, meaning originally "length." Instead of (3) we should accordingly write

$$1^{d}=6{,}0 \text{ uš} \quad \text{or} \quad 1^{H}=1{,}0 \text{ uš}. \tag{3a}$$

Obviously the uš are the units called "*time degrees*" by the Greeks[2] such that

$$1^{d}=6{,}0^{\circ}. \tag{3b}$$

It is important to note that these units are of constant length, in contrast to the "seasonal hours" of the Greeks. The counterpart to the Babylonian time degrees are the "equinoctial hours." It is from the combination of the Babylonian sexagesimal norm (3a) with the 12-division of night and daytime in Egypt[3] that hellenistic astronomy developed the division of the day in 24 equinoctial hours which we are still using today.

Noone will assume that such neat arithmetical data as (1) and (2) are the direct result of observations. Nevertheless one may ask how well actual conditions for Babylon ($\varphi \approx 32;30$) are represented by

$$M=14;24^{h}, \qquad m=9;36^{h}. \tag{2a}$$

Computing without any consideration of refraction and for an ideal horizon we obtain from[4]

$$\tan\varphi=-\cos\frac{M}{2}\cot\varepsilon \tag{4}$$

the extrema

$$M\approx 3{,}32;44^{\circ}\approx 14;11^{h} \quad \text{hence} \quad m\approx 9;49^{h}$$

or conversely, from $M=3{,}36^{\circ}$ (depending on ε) a value between $34;45^{\circ}$ and 35° for φ. This shows that the Babylonian parameters are, as to be expected, far from accurate but still correct within a reasonable order of magnitude.[5]

As we have seen[6] Ptolemy could compute φ from M and conversely by formulae equivalent to (4). Hence, accepting (1), the customary characterization for the latitude of Babylon, he had to obtain, as we do, a value for φ very near to 35°. This is indeed the geographical latitude assigned to Babylon in his "Geography."[7] A value, conveniently rounded for the computation of Babylonian ephemerides by means of arithmetical methods, caused a distortion of the ancient, and hence mediaeval, maps of the Near East by 2 1/2° in latitude, substituting the location of, e.g., Samarra for Babylon. Nothing would have been easier than to measure φ directly; that this had not been done, or at least had not become known to Alexandrian astronomers during the 500 years or so which separate Ptolemy from the beginning of mathematical astronomy in Mesopotamia is a typical example of the absence of a scientific organization in antiquity.

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[2] Cf. above p. 40.

[3] Cf. below p. 561.

[4] Cf. p. 37, (4a).

[5] The actual values are supposedly $M\approx 14;20^{h}$ and $m\approx 10;0^{h}$, hence M: $m\approx 1{,}26$ instead of 1;30 (Schaumberger, Erg., p. 377).

[6] Above p. 38.

[7] Ptolemy, Geogr. V, 20, § 6 (ed. Nobbe, p. 78).

1. Oblique Ascensions

The arithmetical, non-observational, basis of the Babylonian description of the variable length of daylight becomes still more evident when one looks in the ACT material at the methods which lead from a given solar longitude $\lambda_\odot$ to the corresponding length of daylight (henceforth denoted by C). As we have seen before[1] $C=d(\lambda_\odot)$ must be the rising time or oblique ascension of the semicircle of the ecliptic from $\lambda_\odot$ to $\lambda_\odot+180°$. Making use of this relationship one can use the Babylonian values of C to compute the corresponding rising times for consecutive 30° sections of the ecliptic, beginning with the vernal equinox. The resulting values turn out to belong to one of the following two arithmetical progressions which we call "*System A*" and "*System B*", respectively[2]:

	A	B	
$\rho_1=\rho_{12}$	20°	21°	
$\rho_2=\rho_{11}$	24	24	
$\rho_3=\rho_{10}$	28	27	(1)
$\rho_4=\rho_9$	32	33	
$\rho_5=\rho_8$	36	36	
$\rho_6=\rho_7$	40	39.	

In System A the numbers show a constant difference of 4°, in System B the middle difference is 6°, otherwise 3°.

The division of the ecliptic, beginning with the "*vernal equinox*," requires some specification. Both in System A and B the vernal equinox is defined as the point of the ecliptic at which the length of daylight is the same as the length of the night (i.e. 3,0 uš). But in neither System is this point called "♈0°" (as, e.g., in the Almagest[3]) but

$$\text{Vernal Equinox in}\begin{cases}\text{System A: ♈ } 10°\\ \text{System B: ♈ } 8°.\end{cases} \tag{2}$$

In other words: Babylonian "longitudes" are always reckoned by "*zodiacal signs*" ♈, ♉, etc., but the cardinal points of the solar year are not located at the zero points of their respective signs but at 10° in System A, at 8° in System B.

One must not ascribe to this norm any deeper astronomical significance. The zodiacal signs originated, of course, from irregular constellations. At that stage the position of movable celestial objects were expressed by means of relatively small distances with respect to specific stars, a type of procedure still in evidence in GADEx texts.[4] When finally the irregular configurations were replaced by real ecliptic coordinates in signs of equal 30° length the sign "Aries" obtained by some accidental compromise such a position within the constellation Aries that the vernal equinox took place when the sun was at the 10th, respectively 8th, degree of the sign. We do not know what chronological relation existed between

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[1] Above p. 40f.

[2] These schemes must, of course, satisfy the symmetry relations $\rho_1=\rho_{12}$, etc. Cf. above p. 35.

[3] Cf., e.g., above p. 30.

[4] Cf. below p. 545.

these two norms and what caused the difference. We have no evidence from Babylonian sources about a recognition of precession and we have no reason to assume that the difference of zero points in System A and B had anything to do with it, knowingly or unknowingly.

In comparing Babylonian longitudes with modern ones (which always denote the vernal point as $\lambda = 0°$) one cannot simply say that $\lambda_{\text{Bab}} - \lambda_{\text{mod}} = 10°$ or $8°$ as (2) might suggest. Kugler had already found[5] that the Babylonian ephemerides were based on sidereal longitudes such that their "♈ 0°" around −120 had a tropical longitude of about −4;36°. A very careful investigation of additional material[6] by P. Huber[7] showed that in about −100 the relation

$$\lambda_{\text{Bab}} - \lambda_{\text{mod}} = 4;28 \pm 0;20° \qquad (3)$$

holds. Hence we obtain for the ecliptic coordinates a situation represented approximately in Fig. 4. That the vernal point maintained in each of the two systems a fixed sidereal longitude indicates clearly that precession was unknown. It is, however, important to realize that the computation of planetary or solar and lunar longitudes is totally independent of the location assigned to the equinoxes and solstices. The latter play a role only for the determination of the length of daylight or night, i.e. for the moments of sunset or sunrise.

2. Length of Daylight

Table 5 displays the schemes for the length of daylight, being the exact equivalents of the patterns (1), p. 368 for the rising times. The tabulated values correspond to solar longitudes at the endpoints of the 30° sections which begin at the vernal equinox, i.e. at ♈ 10° or ♈ 8°, respectively. Rules which correspond to this table are preserved in procedure texts[8] and are confirmed by their application in lunar ephemerides. For intermediate solar longitudes linear interpolation is used.

Example for System A: $\lambda_\odot = $ ♓ 6;32 = ♒ 10 + 26;32°. According to Table 5 the increase of C in this interval is 12°, hence 0;24° for each degree of solar longitude. Therefore

$$C = 2{,}28 + 26;32 \cdot 0;24 = 2{,}38;36{,}48°$$

as given in the text.[9]

Example for System B: $\lambda_\odot = $ ♌ 3;32,25,38 = ♋ 10 + 23;32,25,38. Table 5 shows between ♋ 10° and ♌ 10° a decrement of 6°, hence 0;12° per degree. Thus

$$C = 3{,}36 - 23;32{,}25{,}38 \cdot 0;12 = 3{,}31;17{,}30{,}52{,}24°$$

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[5] Kugler, Sternk. I, p. 172. Similar results were obtained by van der Waerden [1952], p. 222.

[6] Including the fragment of a catalogue of stars, discovered by Sachs [1952, 2].

[7] Huber [1958].

[8] ACT No. 200, Sect. 2 (p. 187); No. 200b, Sect. 2 (p. 214); both texts for System A. For System B no procedure text is preserved. The corresponding part of Table 5 was first reconstructed from the applications by Kugler, Mondr., p. 99. Its derivation from rising times was given by Neugebauer [1936, 2].

[9] ACT No. 9 obv. III and IV, 12.

Table 5

	System A						System B				
	ρ	$\Sigma\rho$	$\lambda_\odot$	Length of Daylight			ρ	$\Sigma\rho$	$\lambda_\odot$	Length of Daylight	
ρ_1	20°	20°	♈ 10°	3, 0° = 12; 0ʰ		ρ_1	21°	21°	♈ 8°	3, 0° = 12; 0ʰ	
ρ_2	24	44	♉	3,20	13;20	ρ_2	24	45	♉	3,18	13;12
ρ_3	28	1,12	♊	3,32	14; 8	ρ_3	27	1,12	♊	3,30	14; 0
ρ_4	32	1,44	♋	3,36	14;24 = M	ρ_4	33	1,45	♋	3,36	14;24 = M
ρ_5	36	2,20	♌	3,32	14; 8	ρ_5	36	2,21	♌	3,30	14; 0
ρ_6	40	3, 0	♍	3,20	13;20	ρ_6	39	3, 0	♍	3,18	13;12
ρ_7	40	3,40	♎	3, 0	12; 0	ρ_7	39	3,39	♎	3, 0	12; 0
ρ_8	36	4,16	♏	2,40	10;40	ρ_8	36	4,15	♏	2,42	10;48
ρ_9	32	4,48	♐	2,28	9;52	ρ_9	33	4,48	♐	2,30	10; 0
ρ_{10}	28	5,16	♑	2,24	9;36 = m	ρ_{10}	27	5,15	♑	2,24	9;36 = m
ρ_{11}	24	5,40	♒	2,28	9;52	ρ_{11}	24	5,39	♒	2,30	10; 0
ρ_{12}	20	6, 0	♓	2,40	10;40	ρ_{12}	21	6, 0	♓	2,42	10;48

rounded in the text[10] to 3,31°. Rounding is a common practice for the values of C in texts of System B.

Linear change for the length of daylight within each 30° section implies constancy of rising times in each interval. Hence, if we wish to obtain, e.g., the rising times for 10° sections we have only to divide the values given in (1), p. 368 by 3 and obtain within each 30° section

System A:	6;40	8	9;20	10;40	12	13;20
System B:	7	8	9	11	12	13.

In Fig. 5 we compare these rising times with accurately computed values, based on $\varphi = 32;30$ and $\varepsilon = 23;51,20$. The corresponding lengths of daylight are shown in Fig. 6.[11]

That the values for the variable length of daylight were based on the summation of six consecutive values ρ shown in Table 5 cannot be doubted. But in principle one could still assume that we are dealing with a purely arithmetical device designed to produce a function of sinusoidal shape by the summation of a difference sequence. Such an assumption would be by no means without foundation since this method is well attested in different sections of Babylonian mathematical astronomy. Fortunately the numbers ρ do occur not only in connection with the computation of C but also in the problem of computing the time interval between sunset and the setting of the moon's crescent at a given elongation in longitude. Here one faces exactly the problem of finding the oblique ascension[12] of a given arc of the ecliptic and the fact that the corresponding coefficients are the same as the ρ's in the daylight scheme[13] demonstrates that their property as rising times was fully realized.

[10] ACT No. 122 rev. II and III, 10.

[11] In both figures I have omitted symmetric branches in order to avoid overcrowding of the graphs.

[12] Actually we should say "setting time" instead of rising time, but by replacing λ by $\lambda + 180$ one can always transform one problem into the other.

[13] Cf. below II B 10, 2; also Neugebauer [1953].

The historical significance of the Babylonian schemes for the rising times reaches far beyond their applications in the solar and lunar theory. Since Greek mathematical geography characterized the latitude of a locality by its maximum daylight M the Babylonian method of finding the function $C(\lambda)$ of daylight depending on the solar longitude was properly modified, but under preservation of the arithmetical types A or B for the rising times. The geographical system of the "seven climata" preserved vestiges of the Babylonian oblique ascensions until deep into the Middle Ages.[14] On the other hand one finds the unaltered set of Babylonian rising times of System A in Indian astronomy of the sixth century A.D.[15] without any consideration for India's far more southern position. Rising times and related patterns have thus become an excellent indicator of cultural contacts, ultimately originating in Mesopotamia.

§ 5. Solar Motion

The ephemerides of the Seleucid period contain a special column (B) for the longitude of the sun, tabulated at consecutive mean syzygies. These longitudes are derived from one of two types of arithmetical models, henceforth called "System A" and "System B", respectively, according to their basic mathematical structure:

System A: step functions

System B: linear zigzag functions.

Replacing arithmetical functions by continuous graphs Fig. 7 will clarify what we mean by these two "Systems."

The same terms we used before as a description of two types of variation assumed for the length of daylight.[1] Nevertheless this will cause no ambiguity since the cinematic solar models A and B are always associated with the daylight scheme of the same type although there exists no causal coupling between the velocity scheme and the scheme for the variation of the length of daylight.

An early discovery of the anomaly of the lunar motion is not surprising. Within about two weeks the daily progress of the moon can vary by as much as about 3°, i.e. about six lunar diameters, and this fact can be observed directly with respect to the background of the fixed stars. For the solar anomaly, however, the situation is totally different. Not only is it impossible to directly observe the sidereal motion of the sun but the extrema of the daily progress differ only by some 4 minutes of arc at points half a year apart. To detect such a slowly varying inequality and to construct an adequate mathematical model for it is certainly no mean achievement of Babylonian astronomy.

If one looks for plausible data which could have led to the discovery of the solar anomaly it is obvious that one must find a secondary phenomenon which suggests a variation of the solar motion as function of its longitude. This is exactly

[14] Cf. below p. 727ff. and p. 938.
[15] Varâhamihira, Bṛhajjataka I, 19.
[1] Above p. 368.

what we know from Greek astronomy: the time intervals between equinoxes and solstices are found to be unequal and consequently the solar motion cannot amount to exactly 90° on a circle with the earth as center. We know how this then leads to the classical eccenter model and to the determination of its parameters.[2]

In the Babylonian ephemerides it follows from the schemes for the length of daylight[3] that the equinoxes and solstices were related to the solar longitudes in such a way that they fell in System A at ♈10° and subsequent quadrants, at ♈8°, ♋8°, etc., in System B.

In System A the solar velocity is assumed to be constant on two complementary arcs of the ecliptic (cf. Fig. 8):

$$\begin{array}{ll} w_1 = 30^{\circ/m} & \text{from ♍13 to ♓27 ("fast arc")} \\ w_2 = 28;7,30^{\circ/m} & \text{from ♓27 to ♍13 ("slow arc").} \end{array} \qquad (1)$$

In System B, period and mean value of the velocity function can be considered known, though for us it is still difficult to accurately determine the parameters of the model since we do not know the underlying length of the year.

Nevertheless it is possible in both cases to determine within narrow limits the lengths of the seasons, measured in days, necessary to derive from them the parameters of the models for the solar velocity. The resulting conclusion turns out to be clearly negative: nowhere do there appear numbers simple enough to be taken as the result of observations; and as soon as one replaces these data by plausible roundings the parameters of the model would have to be drastically modified. Hence it cannot be the inequality of the seasons that caused the recognition of the solar anomaly.[4]

A new and very promising approach to our problem was opened in a paper by Lis Bernsen [1969] who observed that the spacing of the true syzygies depends mainly on the solar velocity alone, being practically independent of the lunar velocity. This is a consequence of the following situation: a synodic month is only about 2 days longer than the anomalistic month. The effect of the variability of the lunar motion is therefore only felt during the short intervals between anomalistic and synodic months; consequently the spacing of the syzygies is essentially the same as produced by combining the motion of the mean moon with the motion of the true sun. Hence the syzygies will always be far apart near the perigee of the sun and closer together at the solar apogee. Numerical examples show that the model of System A for the monthly solar progress, as defined above by (1), is a very good approximation of empirical data.[5] Hence it can hardly be doubted that we now know the ultimate source of the discovery of the solar anomaly.

Nevertheless the problem is still far from a complete solution which would require the numerical determination of the parameters of the models. System B is

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[2] Above p. 57f.

[3] Above p. 368.

[4] The fact that the "Uruk scheme" for solstices and equinoxes (above p. 361) divides the year into four equal seasons is not decisive since there we are dealing with a purely calendaric scheme. Similarly the 19-year cycle does not reflect the level of the contemporary luni-solar theory.

[5] Cf. Bernsen [1969], p. 27, Fig. 2. An equal fit is obtainable for column A in System B with the parameters given in ACT I, p. 70.

here much simpler to handle since one need only to know the extremal values, the period being the length of the year. No such obvious considerations exist for System A which requires compromises for the distribution of the syzygies on the two arcs. But the main difficulty is common to both systems: how could one find the accurate positions of the true syzygies? It is plausible to assume that the investigation of sequences of lunar eclipses must have played an important role though eclipses do not furnish consecutive syzygies.

§ 6. Mathematical Methodology

The basic methods of Babylonian astronomy are undoubtedly "arithmetical" in character. In marked contrast to the Greek approach the first goal of the Babylonian lunar and planetary theory is not the determination of the longitude λ of a celestial body as a continuous function $\lambda(t)$ of time. Instead the attention is concentrated on specific events, e.g. new moons, or consecutive stationary points for a planet. For these isolated events one tries to determine their spacing on the ecliptic, without reference to the motion which actually brings the moon or the planet from one such phase to the next. We shall call these arcs "*synodic arcs*," a term which should not be taken as referring to conjunctions only. Since all phases under discussion occur at essentially fixed elongations from the sun and hence follow similar patterns we may, in a loose fashion, apply the term "synodic" to all of them.

The idea of making sequences of isolated phenomena the basis of the theory is mathematically reflected in the construction of "*ephemerides*" which tabulate moments and positions of such sequences in their natural order. Here the independent variable is not continuous (e.g. time or longitude) but simply the order number of the events under consideration, or, as we shall say, the number of the "*occurrences*," e.g. of new moons or planetary stations.

The synodic arcs for a given phenomenon fluctuate around a mean value $\overline{\Delta\lambda}$ which can be easily determined as soon as a period for the synodic arcs is known. Here it is an essential assumption that the length of the synodic arcs depends only on the longitude where an event takes place. Only under this condition do we know that the synodic arcs repeat themselves exactly as soon as a phase returns to its initial position in the ecliptic. In modern astronomical interpretation this would mean assuming that the positions of the apsidal lines are fixed in the (sidereal) ecliptic, but we have no right to ascribe such concepts to the Babylonian astronomers.

The ephemerides apply two different methods in describing the variation of the synodic arcs. One, called "*System A*," uses synodic arcs which have constant values on fixed arcs of the ecliptic, changing these values discontinuously at the boundaries of these arcs. The second method, "*System B*," operates with synodic arcs which form alternatingly increasing and decreasing arithmetical progressions such that their values lie on a linear zigzag function when plotted against equidistant points which count the number n of consecutive occurrences.

We shall eventually discuss the astronomical significance and interrelation of these two models in great detail and for each planet as well as for sun and moon. At the moment, however, we simply take these two "Systems" for granted and restrict ourselves to deriving some rules which are useful in operating with such functions. Also the question of units (e.g. degrees or days, etc.) is at present of little interest and it usually suffices to consider all numbers as integers in units of the last tabulated digit.

1. System B

Linear zigzag functions $y(n)$ are very easy to handle (cf. Fig. 9). We call the slope

$$d = y(n+1) - y(n) \tag{1}$$

which is stretchwise constant its difference, its maximum M, its minimum m, hence

$$1/2(M+m) = \mu, \qquad M-m = \Delta \tag{2}$$

mean value and amplitude, respectively. If $y(n)$ is a value preceding a maximum or minimum, $y(n+1)$ a value following it, then

$$y(n)+y(n+1) = \begin{cases} 2M-d & \text{at maximum} \\ 2m+d & \text{at minimum.} \end{cases} \tag{3}$$

If we construct a continuous function $y(x)$ which connects linearly all values $y(n)$ then we can speak about a "*period*" P of $y(x)$ that is not, in general, of integer length, reaching, e.g., from minimum to minimum which need not (and in general will not) belong to the set of values $y(n)$. Obviously this period is given by

$$P = \frac{2\Delta}{d}. \tag{4a}$$

If we represent this number as quotient of the two smallest integers Π and Z such that

$$P = \frac{\Pi}{Z} \tag{4b}$$

then Π is the smallest integer for which for all integers k

$$y(k) = y(k+\Pi) \tag{5}$$

while Z indicates that the interval of length Π contains Z periods P. We call Π the "*number period*," Z the "*wave number*."

Instead of representing a linear zigzag function $y(x)$ as a broken line of amplitude Δ one can also use a system of parallel strips of width Δ, alternatingly containing ascending and descending branches (cf. Fig. 10). Such a representation is helpful, e.g., when one wishes to see what happens when a continuous zigzag function $y(x)$ of period p is tabulated at intervals of length 1 where[1]

$$1 > p > 1/2. \tag{6a}$$

[1] The restriction $p > 1/2$ is only made for the sake of simplicity. It is satisfied in the most important cases which actually occur.

We call the underlying continuous function of period p the "*true function*" in contrast to the "*tabulation function*" which is based on the set of values $y(n)$. Then it is easy to see (cf. Fig. 11) that the values $y(n)$ can also be interpreted as belonging to a zigzag function (i.e. our tabulation function) of the same amplitude Δ as the true function but with smaller difference d, hence greater period P. One finds for these new parameters that

$$P=\frac{p}{1-p} \quad \text{or} \quad p=\frac{P}{P+1}=\frac{2\Delta}{2\Delta+d}. \tag{6b}$$

As an example may be mentioned the velocity of the moon. If its variation is represented by a linear zigzag function of period p = the anomalistic month $\approx 27\,1/2$ days, a tabulation function of much greater period appears if one tabulates the velocities at equidistant, i.e. mean, conjunctions (about 29 1/2 days apart).[2] In this case the astronomical meaning of the two functions is fully known; in general, however, a linear zigzag function always leaves the question open as to whether one is dealing with a true function or only with a tabulation function whose period by itself is not astronomically significant.

Sometimes it is advantageous to have at one's disposal a representation of a zigzag function that makes full use of the existence of a number period Π. Indeed a set of Π points on a circle can be made to represent all possible values of a zigzag function, e.g. by rolling the straight line which crosses infinitely many strips of width Δ (Fig. 10) onto a circle of a circumference of exactly the length as the line within two strips, thus being the equivalent of one wave of length P. In this way the total zigzag function will be represented by a regular Π-gon.[3] It must be remembered, however, that consecutive points on the circle do not represent consecutive points of the original function since Z waves are superimposed on the same circle.

2. System A

We divide the total length of the ecliptic (6,0°) into k arcs – k being at least two, but there exist models with up to six arcs – schematically represented in Fig. 12 by $\alpha_1, \alpha_2, \ldots$. On each of these arcs, the synodic arc for the phenomenon under consideration, is constant, $w_1, w_2, \ldots$ respectively.

Suppose this phenomenon takes place at λ_0 in α_1. The next occurrences will be at the longitudes $\lambda_1=\lambda_0+w_1$, $\lambda_2=\lambda_1+w_1$, etc., until one approaches the boundary between α_1 and α_2. We shall now formulate the rules which must be followed at the crossing of such a boundary.

Let w_i and w_{i+1} be the values for the synodic arc valid before and after the discontinuity, r and s the corresponding distances of the nearest occurrences (cf. Fig. 13). Obviously the synodic arc

$$w'=r+s \tag{1}$$

must lie between w_i and w_{i+1}. The rule according to which s in the new section is determined from r prescribes that the remainder w_i-r is changed in the ratio

[2] The value of P is about 14 mean synodic months; cf. below p. 476.

[3] Cf. below p. 384 and Fig. 15 there.

of the synodic arcs after and before the boundary:

$$w' = r + (w_i - r)\frac{w_{i+1}}{w_i} \tag{2}$$

or

$$w' = w_{i+1} + r\left(1 - \frac{w_{i+1}}{w_i}\right) = w_{i+1} - c_{i,\,i+1}\cdot r \tag{3a}$$

where

$$c_{i,\,i+1} = \frac{w_{i+1}}{w_i} - 1 \tag{3b}$$

is the "*transition coefficient*" for crossing from α_i to α_{i+1}. It follows from (1) and (3a) that

$$r + s = w_{i+1} + r\left(1 - \frac{w_{i+1}}{w_i}\right)$$

or

$$\frac{r}{w_i} + \frac{s}{w_{i+1}} = 1, \tag{4}$$

a relation which holds at each boundary for the parts r and s which make up the arc w' of transition from w_i to w_{i+1}.

Conversely, these relations allow us to determine in a given ephemeris from the values of the w' the location of the points of discontinuity (the so-called "*jumps*"). Indeed one has from (3a)

$$r = \frac{w' - w_{i+1}}{-c_{i,\,i+1}}. \tag{5}$$

This amount has to be added to the last longitude inside α_i to find the longitude of the jump between α_i and α_{i+1}.

We now ask how many occurrences take place, one after the other, on the ecliptic, or rather, how many synodic arcs can be placed on 6,0°.

On the arc α_i all complete synodic arcs are of the length w_i, thus $[\alpha_i/w_i]$ is their number.[4] In general this leaves out two arcs shorter than w_i, one at the beginning and one at the end of α_i. We now form a number

$$[P] = \sum_1^k{}' \frac{\alpha_i}{w_i} \tag{6}$$

where the summation $\sum'$ is extended over all k arcs α_i but omitting an arc $s_1 < w_1$ at the beginning of α_1 and $r_1 < w_k$ at the end of α_k. Then (4) tells us that $[P]$ is indeed an integer, counting not only the complete synodic arcs on each α_i but also the composite synodic arcs w' for each interior jump.

Again the fundamental assumption is made that the length of the synodic arcs depends exclusively on their position on the ecliptic.[5] Hence there must exist a smallest number Π such that after Π occurrences all synodic arcs are

[4] [A] denotes the greatest integer contained in A.

[5] Cf. above p. 373.

exactly repeated.[6] Any sequence of Π consecutive synodic arcs must therefore correspond to an integer number Z of passages through the ecliptic. Again because of (4) we know that the number Π must be exactly the result of adding all synodic arcs, complete ones as well as those divided into two smaller parts at a discontinuity. In doing this we have traversed the ecliptic exactly Z times. Hence we see that

$$\Pi = Z \cdot \sum_{1}^{k} \frac{\alpha_i}{w_i}. \tag{7}$$

As before we call Π the "*number period*," Z the "*wave number*" and we also introduce[7] a "*period*" P by defining with (7)

$$P = \frac{\Pi}{Z} = \sum_{1}^{k} \frac{\alpha_i}{w_i}. \tag{8}$$

Since Π counts the number of synodic arcs which cover the ecliptic Z times before repeating their values periodically it is natural to define a "*mean synodic arc*" $\overline{\Delta\lambda}$ by

$$\overline{\Delta\lambda} = \frac{6{,}0 \cdot Z}{\Pi} \tag{9a}$$

or, using (8)

$$\overline{\Delta\lambda} = \frac{6{,}0}{P}. \tag{9b}$$

Hence we can also define P as the number (integer plus fraction) which indicates how many mean synodic arcs can be placed on 360°. The equation (8), however, is of special importance since it allows us to find the characteristic parameters Π and Z of any model of System A from the α_i and w_i used in an ephemeris. Conversely, given periods Π and Z impose arithmetical conditions on the distribution and values of the synodic arcs in such a model.

Example[8]

We consider System A of the lunar theory. From ephemerides[9] one finds for the synodic arcs which separate consecutive syzygies of the same type from one another

$$\begin{aligned} &w_1 = 30 && \text{from } ♍13 \text{ to } ♓27 \text{ thus } \alpha_1 = 3{,}14° \\ &w_2 = 28;7{,}30 && \text{from } ♓27 \text{ to } ♍13 \text{ thus } \alpha_2 = 2{,}46°. \end{aligned} \tag{10}$$

Hence we have the transition coefficients

$$\begin{aligned} c_{1,2} &= \frac{w_2}{w_1} - 1 = 0;56{,}15 - 1 = -0;3{,}45 = -1/16 \\ c_{2,1} &= \frac{w_1}{w_2} - 1 = 1;4 - 1 = 0;4 = 1/15. \end{aligned} \tag{11}$$

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[6] We make here use of the fact that we are dealing with strictly arithmetical patterns which contain no approximations or roundings.

[7] Cf. (4a), p. 374.

[8] For other examples cf. below p. 392ff.

[9] ACT, p. 86ff.

Hence for a syzygy $r°$ before a jump ($\downarrow$ or $\uparrow$ respectively) the next synodic arc will be

$$\begin{aligned} w'(\downarrow) &= 28;7,30 + r \cdot 0;3,45 = 28;7,30 + \frac{r}{16} \\ w'(\uparrow) &= 30 - r \cdot 0;4 = 30 - \frac{r}{15}. \end{aligned} \tag{12}$$

As period one finds

$$P = \frac{3,14}{30} + \frac{2,46}{28;7,30} = 6;28 + 5;54,8 = 12;22,8 = \frac{46,23}{3,45} = \frac{\Pi}{Z}. \tag{13}$$

The values

$$\Pi = 46,23, \qquad Z = 3,45 \tag{14}$$

allow an astronomical interpretation. We know that P indicates that 12;22,8 mean synodic arcs cover the ecliptic exactly once. For the sun each coverage of the ecliptic means one year, hence $Z=3,45$ means that 3,45 years have elapsed during the completion of $\Pi=46,23$ mean synodic months. Hence the model is based on

$$3,45 \text{ years} = 46,23 \text{ mean synodic months} \tag{15a}$$

or

$$1 \text{ year} = 12;22,8 \text{ mean synodic months.} \tag{15b}$$

We will find the same norm also in the planetary theory.[10]

Supplementary Remarks

From the fact that neither P nor $[P]$ depends on the specific location of the synodic arcs it follows that

$$P - [P] = \frac{s}{w_1} + \frac{r}{w_k} \qquad 0 < s < w_1,\ 0 < r < w_k \tag{16}$$

has the same value for each of the Z distributions of the Π synodic arcs on 360° of the ecliptic, s being the fraction of one synodic arc at the beginning, r at the end of each run through 360° of the ecliptic. Consequently, if the values s form an arithmetic progression (of course mod. w_1) with a difference σ then also the values r form such a progression (mod. w_k) with the difference

$$\rho = -\frac{w_k}{w_1}\sigma \tag{17}$$

and vice versa.

Indeed, since r_n and s_{n+1} are adjacent arcs at the boundary between the nth and $n+1$st coverage of the circle, and similarly r_{n+1} and s_{n+2} at the next boundary, we have from (4)

$$\frac{r_n}{w_k} + \frac{s_{n+1}}{w_1} = 1 \qquad \frac{r_{n+1}}{w_k} + \frac{s_{n+2}}{w_1} = 1,$$

therefore

$$\frac{r_{n+1} - r_n}{w_k} + \frac{s_{n+2} - s_{n+1}}{w_1} = 0$$

[10] Cf., e.g., below p. 396.

which is the same as (17) if either the r or the s are known to form an arithmetic progression.

But it is easy to see that, e.g., the values r form an arithmetic progression. Suppose we change r, at the end of the arc α_i, to $r+\delta$. Then s, at the beginning of the arc α_{i+1}, is changed to $s-\varepsilon$ such that, because of (4),

$$\frac{r+\delta}{w_i}+\frac{s-\varepsilon}{w_{i+1}}=1.$$

Previously we had

$$\frac{r}{w_i}+\frac{s}{w_{i+1}}=1;$$

hence we find for the decrease ε of s

$$\varepsilon=\frac{w_{i+1}}{w_i}\,\delta. \tag{18}$$

This decrease of s at the beginning of the arc α_{i+1} again causes an increase of r by the same amount ε at the end of α_{i+1} because the intermediate synodic arcs all have the length w_{i+1}. Hence the next s decreases by the amount

$$\frac{w_{i+2}}{w_{i+1}}\,\varepsilon=\frac{w_{i+2}}{w_i}\,\delta,$$

producing the same amount of increase for the next r. Consequently, if in this fashion we go through all k arcs, an initial increase of r to $r+\rho$ results in a new increase by $\frac{w_k}{w_k}\rho=\rho$. Hence we have a sequence of values r, $r+\rho$, $r+2\rho$, etc., q.e.d.

In order to determine the value of ρ we note that with (16)

$$\frac{s_{n+1}}{w_1}+\frac{r_{n+1}}{w_k}=P-[P]$$

and with (4)

$$\frac{s_{n+1}}{w_1}+\frac{r_n}{w_k}=1$$

thus

$$\rho=r_{n+1}-r_n=w_k(P-([P]+1)) \tag{19a}$$

and with (17)

$$-\sigma=s_n-s_{n+1}=w_1(P-([P]+1)). \tag{19b}$$

Both the values of r and of s lie on a "*linear saw function*"[11] (cf. Fig. 14) of amplitude w_k and w_1 respectively, hence of a period

$$P_0=1/(P-([P]+1)), \tag{20}$$

counted in units which represent single rotations through the ecliptic. The number period Π_0 of this function is, of course, Z.

[11] For examples cf. below p. 392 (2).

A. Planetary Theory

The first six sections of this chapter are meant to be a general introduction to the methods of Babylonian planetary theory. Many problems which are specific for each planet will be discussed separately.

§ 1. Basic Concepts

Ephemeris for Saturn

The first column shown in the text transcribed in Table 6[1] gives consecutive years of the Seleucid era as is obvious from the pattern of intercalations.[2] From a colophon at the end of the tablet[3] we know that it was written in S.E. 124 (−187/186) in Uruk, probably covering a period of 60 years, from S.E. [123] to 182.

Column II gives the differences for column III which contains months, days, and their fractions, obviously based on tithis as the reckoning modulo 30 shows. For the sequence of the months the character of the year, intercalary or ordinary, is of course taken into consideration. Column IV is the difference column for column V which gives the longitudes to seconds of degrees. Since $\Delta\lambda$ in IV is of the order of magnitude of 12° and since each line represents about one year it takes about 30 years to complete one rotation through the ecliptic.[4] Hence we are dealing with Saturn. From obv. 5/6 we see that the planet was in Libra during month I when the sun must be near Aries. This shows that the dates and longitudes refer to oppositions. That much suffices for the moment as characterization of the text.

The mathematical structure of the difference columns II and IV is obvious: both form linear zigzag functions; hence we classify the text under System B. The characteristic parameters are

	Column II	Column IV	
d	0;12$^{\tau}$	0;12°	
m	22;41,23,7,30	11;14, 2,30	
M	25;32, 3,7,30	14; 4,42,30	(1a)
Δ	2;50,40	2;50,40	
μ	24; 6,43,7,30	12;39,22,30.	

[1] Table 6 is a transliteration of ACT No. 702 but omitting all details concerning readings and restorations for which see ACT II, p. 357 f. and III, Pl. 207 and Pl. 249 (photo).

[2] The year 151** in obv. 11 is the year [7:19] in the 19-year cycle (cf. above p. 356).

[3] Cf. ACT I, p. 20, colophon Z.

[4] Cf., e.g., obv. V, 12 and rev. V, 7 with ♑ 11° and ♑ 18°, respectively.

Table 6

		I	II	III	IV	V
Obv.	0.	139	23,55,40	XI 19, 5,28,45	12,35,20	19,18,25
		140*	23,43,40	XII 12,49, 8,45	12,23,20	1,41,45 ♍
		141	23,31,40	XII 6,20,48,45	12,11,20	13,53, 5
		142*	23,19,40	XII 29,40,28,45	11,59,20	25,52,25
		143	23, 7,40	XII 22,48, 8,45	11,47,20	7,39,45 ♎
	5.	145*	22,55,40	I 15,43,48,45	11,35,20	19,15, 5
		146	22,43,40	I 8,27,28,45	11,23,20	30,38,25
		147	22,51, 6,15	II 1,18,35	11,16,45	11,55,10 ♏
		148*	23, 3, 6,15	II 24,21,41,15	11,28,45	23,23,55
		149	23,15, 6,15	II 17,36,47,30	11,40,45	5, 4,40 ♐
	10.	150	23,27, 6,15	III 11, 3,53,45	11,52,45	16,57,25
		151**	23,39, 6,15	IV 4,43	12, 4,45	29, 2,10
		152	23,51, 6,15	III 28,34, 6,15	12,16,45	11,18,55 ♑
		153*	24, 3, 6,15	IV 22,37,12,30	12,28,45	23,47,40
		154	24,15, 6,15	IV 16,52,18,45	12,40,45	6,28,25 ♒
	15.	155	24,27, 6,15	V 11,19,25	12,52,45	19,21,10
		156*	24,39, 6,15	VI 5,58,31,15	13, 4,45	2,25,55 ♓
		157	24,41, 6,15	V 30,49,37,30	13,16,45	15,42,40
		158	25, 3, 6,15	VI 25,52,43,45	13,28,45	29,11,25
		159*	25,15, 6,15	VII 21, 7,50	13,40,45	12,52,10 ♈
Rev.	−14.	160	25,27, 6,15	VII 16,34,56,15	13,52,45	26,44,55
		161*	25,25	VIII 11,59,56,15	14, 4,40	10,49,35 ♉
		162	25,13	VIII 7,12,56,15	13,52,40	24,42,15
		163	25, 1	IX 2,13,56,15	13,40,40	8,22,55 ♊
	−10.	164*	24,49	IX 27, 2,56,15	13,28,40	21,51,35
		165	24,37	IX 21,39,56,15	13,16,40	5, 8,15 ♋
		166	24,25	X 16, 4,56,15	13, 4,40	18,12,55
		167*	24,13	XI 10,17,56,15	12,52,40	1, 5,35 ♌
		168	24, 1	XI 4,18,56,15	12,40,40	13,46,15
	−5.	169	23,49	XI 28, 7,56,15	12,28,40	26,14,55
		170**	23,37	XI 21,44,56,15	12,16,40	8,31,35 ♍
		171	23,25	XII 15, 9,56,15	12, 4,40	20,36,15
		172*	23,13	XII$_2$ 8,22,56,15	11,52,40	2,28,55 ♎
		174	23, 1	I 1,23,56,15	11,40,40	14, 9,35
	0.	175*	22,49	I 24,12,56,15	11,28,40	25,38,15
		176	22,45,46,15	I 16,58,42,30	11,16,40	6,54,55 ♏
		177	22,57,46,15	II 9,56,28,45	11,23,25	18,18,20
		178*	23, 9,46,15	III 3, 6,15	11,35,25	29,53,45
		179	23,21,46,15	II 26,28, 1,15	11,47,25	11,41,10 ♐
	5.	180*	23,33,46,15	III 20, 1,47,30	11,59,25	23,40,35
		181	23,45,46,15	III 13,47,33,45	12,11,25	5,52 ♑
		182	23,57,46,15	IV 7,45,20	12,23,25	18,15,25

Both columns have the same period

$$P = 28;26,40 = \frac{4,16}{9} \qquad \Pi = 4,16 \qquad Z = 9. \tag{1b}$$

An accuracy of three or four sexagesimal digits is, of course, not the result of highly accurate measurements but the consequence of arithmetical conditions as will soon become evident (cf. p. 386).

Significance of P

As we have seen, column IV of our text lists the differences $\Delta\lambda$ between the positions λ, given in column V, of consecutive oppositions of Saturn. Using the previously adopted terminology[5] we can say that column IV gives the "*synodic arcs*" in their proper order. We also know that these values form a linear zigzag function for equidistant values of the independent variable. Consequently the independent variable here is the serial number n of the consecutive events to which the synodic arcs belong. Similarly we can say that column II gives the corresponding "*synodic times*" $\Delta\tau$ and we find that also the $\Delta\tau$ form a linear zigzag function with respect to the number n of occurrences.

These "*true*" synodic arcs and times vary around mean values given by μ in (1 a) and henceforth called "*mean*" synodic arcs and times, in our case

$$\overline{\Delta\lambda}=12;39,22,30^\circ, \qquad \overline{\Delta\tau}=24;6,43,7,30^{\mathrm{r}}. \tag{2}$$

If we assume that the value of a true synodic arc depends only on its position in the ecliptic then it is clear that the addition of Π consecutive true synodic arcs, i.e. $\sum_1^{\Pi}\Delta\lambda_i$, must result in the return to the same λ. The same assumption implies that each of the Z waves contained in a sequence of Π synodic arcs corresponds to one circuit of the ecliptic. Hence

$$\sum_1^{\Pi}\Delta\lambda_i=Z\cdot 6,0. \tag{3}$$

It is a plausible conjecture that the same result would be reached if we simply add Π mean synodic arcs, i.e., that

$$\sum_1^{\Pi}\Delta\lambda_i=\Pi\overline{\Delta\lambda}. \tag{4}$$

If we can show that (4) is correct then (3) and (4) would give

$$\Pi\overline{\Delta\lambda}=Z\cdot 6,0 \tag{4a}$$

or with (1 b)

$$P=\frac{6,0}{\overline{\Delta\lambda}}. \tag{5}$$

This is exactly the same relation which we found for the period P of the synodic arcs in System A.[6] Hence this parallelism between the fundamental concepts in the two systems hinges on the validity of the equation (4).

We shall now prove that (4) is indeed always correct if and only if Π is an even number. The proof is such that in its major part it lies fully within the methodology of Babylonian astronomy. Only in its last part which concerns the influence of an asymmetry must we proceed in a typically modern fashion. A little experimenting with the actual numerical material would, however, also show the ancient computer that the deviation from the exact relation (4) would be negligible.

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[5] Above p. 373.

[6] Cf. p. 377 (9).

We assume first that

(a) Π is even: $\Pi = 2a$

and in addition that one of the Π different true synodic arcs $\Delta\lambda_i$ actually has the value of the mean synodic arc $\overline{\Delta\lambda} = \mu$, where μ is the mean value $1/2(M+m)$ of the linear zigzag function. Since we may start the counting of the occurrences with this specific arc our second assumption can be formulated as

(b) $\Delta\lambda_1 = \Delta\lambda_{\Pi+1} = \overline{\Delta\lambda}$.

If d is the difference of our zigzag function then we know that all synodic arcs in positions symmetric to $\Delta\lambda_1 = \Delta\lambda_{\Pi+1}$ deviate from $\overline{\Delta\lambda}$ by the same absolute amount but in opposite direction:

$$\Delta\lambda_2 = \overline{\Delta\lambda} \pm d \qquad \Delta\lambda_\Pi = \overline{\Delta\lambda} \mp d$$

etc. Hence

$$\Delta\lambda_{k+2} + \Delta\lambda_{\Pi-k} = 2\overline{\Delta\lambda}$$

for $k = 0, 1, \ldots, a-2$. The last of these $a-1$ pairs which add up to $2\overline{\Delta\lambda}$ is

$$\Delta\lambda_a + \Delta\lambda_{a+2} = 2\overline{\Delta\lambda}$$

leaving as single arc $\Delta\lambda_{a+1}$ which must, by reason of symmetry, satisfy

$$\Delta\lambda_{a+1} = \overline{\Delta\lambda}$$

(of course on a down-going branch if $\Delta\lambda_1 = \overline{\Delta\lambda}$ was on an upgoing branch, and vice versa). Hence we find for the total

$$\sum_1^\Pi \Delta\lambda_i = \overline{\Delta\lambda} + (a-1)2\overline{\Delta\lambda} + \overline{\Delta\lambda} = 2a\overline{\Delta\lambda} = \Pi\overline{\Delta\lambda}$$

which shows that (4) is correct under the assumptions (a) and (b).

Secondly we consider the case

(c) Π is odd: $\Pi = 2a+1$,

again combined with

(b) $\Delta\lambda_1 = \Delta\lambda_{\Pi+1} = \overline{\Delta\lambda}$.

As before

$$\Delta\lambda_{k+2} + \Delta\lambda_{\Pi-k} = 2\overline{\Delta\lambda}$$

but now with $k = 0, 1, \ldots, a-1$. Hence the final pair is

$$\Delta\lambda_{a+1} + \Delta\lambda_{a+2} = 2\overline{\Delta\lambda}$$

which completes the sequence of the $\Delta\lambda_i$. Hence

$$\sum_1^\Pi \Delta\lambda_i = \overline{\Delta\lambda} + a \cdot 2\overline{\Delta\lambda} = (2a+1)\overline{\Delta\lambda} = \Pi\overline{\Delta\lambda}$$

again in agreement with (4), assuming that (b) and (c) holds.

Our result is: the addition of Π consecutive true synodic arcs always yields the same value as $\Pi\overline{\Delta\lambda}$, provided that the value $\overline{\Delta\lambda} = 6{,}0/P$ actually occurs among the true synodic arcs. Our proof used only the obvious property of linear zigzag

functions that two elements in equidistant position with respect to the mean value μ yield a sum 2μ. All this was, of course, evident to a Babylonian computer.

What remains, however, is the less obvious statement that (4) remains valid also if $\overline{\Delta\lambda}$ does not occur among the Π true synodic arcs if Π is even and that the error remains very small if Π is odd.

In order to discuss this problem we introduce for the sake of simplicity values

$$y_i = \Delta\lambda_i - m$$

which form a linear zigzag function with the minimum 0, the maximum Δ, and with the mean value $1/2\,\Delta$. Hence

$$\sum_1^{\Pi} y_i = \sum_1^{\Pi} \Delta\lambda_i - \Pi m.$$

So far we have shown that $\sum_1^{\Pi} y_i = 1/2\,\Pi\Delta$ if, e.g., $y_1 = 1/2\,\Delta$. We shall start from this symmetric situation but it is our goal to evaluate the effect of a deviation from it.

Again we assume first that $\Pi = 2a$ and that $y_1 = 1/2\,\Delta$ occurs, e.g., on an upgoing branch.[7] We construct a circle of circumference 2Δ and reckon values y_i on upgoing branches as arc lengths on the semicircle ABC (cf. Fig. 15); thus the point B represents $y_1 = 1/2\,\Delta$. The subsequent values $y_2, y_3, \ldots$ are represented by points always separated by an arc d. Until C these points belong to values from an upgoing branch (AC $= \Delta$). If we go with the same steps of length d beyond C we obtain representations of values from the next downgoing branch, provided that we measure the arc lengths again from A but now in the direction ADC.

If we do this with all Π values y_i in one number period we cover the whole circumference Z times before we return to $y_1 = y_{\Pi+1}$. It is also clear that these points define a regular polygon with Π sides on our circle with a distance $\frac{2\Delta}{\Pi} = \frac{2\Delta}{PZ} = \frac{d}{Z}$ between adjacent (but no longer consecutively numbered) points.

Let us now assume that $y_1 = 1/2\,\Delta + \varepsilon$. Obviously only such values of ε are of interest for which $0 < |\varepsilon| \leqq \frac{d}{2Z}$. The effect of this increment can be visualized by rotating the polygon in Fig. 15 by the amount ε, e.g. from B toward C if $\varepsilon > 0$. Every element y_r on the upgoing semicircle (ABC) gains ε by this rotation, but every y_s on the downgoing semicircle (ADC) loses ε. If Π is even, there are always equally many points on each semicircle and the rotation through ε has no effect on $\sum_1^{\Pi} y_i$ even if one point near C changes sides, because simultaneously another point crosses A in the opposite direction. Hence for even Π the relation (4) remains valid, regardless of whether or not $\overline{\Delta\lambda}$ is also one of the true synodic arcs.

If, however, Π is odd one semicircle must contain one more point than the other. It is therefore possible that a rotation which changes $y_1 = 1/2\,\Delta$ to $1/2\,\Delta + \varepsilon$ causes one point to change sides without a compensation at the diametrically

[7] Obviously $y_{a+1} = 1/2\,\Delta$ on a downgoing branch.

opposite boundary. Hence $\sum_{1}^{\Pi} y_i$ can be modified by an amount ε and (4) must be replaced by

$$\Pi\overline{\Delta\lambda}-\frac{d}{2Z}\leqq\sum_{1}^{\Pi}\Delta\lambda_i\leqq\Pi\overline{\Delta\lambda}+\frac{d}{2Z} \tag{6}$$

and similarly $P=2\Delta/d=\Pi/Z$ is not necessarily exactly the same as $6{,}0/\mu = 12{,}0/(M+m)$. But the maximal possible error $d/2Z$ is usually small since already d will be small in comparison with $\overline{\Delta\lambda}$.

If, however, (4) is valid (e.g. because Π is even) then we have not only per definition

$$P=\frac{2\Delta}{d}=\frac{2(M-m)}{d}$$

but also from (4)

$$P=\frac{6{,}0}{\overline{\Delta\lambda}}=\frac{6{,}0}{\mu}=6{,}0\,\frac{2}{M+m}$$

hence

$$d=\frac{1}{6{,}0}(M+m)(M-m). \tag{7}$$

In the case of our ephemeris for Saturn $\Pi=4{,}16$ is even (cf. above p. 381) and with (1a) substituted in (7)

$$\frac{1}{6{,}0}\,25;18{,}45\cdot 2;50{,}40=\frac{1{,}12}{6{,}0}=0;12=d$$

as it should be.

The fact that the parameters of our model satisfy exactly the identity (7) allows us some insight into the way that this model of the System B type could have been constructed. Obviously the most fundamental empirical data must have been Π and Z, based on direct countings during one or more sidereal periods of the planet.[8] Consequently the mean synodic arc

$$\overline{\Delta\lambda}=Z\cdot 6{,}0/\Pi$$

is fixed and therefore also

$$M+m=2\overline{\Delta\lambda} \quad\text{and}\quad P=\Pi/Z.$$

Then (7) tells us that only one more parameter can be chosen freely: either the amplitude $M-m$ for the synodic arcs or the difference d of the zigzag function.

Obviously it is of great practical importance for the computation of an ephemeris that d is a convenient round number. On the other hand $M-m$ must reflect at least a general experience about the range within which the synodic arcs can vary. It is evident that both conditions can only be reconciled by properly modifying the numbers in question.

Let us, for example, assume that one had established that $Z=9$ complete rotations of Θ through the ecliptic contained $\Pi=4{,}16$ synodic intervals (i.e. 257 oppositions, counting both ends); hence $P=4{,}16/9=28;26{,}40$ and $\overline{\Delta\lambda}=6{,}0/P \approx 12;40°$. Suppose furthermore that one had found that the true synodic arcs

[8] For the details of this part of the procedures cf. below p. 388.

could deviate from $\overline{\Delta\lambda}$ by about $\pm 1;30°$. Then $M-m=3°$ would lead with (7) to

$$d=\frac{3}{6,0}(M+m)=\frac{3}{3,0}\cdot\frac{M+m}{2}=\frac{1}{1,0}\overline{\Delta\lambda}=0;12,39,22,30°$$

using for $\overline{\Delta\lambda}$ the accurate value for $6,0/P$ as is necessary for the computation of an ephemeris. Such a value of d is, however, most inconvenient. But since

$$M-m=1/2\,Pd=14;13,20\,d$$

one has only to modify $M-m\approx 3°$ to $M-m=2;50,40°$ in order to get such a convenient difference as $d=0;12$. With this choice made, all parameters are now exactly fixed to the values shown in (1a) of p. 380.

The apparent high accuracy of the limits of the synodic arcs

$$m=11;14,2,30°, \qquad M=14;4,42,30°,$$

down to half seconds of arc, is clearly not the result of observations but the necessary result of adjustments made in order to satisfy arithmetical conveniences. In general it is a sound principle to suspect low observational accuracy whenever very refined numbers appear in the parameters of a model.

Finally let us compare a model of the System B type with System A. In System A the mean synodic arc is always exactly $6,0/P$ regardless whether Π is even or odd.[9] Furthermore the degree of freedom in the design of step functions is greater by far with $2k$ parameters α_i and w_i at one's disposal. It is therefore by no means evident which one of the two main planetary models is historically more "primitive," even if in the lunar theory System B seems to be a refinement of System A.

The "Greek-Letter Phenomena"

The ephemeris which served us as an example in the preceding discussion is only concerned with longitudes and moments of one specific planetary phase, the so-called opposition Θ. The rest of the planetary motion, e.g. retrogradations or stretches of invisibility, are of no concern to such an ephemeris. This situation is characteristic for large classes of Babylonian planetary texts. They are concerned solely with one of the phases enumerated here and designated by Greek letters (cf. also Fig. 16):

$$\text{outer planets}\begin{cases}\Gamma & \text{reappearance after invisibility}\\ \Phi & \text{first station}\\ \Theta & \text{opposition}\\ \Psi & \text{second station}\\ \Omega & \text{disappearance}\end{cases}$$

$$\text{inner planets}\begin{cases}\Xi & \text{reappearance}\\ \Psi & \text{first station}\\ \Omega & \text{disappearance}\end{cases}\text{evening star}\qquad\begin{cases}\Gamma & \text{reappearance}\\ \Phi & \text{second station}\\ \Sigma & \text{disappearance}\end{cases}\text{morning star}$$

[9] Cf. above p. 377.

The coordinates of these phases provide the skeleton for the computation of intermediate positions of the planet, e.g. by means of interpolatory devices.[10] The first goal, however, is undoubtedly the determination of the "*Greek-letter phenomena*" as we shall call henceforth the above listed five, respectively six, phases.[11]

This central position of the Greek-letter phenomena is the most characteristic feature of Babylonian planetary theory. Each of the phases is handled as if it were a celestial body of its own – very much reminiscent of the Indian attitude toward the lunar nodes which are put in the same category with the other planets.[12] This approach has the great advantage that each Greek-letter phenomenon progresses with fair regularity in the ecliptic, free from the retrogradations of the planet itself.

It is noteworthy that these phases, except for the stations, are horizon phenomena,[13] which is quite in line with the general trend in early astronomy. One need only recall first and last visibility of the moon in connection with the lunar calendar,[14] or Γ, Θ, Ω of Sirius,[15] the use of seasonal hours or the role of the length of daylight,[16] ortive amplitudes,[17] etc.

We have already commented on the deceiving accuracy of numerical data.[18] Similarly deceiving is the inference from large numbers for planetary periods to a great extent of recorded observations. We shall see soon how larger periods were derived from linear combinations of smaller ones[19] without the use of far distant observations.[20] It should be added, however, that even in ancient Mesopotamia it must not have been trivial to establish exact time differences from dates in a true lunar calendar, especially with the absence of any era before the Seleucids.[21]

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[10] Cf. below p. 397.

[11] I have chosen ([1954]) this notation because it is short, adapted to mathematical symbols (e.g. $\lambda(\Gamma)$), and independent of the writer's language. Purposely I did not follow Schoch (Ammiz., p. 103) who called Ω and Γ of Venus "*e* last" and "*m* first", respectively. Van der Waerden took up Schoch's notation and speaks, e.g., about Mk = Morgenkehrpunkt for Φ. He also calls ([1957]) the Greek-letter phenomena "Kardinalpunkte" and replaces the commonly used term ephemeris by "Kardinaltafel" (probably because ephemeris literally should mean day by day positions).

[12] Together they are represented as the Navagrahas (i.e. the Nine Demons). Cf., e.g., the sculptures shown in Sivaramamurti [1950], Pl. VIII C and IX A.

[13] This also holds for Θ; cf. p. 363.

[14] Cf. II Intr. 3.

[15] Cf. II Intr. 3, 3.

[16] Cf. II Intr. 4, 2.

[17] Cf. the "Gates" of primitive (Palestinian) lunar theory: Neugebauer [1964], p. 51–58.

[18] Above p. 386.

[19] Below p. 391.

[20] Even at the great Islamic observatories of the latest period observational programs were not extended much beyond the shortest planetary periods, e.g., in Marâgha, in the 13th century, periods of 30 and 12 years, respectively; cf. Sayili, Observ. p. 204 and p. 276.

[21] The "Era Nabonassar" is nowhere attested in cuneiform sources and is in all probability the invention of Greek astronomers (Hipparchus?) for purely astronomical purposes – much like the "julian days" in modern astronomy.

§ 2. Periods and Mean Motions

In II A 1 (p. 382) we made the distinction between variable "true" synodic arcs and the "mean" synodic arc

$$\overline{\Delta\lambda}=\frac{6{,}0\cdot Z}{\Pi}=\frac{6{,}0}{P}. \tag{1}$$

In later sections we will have to discuss a variety of ways to form true synodic arcs. Also the relation between $\Delta\lambda$ and $\Delta\tau$ needs to be analyzed as well as the connection between different Greek-letter phenomena, e.g. between Φ, Θ, and Ψ which describe the retrograde motion of an outer planet.

At the moment all this will be left aside and the discussion will be restricted to the mean synodic arc of one specific phase. The latter we simply denote by F for which any of the Greek letters may be substituted.

Let $F_0, F_1, F_2, \ldots$ denote a sequence of consecutive occurrences of the phase F, separated from one another by the arc $\overline{\Delta\lambda}$. We disregard the way how the planet moves from F_k to F_{k+1} and consider only the resultant gain in longitude. That means that always

$$0<\overline{\Delta\lambda}<6{,}0 \quad \text{hence} \quad P>1. \tag{2}$$

We express this by saying that F (not the planet) moves in each step by the amount $\overline{\Delta\lambda}$, reaching after Π steps for the first time again the initial longitude of F_0, having traversed the ecliptic exactly Z times.

The astronomical significance of the Greek-letter phenomena lies, of course, in the relation of the planet to the sun. Restricting ourselves to mean motions we can say that each occurrence of F reproduces the same elongation of the planet from the mean sun. In whatever fashion the planet and the sun may move while F progresses by $\overline{\Delta\lambda}$, we know that the actual motion of the two bodies can only contain (if any) complete rotations beyond $\overline{\Delta\lambda}$. Calling the actual travel of sun and planet $\bar{S}$ and $\bar{\Lambda}$, respectively, we have

$$\left.\begin{aligned}\bar{S}&=\overline{\Delta\lambda}+i\cdot 6{,}0\\ \bar{\Lambda}&=\overline{\Delta\lambda}+k\cdot 6{,}0\end{aligned}\right\} i, k \text{ integers } \geqq 0. \tag{3}$$

The outer planets, which move slower than the sun, must be overtaken by the sun in order to repeat a specific phase, hence

$$i=k+1.$$

The inner planets always remain within a limited elongation from the sun, thus

$$i=k.$$

For Saturn, Jupiter, and Mercury we obviously have $k=0$. Mars, however, moves so fast that the sun overtakes the planet not before two years have passed; hence $i=2$, $k=1$. Similarly for Venus with 5 repetitions of the same phase in

8 years $i=k=1$. Thus

$$\begin{array}{lll} i=2, & k=1 & \text{for Mars} \\ i=1, & k=1 & \text{for Venus} \\ i=1, & k=0 & \text{for Saturn and Jupiter} \\ i=0, & k=0 & \text{for Mercury.} \end{array} \tag{4}$$

With i and k known we can now formulate relations which must hold for the characteristic periods.

Without restriction of generality we may assume that the phase F under consideration is a conjunction of the planet with the sun and that F_0 occurs at $\lambda=0$. Fig. 17 illustrates the case in which i years elapse between F_0 and the next occurrence F_1, assuming that the graph for the sun reaches i times the line $\lambda=360$. We can consider F_0A as the measure for one year. Since $\overline{\Delta\lambda}=6{,}0/P$ we also know that DF_1 represents the Pth part of one year. Hence

$$F_0F_1=\left(i+\frac{1}{P}\right)\text{ years}=\left(i+\frac{Z}{\Pi}\right)\text{ years.}$$

Since at F_Π the phase F occurred Π times between F_0 (excluded) and F_Π (included) we can say

$$\Pi \text{ occurrences of } F=(i\Pi+Z)\text{ years.} \tag{5}$$

We know from (3) that the planet traversed in the ecliptic a distance $\bar{\Lambda}=\overline{\Delta\lambda}+k\cdot 6{,}0$ between F_0 and F_1. Hence we have

$$\Pi\bar{\Lambda}=\Pi\overline{\Delta\lambda}+k\Pi\cdot 6{,}0°$$

from F_0 to F_Π. But with (1)

$$\Pi\overline{\Delta\lambda}=Z\cdot 6{,}0°$$

and thus

$$\Pi\bar{\Lambda}=(k\Pi+Z)\cdot 6{,}0°=(k\Pi+Z)\text{ sidereal rot. of the planet.}$$

Combining this with (5) we have shown that

$$\Pi \text{ occur. of } F=(i\Pi+Z)\text{ years}=(k\Pi+Z)\text{ sid. rot. of planet} \tag{6}$$

where all numbers are integers.

If we add to the number Π of occurrences the number $k\Pi+Z$ of the corresponding sidereal rotations of the planet we obtain a number $(k+1)\Pi+Z$ which, for an outer planet according to (4), is the same as $i\Pi+Z$, i.e., according to (6), the number of years. Thus it holds for an outer planet

$$\text{number of occurrences}+\text{number of sidereal rotations}=\text{number of years.} \tag{7}$$

This is the well-known relation which underlies the Greek epicyclic models for the outer planets.[1]

One can use (6) to derive additional relations which may serve to find the mean synodic arc $\overline{\Delta\lambda}$ or the period $P=6{,}0/\overline{\Delta\lambda}$ from the simple counting of occurrences of a given phase, of years, and of the number of the corresponding sidereal rotations of the planet between occurrences at the same place of the ecliptic.

[1] Cf. above p. 170 (1) and Fig. 158.

Assuming exact returns we know that the number of occurrences represents Π. Then (6) tells us that in this interval

$$\text{number of sidereal years} = Z + i\Pi$$
$$\text{number of sid. rot. of the planet} = Z + k\Pi.$$

Since both i and k are known for each planet (cf. (4)) we have two ways to express Z for each planet, hence also two ways to find $\overline{\Delta\lambda} = 6{,}0 \cdot Z/\Pi$:

$$\overline{\Delta\lambda} = 6{,}0 \frac{\text{no. of sid. rot.}}{\text{no. of occur.}} - i \tag{8a}$$

or

$$\overline{\Delta\lambda} = 6{,}0 \frac{\text{no. of sid. rot. of plan.}}{\text{no. of occur.}} - k. \tag{8b}$$

Similarly, if one wishes to determine the period P – e.g. for the construction of a linear zigzag function as described above p. 380 – one has

$$P = \frac{\text{no. of occur.}}{\text{no. of sid. y.} - i \cdot \text{no. of occur.}} \tag{9a}$$

or

$$P = \frac{\text{no. of occur.}}{\text{no. of sid. rot. of plan.} - k \cdot \text{no. of occur.}}. \tag{9b}$$

For the inner planets (8) and (4) give for $P = 6{,}0/\overline{\Delta\lambda}$

$$\begin{aligned} P &= \frac{\text{no. of occur.}}{\text{no. of years}} \quad \text{for Mercury} \\ P &= \frac{\text{no. of occur.}}{\text{no. of years} - \text{no. of occur.}} \quad \text{for Venus.} \end{aligned} \tag{9c}$$

This shows that, e.g., the mere counting of the number of times a planet began its retrograde loops until a certain phase returns to the same (or nearly the same) place of the ecliptic suffices for the determination of the mean synodic arc, and hence of P, without the use of any instruments and without needing to determine accurately the moment when the planet is stationary since one knows beforehand that the number of elapsed years must be an integer.

Numerical Data

For the outer planets (cf. (6), p. 389):

$$\begin{aligned} &\text{Saturn: } 4{,}16\ (=256)\ \text{occur.} = 9\ \text{sid. rot.} = 4{,}25\ (=265)\ \text{years} \\ &\text{Jupiter: } 6{,}31\ (=391)\ \text{occur.} = 36\ \text{sid. rot.} = 7,\ 7\ (=427)\ \text{years} \\ &\text{Mars: } 2{,}13\ (=133)\ \text{occur.} = 2{,}31\ (=151)\ \text{sid. rot.} = 4{,}44\ (=284)\ \text{years.} \end{aligned} \tag{10a}$$

For the inner planets:

$$\begin{aligned} &\text{Venus: } 12,\ 0\ (=720)\ \text{occur.} = 19{,}11\ (=1151)\ \text{years} \\ &\text{Mercury } (\Xi)\text{: } 25{,}13\ (=1513)\ \text{occur.} = 8,\ 0\ (=480)\ \text{years.} \end{aligned} \tag{10b}$$

From these numbers one finds with (9) and (4):

	P	$\overline{\Delta\lambda}$	
Saturn	$\frac{4,16}{9} = 28;26,40$	$\frac{6,45}{32} = 12;39,22,30°$	
Jupiter	$\frac{6,31}{36} = 10;51,40$	$\frac{3,36,0}{6,31} \approx 33;\ 8,44,48,\ldots$	
Mars	$\frac{2,13}{18} = \ 7;23,20$	$\frac{1,48,0}{2,13} \approx 48;43,18,29,\ldots$	(11)
Venus	$\frac{12,0}{7,11} \approx \ 1;40,13,\ldots$	$\frac{7,11}{2} = \ 3,35;30$	
Mercury	$\frac{25,13}{8,0} = \ 3;\ 9,\ 7,30$	$\frac{48,0,0}{25,13} \approx \ 1,54;12,36,38,\ldots$	

The periods (10) will be called the "*exact periods.*" These numbers seem to imply that the corresponding ephemerides should not produce an accumulative error for centuries. However, such huge intervals do not presuppose correspondingly long periods of observations. We know, e.g., that the 427-year period of Jupiter was derived from two much shorter ones which are admittedly only approximative. Procedure texts enumerate several different periods of Jupiter, the first two of 12 years and 71 years which are said to require corrections of $+5°$ and $-6°$, respectively, for a return to the same sidereal longitude[2]. Consequently a perfect return should be expected after

$$6 \cdot 12 + 5 \cdot 71 = 427 \text{ years.} \tag{12}$$

This shows how two periods which need less than one century of recorded planetary positions, e.g. with respect to fixed stars,[3] were utilized to construct a period which should be "exact" for more than four centuries[4].

Similar smaller periods are also known for the other planets and will be discussed in each individual case[5]. They also appear repeatedly in Greek astronomy[6] and astrology and constitute the most definite proof for Babylonian influence during the hellenistic period. For example the exact periods (10) are quoted by Psellus[7] ($\approx$A.D. 1018 to 1078) who took them from Rhetorius[8] (an astrological author of the beginning of the 6th cent.) who took them from Antiochus[9] (another astrologer, about 200 A.D.).

[2] We shall return to the details below p. 442.
[3] Extant in "Normal Star Almanacs"; cf. below p. 555.
[4] We shall see that these smaller periods need not to be exactly of the same character; the 71-year period, e.g., is used in the "Goal year texts" as restoring synodic phenomena, while the 83-year period is more accurate for sidereal returns; cf. below p. 554f.
[5] Cf. also ACT II, p. 283.
[6] Cf. p. 151.
[7] Boll [1898].
[8] CCAG 1, p. 163, 19.
[9] Cf. Tannery, Mém. Sci. 4, p. 265.

§ 3. System A

In our general introduction[1] we have given a description of the arrangement for models of "System A," i.e. models for which the synodic arcs and times are stretchwise constant on the ecliptic. As an example we analyzed the solar motion based on a two zone model. Similarly we now give data for Jupiter, one model ("A") operating with two zones, another ("A′") with four zones.

Jupiter, System A

The ephemeris which is transliterated in Table 7[2] covers the years S.E. 113 to 173 (−198 to −138) and was written in Uruk in S.E. 118 (−193).[3] The phenomenon in question is the first station (Φ) as is explicitly stated in the first three lines of obv. IV and easily confirmed by comparison with modern tables.

The longitudes in column IV suggest a System A pattern with

$$w_1 = 36°, \quad w_2 = 30° \tag{1a}$$

on two arcs α_1 and α_2 covering the ecliptic. What needs to be shown is only the existence of discontinuities at fixed longitudes. Indeed, to (1a) belong as transition coefficients

$$c_{1,2} = -0;10 = -1/6, \quad c_{2,1} = 0;12 = 1/5. \tag{1b}$$

From the text we take the synodic arcs w' which separate w_1 from w_2 and compute the arcs

$$r = (w_2 - w')/c_{1,2}$$

which still belong to α_1.[4] If we add r to the longitude $\lambda(\Phi)$ of the endpoint of w' in α_1 we obtain[5] $\lambda(\downarrow)$:

	w'	$w'-w_2$	r	$\lambda(\Phi)$	$\lambda(\downarrow)$	
Obv. 5/6	33;49	3;49	22;54	♊ 2;6	♊25	
16/17	32;59	2;59	17;54	7;6	25	
27/28	32; 9	2; 9	12;54	12;6	25	(2)
Rev. 3/4	31;19	1;19	7;54	17;6	25	
14/15	30;29	0;29	2;54	22;6	25	

i.e. always ♊25. Similarly one shows that $\lambda(\uparrow) = $ ♐0. Hence

$\alpha_1 = 3{,}25°$	from ♐ 0 to ♊25	$w_1 = 36°$	fast arc	(3)
$\alpha_2 = 2{,}35$	from ♊25 to ♐ 0	$w_2 = 30$	slow arc	

and

$$P = \frac{\alpha_1}{w_1} + \frac{\alpha_2}{w_2} = 5;41{,}40 + 5;10 = 10;51{,}40 = \frac{6{,}31}{36} \qquad \Pi = 6{,}31 \quad Z = 36. \tag{4}$$

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[1] Above p. 375ff.

[2] ACT 600; cf. for details ACT II, p. 339 and III, Pl. 176. The transliteration in Table 7 follows the same principle as in Table 6 for which cf. above p. 380, note 1.

[3] ACT, p. 17 colophon L.

[4] Cf. p. 376 (3b).

[5] Note the "linear saw functions" formed by w', r, and $\lambda(\Phi)$. Their period is found to be $P_0 = 7;12 = 36/5$.

Table 7

Obv.	I	II	III		IV
1.	113**	48, 5,10	I	28,41,40 ♃	8, 6 ♑ Φ
	114	48, 5,10	II	16,46,50 ♃	14, 6 ♒ Φ
	115*	48, 5,10	IV	4,52	20, 6 ♓ Φ
	116	48, 5,10	IV	22,57,10	26, 6 ♈
5.	117	48, 5,10	VI	11, 2,20	2, 6 ♊
	118*	45,54,10	VII	26,56,30	5,55 ♋
	119	42, 5,10	VIII	9, 1,40	5,55 ♌
	120	42, 5,10	IX	21, 6,50	5,55 ♍
	121*	42, 5,10	XI	3,12	5,55 ♎
10.	122	42, 5,10	XI	15,17,10	5,55 ♏
	123*	43,16,10	XII	28,33,20	7, 6 ♐
	125	48, 5,10	I	16,38,30	13, 6 ♑
	126*	48, 5,10	III	4,43,40	19, 6 ♒
	127	48, 5,10	III	22,48,50	25, 6 ♓
15.	128	48, 5,10	V	10,54	1, 6 ♉
	129*	48, 5,10	VI	28,59,10	7, 6 ♊
	130	45, 4,10	VII	14, 3,20	10, 5 ♋
	131	42, 5,10	VIII	26, 8,30	10, 5 ♌
	132**	42, 5,10	IX	8,13,40	10, 5 ♍
20.	133	42, 5,10	X	20,18,50	10, 5 ♎
	134*	42, 5,10	XII	2,24	10, 5 ♏
	135	44, 6,10	XII	16,30,10	12, 6 ♐
	137*	48, 5,10	II	4,35,20	18, 6 ♑
	138	48, 5,10	II	22,40,30	24, 6 ♒
25.	139	48, 5,10	IV	10,45,40	30, 6 ♓
	140*	48, 5,10	V	28,50,50	6, 6 ♉
	141	48, 5,10	VI	16,56	12, 6 ♊
	142*	44,14,10	VIII	1,10,10	14,15 ♋
	143	42, 5,10	VIII	13,15,20	14,15 ♌
30.	144	42, 5,10	IX	25,20,30	14,15 ♍
	145*	42, 5,10	XI	7,25,40	14,15 ♎
	146	42, 5,10	XI	19,30,50	14,15 ♏
	148*	44,56,10	I	4,27	17, 6 ♐
	149	48, 5,10	I	22,32,10	23, 6 ♑
35.	150	48, 5,10	III	10,37,20	29, 6 ♒

Rev.	I	II	III		IV
1.	151**	48, 5,10	IV	28,42,30	5, 6 ♈
	152	48, 5,10	V	16,47,40	11, 6 ♉
	153*	48, 5,10	VII	4,52,50	17, 6 ♊
	154	43,24,10	VII	18,17	18,25 ♋
5.	155	42, 5,10	VIII	30,22,10	18,25 ♌
	156*	42, 5,10	X	12,27,20	18,25 ♍
	157	42, 5,10	X	24,32,30	18,25 ♎
	158	42, 5,10	XII	6,37,40	18,25 ♏
	159*	45,46,10	XII_2	22,23,50	22, 6 ♐
10.	161*	48, 5,10	II	10,29	28, 6 ♑
	162	48, 5,10	II	28,34,10	4, 6 ♓
	163	48, 5,10	IV	16,39,20	10, 6 ♈
	164*	48, 5,10	VI	4,44,30	16, 6 ♉
	165	48, 5,10	VI	22,49,40	22, 6 ♊
15.	166	42,34,10	VIII	5,23,50	22,35 ♋
	167*	42, 5,10	IX	17,29	22,35 ♌
	168	42, 5,10	IX	29,34,10	22,35 ♍
	169	42, 5,10	XI	11,39,20	22,35 ♎
	170**	42, 5,10	XI	23,44,30	22,35 ♏
20.	172*	46,36,10	I	10,20,40	27, 6 ♐
	173	48, 5,10	I	28,25,50	3, 6 ♒

As it should be, this value of P is the same as the value obtained from the mean synodic arc.[6]

Column II gives the differences for the dates in column I and III, in particular

$$\begin{aligned} \Delta\tau_1 &= 48;5,10^{\tau} \quad \text{on the fast arc} \\ \Delta\tau_2 &= 42;5,10^{\tau} \quad \text{on the slow arc.} \end{aligned} \tag{5}$$

Both values are numerically obtainable from w_1 and w_2 by adding 12;5,10. The same holds for the transition values, as is seen, e.g., from (2):

Obv. 5/6	$w'=33;49$	$\Delta\tau=45;54,10$
16/17	32;59	45; 4,10

[6] Above p. 391 (11).

etc. Hence we have the general rule

$$\Delta\tau = \Delta\lambda^{\tau} + 12;5,10^{\tau} \tag{6}$$

where $\Delta\lambda^{\tau}$ indicates the same number of tithis as $\Delta\lambda$ contains degrees. A justification for this rule will be given in § 4 (below p. 396).

Jupiter, System A′

It is evident that an ephemeris computed with a System A pattern will show greatest deviations from observable synodic arcs in the neighbourhood of the discontinuities. By introducing arcs with intermediate values for the synodic arcs a smoother distribution of the true $\Delta\lambda$ can be achieved. This is obviously the purpose of a four-zone model for Jupiter which is based on the following parameters

$$\begin{array}{llllll} \alpha_1 = 2,0° & \text{from ♋ 9} & \text{to} & \text{♏ 9} & w_1 = 30° \\ \alpha_2 = 53 & \text{from ♏ 9} & \text{to} & \text{♑ 2} & w_2 = 33;45 \\ \alpha_3 = 2,15 & \text{from ♑ 2} & \text{to} & \text{♉ 17} & w_3 = 36 \\ \alpha_4 = 52 & \text{from ♉ 17} & \text{to} & \text{♋ 9} & w_4 = 33;45. \end{array} \tag{7}$$

The transition coefficients are

$$\begin{array}{ll} c_{1,2} = \dfrac{3;45}{30} = 0;7,30 = 1/8 & c_{3,4} = -\dfrac{2;15}{36} = -0;3,45 = -1/16 \\ c_{2,3} = \dfrac{2;15}{33;45} = 0;4 = 1/15 & c_{4,1} = -\dfrac{3;45}{33;45} = -0;6,40 = -1/9. \end{array} \tag{8}$$

The period is the same as in System A (cf. (4)):

$$P = \sum \frac{\alpha_i}{w_i} = \frac{2,0}{30} + \frac{1,45}{33;45} + \frac{2,15}{36} = 4 + 3;6,40 + 3;45 = 10;51,40. \tag{9}$$

There is a slight asymmetry in the distribution of the zones as given by (7), certainly caused only by arithmetical reasons in order to obtain exactly the proper value of P. Equally important for the actual computation of an ephemeris is the representation of all synodic arcs by finite sexagesimal expressions. This requires the same condition not only for the w' but also for the $c_{i,i+1}$; hence the w_i must be "regular" numbers in order that their reciprocals be finite. This is indeed the case for the parameters in (7) and (3).

§ 4. Dates

We have seen (II A 2) how one can find by mere counting of planetary phases a "mean synodic arc" $\overline{\Delta\lambda}$ which represents the mean value of the progress in longitude of the phenomenon in question. It is natural to associate with this $\overline{\Delta\lambda}$ a corresponding "*mean synodic time*" $\overline{\Delta t}$ such that $\Pi \cdot \overline{\Delta t}$ covers exactly the time between the occurrences F_0 and F_{Π}. Since this time is known to amount to $Z + i\Pi$

years[1] one would simply have

$$\overline{\Delta t} = \frac{Z}{\Pi} + i = \left(\frac{1}{P} + i\right) \text{ years.} \tag{1}$$

Actually, however, the dates of the ephemerides are expressed in terms of the lunar calendar, years, months, and tithis, and this compels us to use the same units also for the synodic times. Hence we shall say that

$$1 \text{ year} = (6{,}0 + e)^{\tau} \tag{2}$$

where we call e the "epact," i.e. the excess of the solar year over the lunar year, i.e. over $12^{m} = 6{,}0^{\tau}$. Measuring consequently also the synodic time in tithis we shall write $\overline{\Delta\tau}$ instead of $\overline{\Delta t}$; hence from (1):

$$\overline{\Delta\tau} = \left(\frac{1}{P} + i\right)(6{,}0 + e)^{\tau} = \frac{6{,}0}{P} + \frac{e}{P} + i(6{,}0 + e),$$

all terms counted in tithis. But we have shown (II A 1) that either accurately or very nearly

$$\frac{6{,}0}{P} = \overline{\Delta\lambda}$$

thus

$$\overline{\Delta\tau} = \overline{\Delta\lambda}^{\tau} + \left(\frac{e}{P} + i \text{ years}\right)^{\tau} \tag{3}$$

with $\overline{\Delta\lambda}^{\tau}$ numerically the same as $\overline{\Delta\lambda}$ in degrees.

For the ephemeris of Saturn (Θ) we have found[2] as mean value for the synodic arc

$$\overline{\Delta\lambda} = 12;39,22,30 = \mu \tag{4a}$$

which is based on the exact period relations for this planet.[3] The corresponding mean value for the dates[4] will therefore be the mean synodic time

$$\overline{\Delta\tau} = 6{,}0^{\tau} + 24;6,43,7,30^{\tau} \tag{4b}$$

where the term $6{,}0^{\tau}$ takes into account the increase of the year number in column I of the ephemeris. According to (3) and with $i = 1$[5] we should now have

$$\frac{e}{P} + 6{,}0 + e = \overline{\Delta\tau} - \overline{\Delta\lambda}^{\tau} = 6{,}0^{\tau} + 11;27,20,37,30^{\tau}.$$

Since[6]

$$P = 28;26,40 = \frac{4{,}16}{9}$$

we find from

$$e\left(\frac{1}{P} + 1\right) = 11;27,20,37,30 = e\,\frac{4{,}25}{4{,}16}$$

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[1] Cf. above p. 389.

[2] Above p. 380 (1a).

[3] Above p. 381 (1b) and p. 390f. (10a) and (11).

[4] Again p. 380 (1a).

[5] Cf. p. 389 (4).

[6] Cf. note 3.

that (exactly)

$$e = 11;4^{\mathrm{r}}. \tag{5a}$$

This shows that the determination of the parameter $\overline{\Delta\tau}$ was based on the following definition of "year"

$$1 \text{ year} = 6,11;4^{\mathrm{r}} = 12;22,8^{\mathrm{m}}. \tag{5b}$$

This is exactly the same parameter which we have found as characterizing the model of the solar motion in the lunar System A.[7] We have here an important link between the luni-solar theory and the planetary theory, regardless of the "Systems".

We now turn from the mean values $\overline{\Delta\lambda}$ and $\overline{\Delta\tau}$ to the variable arcs and intervals, $\Delta\lambda$ and $\Delta\tau$, respectively. In an ephemeris for Jupiter we have seen[8] that always

$$\Delta\tau - \Delta\lambda^{\mathrm{r}} = 12;5,10^{\mathrm{r}} = \text{const.} \tag{6}$$

For the mean values we have found the same in (3):

$$\overline{\Delta\tau} - \overline{\Delta\lambda}^{\mathrm{r}} = c \qquad \text{with } c = \frac{e}{P} + i(6,0+e)^{\mathrm{r}}. \tag{7}$$

Obviously (6) is a generalization of (7) to the case of variable quantities.

It is not difficult to motivate (6). The time interval between two consecutive phases of the same type is the time of the solar travel,[9] hence $\left(i + \frac{\Delta\lambda}{6,0}\right)$ years, hence

$$\Delta\tau = \left(i + \frac{\Delta\lambda}{6,0}\right)(6,0+e)^{\mathrm{r}} = \Delta\lambda^{\mathrm{r}} + \frac{e\Delta\lambda}{6,0} + i(6,0+e)^{\mathrm{r}}.$$

Since $\Delta\lambda/6,0$ is in general different from $\overline{\Delta\lambda}/6,0 = 1/P$ it is not strictly correct to call $\Delta\tau - \Delta\lambda^{\mathrm{r}}$ constant, but the error is only

$$\frac{e}{6,0}|\Delta\lambda - \overline{\Delta\lambda}| = \frac{11;4}{6,0}|\Delta\lambda - \overline{\Delta\lambda}| = 0;1,50,40|\Delta\lambda - \overline{\Delta\lambda}|.$$

But

$$|\Delta\lambda - \overline{\Delta\lambda}| \leq |M - \overline{\Delta\lambda}| = \begin{cases} 1;25,20^{\mathrm{r}} & \text{for Saturn}^{10} \\ 4;53,15^{\mathrm{r}} & \text{for Jupiter}^{11}. \end{cases}$$

Hence the error for Saturn is at most $\approx 0;3^{\mathrm{r}}$, for Jupiter $\approx 0;10^{\mathrm{r}} \approx 4^{\mathrm{h}}$. Such an error is of no significance, in particular when one eventually identifies tithis with calendar dates.

Although the general principle for the computation of the dates is now clear and generally applied in System A we do not understand a modification of it in System B. In the ephemeris for Saturn, discussed above in II A 1 (cf. Table 6, p. 381), we find that the zigzag functions for $\Delta\lambda$ and $\Delta\tau$ have the same period[12]

[7] Above p. 378 (15 b).
[8] Above p. 394 (6).
[9] We always assume, of course, a mean velocity for the sun; the solar anomaly plays no role in the planetary theory.
[10] Above p. 380 (1 a), for $M-\mu$.
[11] Cf. below (p. 446) Jupiter System B with $M = 38;2$ $\mu = 33;8,45$ for $\overline{\Delta\lambda}$.
[12] Differences and amplitudes are the same.

as it should be. But a glance at Rev. II and IV lines 0 to 2 shows that the two functions are different in phase and computation shows indeed that $\Delta\tau$ is about 0;35 of an interval ahead of $\Delta\lambda$. This results in

$$\Delta\tau - \Delta\lambda^{\tau} = \begin{cases} 11;34,21,15^{\tau} & \text{on increasing branches} \\ 11;20,20^{\tau} & \text{on decreasing branches}^{13}. \end{cases}$$

The practical effect of this distinction between branches is again negligible but it is difficult to see why the two zigzag functions were not exactly coordinated, nor is it clear why different ephemerides operate with different phase differences.[14]

§ 5. Subdivision of the Synodic Arc; Daily Motion

Every ephemeris for any Greek-letter phenomenon can be extended indefinitely whenever one single pair λ and τ is given. Two consecutive lines are always separated by a whole synodic interval, $\Delta\lambda$ or $\Delta\tau$, respectively. Each such interval[1] contains, however, other Greek-letter phenomena, e.g. for an outer planet Φ, Θ, Ψ, and Ω between Γ_n and Γ_{n+1}. The question arises: what are the relations between the Greek-letter phenomena in their natural order within one synodic cycle? If there exist definite arithmetical rules for the transition from one phenomenon to the nearest of another kind we would have a situation similar to the one which we encountered with the solstice-, equinox-, and Sirius-dates[2]: the whole matrix of all longitudes and dates for all planetary phases would be known as soon as one single pair λ, τ is given. We shall see in the following (cf. below II A 7, 3 C) to what extent such a pattern was actually realized in the planetary ephemerides.

Finally a problem of a very different character presents itself. Suppose one has the complete list of all Greek-letter phenomena for any length of time. If one wishes to know, however, $\lambda(t)$ for arbitrarily given moments one has to bridge the gaps between the isolated Greek-letter phenomena. Though our sources are only of a very fragmentary character they fully suffice to demonstrate the existence of methods of interpolation which provide a transition between consecutive Greek-letter phenomena.

In principle one has thus reached the same completeness of information as obtained by the Greek planetary theory. But the emphasis of the latter lies in the accurate description of the continuous motion of the planet and the phase are only dealt with in a subsidiary fashion.[3] In contrast herewith the Babylonian theory produces as accurate elements the phases and leaves the intermediate points to interpolation. This alone suffices to show that the actual contact between the two major schools of ancient astronomy cannot have been very close, in spite of geographical and chronological proximity.

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[13] The mean value of these two differences is the expected $\overline{\Delta\tau} - \overline{\Delta\lambda}^{\tau} = 11;27,20,37,30^{\tau}$ (cf. above p. 395).

[14] ACT 704, e.g., has a $\Delta\tau$ that follows $\Delta\lambda$ at a distance of 0;1,42,30 of an interval.

[1] For Mars and Venus this should be understood as $\Delta\lambda + 6,0°$.

[2] Cf. p. 365.

[3] Cf. I C 6 and I C 8.

1. Subdivision of the Synodic Arc

A. Jupiter

We have ample textual evidence for ephemerides which concern the whole set of Greek-letter phenomena in their natural order. For example the Jupiter ephemeris ACT No. 611, written in Babylon, covered in each line the data for consecutive Γ, Φ, Θ, Ψ, Ω,[1] extending from S.E. 180 to 252, i.e. over a full 71-year period.[2] The computation follows System A′, described above p. 394.

There exist also ephemerides for individual phases, written on single tablets, e.g. a group of tablets from Uruk[3] with ephemerides of Jupiter for Ω, Φ, Θ, Ψ, respectively (System A) for the 60 years from S.E. 113 to 173 or beyond; lost is only the tablet for Γ which would complete the set.

Finally one has lists which give no dates but longitudes extended over a whole number period. Fragments of texts are preserved[4] which covered (on two tablets) 391 occurrences[5] for each of the five phases of Jupiter Ω, Γ, Φ, Θ, Ψ, using System A. This list probably started with Ω at ♊25, i.e. exactly at the beginning of the slow arc.[6]

These texts, and similarly the procedure texts, provide us with information about the subdivision of the synodic arcs. From the last mentioned list, e.g., one can directly read off the distances between the phases:

	♊25 to ♏30	♏30 to ♊25	
Ω to Γ	$+6°$	$+7;12°$	
Γ to Φ	$+16;15$	$+19;30$	
Φ to Θ	-5	-6	(1)
Θ to Ψ	-5	-6	
Ψ to Ω	$+17;45$	$+21;18$	
w	$30°$	$36°$	

The resulting synodic motion, e.g. from Ω to Ω, agrees with the parameters $w_1 = 30°$, $w_2 = 36°$ which are characteristic for System A. Nevertheless the ephemerides do not strictly follow this pattern. If one computes the above mentioned ephemerides from Uruk back to S.E. 108/109 one finds integer day numbers and simple fractions for the longitudes[7]:

Ω	S.E. 108 IV 4	♌ 9;30		
Φ	IX 4	♍ 1;45		
Θ	XI 5	♌ 27;20		(2)
Ψ	109 I 6	♌ 21;45.		

[1] The last two are now almost completely broken away.
[2] Cf. above p. 391 (12) and note 4 there.
[3] ACT No. 606, 600, 604, and 601, respectively.
[4] Published by Aaboe-Sachs [1966], Text D.
[5] Cf. p. 390 (10a).
[6] Cf. p. 392 (3). Restoration by Aaboe-Sachs [1966], p. 16. If the text began one line later it would be Γ at ♋1.
[7] This was discovered by P. Huber [1957], p. 277.

All these points belong to the slow arc; from Ω to Φ the progress is 22;15° in agreement with (1) which prescribes $6° + 16;15°$ for this arc. The retrogradation, however, is $-4;25° - 5;35° = -10°$ against $-5° - 5°$ in (1). In ACT No. 611 (a text following System A′) one has in the first line

Γ	S.E. 180*	VI 13	♍10	
Φ		X 16	♍26;15	(3)
Θ		XII 18	♍22;15	

hence again 16;15° from Γ to Φ as in (1) but now $-4°$ from Φ to Θ (and hence probably $-6°$ from Θ to Ψ).

Finally, to also quote rules from procedure texts, we give the following pattern for System A′[8]:

	♋9 to ♏9	♏ 9 to ♑2 ♉17 to ♋9	♑2 to ♉17	
Ω to Γ	+6;15°	+7; 1,52,30°	+7;30°	
Γ to Φ	+16;15	+18;16,52,30	+19;30	(4)
Φ to Ψ	−8;20	−9;22,30	−10	
Ψ to Ω	+15;50	+17;48,45	+19	
w	30°	33;45°	36°	

For reasons unknown the retrograde arcs shorter here than in System A.

The asymmetry of Θ with respect to Φ and Ψ in (2) and probably in (3)[9] shows that Θ is not the "opposition" in the strict sense of Greek or modern astronomy but that it corresponds to the "*akronychal rising*" of the planet. The planet is then just visible in the east shortly after sunset; the Babylonian term means in fact "opposition in the east."[10] It follows from this situation that Θ is nearer to Φ than to Ψ (cf. Fig. 18 which uses conveniently an epicycle model).

B. Mars

The ephemerides of this planet are based on the relations[1]

$$2,13 \text{ occurrences} = 2,31 \text{ sidereal rotations} = 4,44 \text{ years} \tag{1}$$

i.e.

$$\Pi = 2,13, \qquad Z = 18 \tag{2a}$$

or[2]

$$\Pi \text{ occurrences} = (2\Pi + Z) \text{ years.} \tag{2b}$$

Consequently

$$P = \frac{\Pi}{Z} = \frac{2,13}{18} = 7;23,20. \tag{2c}$$

[8] For the reconstruction of this pattern cf. ACT II, p. 312.

[9] We shall also find it for Mars; cf. below p. 401 (7).

[10] *Ana* ME-a *ina* kur, or similar (Sachs).

[1] Above p. 390 (10a).

[2] Above p. 389 (4) and (6).

Except for one fragment of an ephemeris of a System B type[3] (with the same periods) the texts are based on a System A model with six zones, each exactly two zodiacal signs in length (cf. Fig. 19):

i	α_i	w_i
1	♉ ♊	45°
2	♋ ♌	30
3	♍ ♎	40
4	♏ ♐	1,0
5	♑ ♒	1,30
6	♓ ♈	1,7;30.

(3)

The resulting period $P = \sum 1{,}0/w_i$ has the value (2c). The transition coefficients are[4]

$$\begin{array}{ll} -0;20 & \text{if entering ♉ or ♋} \\ +0;20 & \text{if entering ♍} \\ +0;30 & \text{if entering ♏ or ♑} \\ -0;15 & \text{if entering ♓} \end{array} \tag{4}$$

The subdivision of the synodic arcs as far as Ω, Γ, and Φ are concerned[5] follows the same principles which we have described for Jupiter and which hold also for Saturn. For Mars (and similarly for Mercury) a new type of connection between the remaining phases has been adopted. In the case of Mars, after Ω, Γ, and Φ are found in the usual fashion, an opposition Θ is derived not from the preceding Θ but from the longitude of the immediately preceding first station Φ. Similarly a second station Ψ is found from the preceding Θ and hence also depends on Φ. In a schematic form we can thus represent the relations between the phases of Mars as follows:

$$\begin{array}{lll} \Omega_1 & \Gamma_1 & \Phi_1 \rightarrow \Theta_1 \rightarrow \Psi_1 \\ \downarrow & \downarrow & \downarrow \\ \Omega_2 & \Gamma_2 & \Phi_2 \rightarrow \Theta_2 \rightarrow \Psi_2 \\ \downarrow & \downarrow & \downarrow \\ & \text{etc.} & \end{array} \tag{5}$$

In a pattern of this type we call Θ and Ψ "*satellites*" of Φ because Θ_n and Ψ_n depend only on Φ_n and not on Θ_{n-1} and Ψ_{n-1}.

For Mars we know of as many as four variants for the dependence of Θ on Φ although only one is actually attested in use among the few extant ephemerides. The rule for this procedure (called scheme "S") are as follows:

$$\begin{array}{lll} \text{if } \Phi \text{ lies in ♋} & \text{then } \Phi \rightarrow \Theta \text{ is} & -7;12^\circ \\ \text{in ♍ or ♉} & & -6;48 \\ \text{in ♏ or ♓} & & -6;24 \\ \text{in ♑} & & -6. \end{array} \tag{6a}$$

[3] Cf. below II A 7, 4 B.

[4] For the corresponding synodic arcs as function of λ cf. below Fig. 40, p. 1332.

[5] Important details for this part of the theory of Mars will be discussed below p. 406ff.

For the remaining signs linear interpolation has to be used. Hence

$$\begin{array}{lll} \text{if } \Phi \text{ occurs at } k^\circ \text{ of } ♌ \text{ then } \Phi \to \Theta \text{ is} & -(7;12-k\cdot 0;0,48)^\circ & \\ \qquad ♎ & -(6;48-k\cdot 0;0,48) & \\ \qquad ♐ & -(6;24-k\cdot 0;0,48) & \\ \qquad ♒ & -(6;\ 0+k\cdot 0;0,48) & (6\text{b}) \\ \qquad ♈ & -(6;24+k\cdot 0;0,48) & \\ \qquad ♊ & -(6;48+k\cdot 0;0,48). & \end{array}$$

Fig. 20 gives the graphical representation for these rules.[6]

For the second part of the retrograde arc we know, at least for scheme "S", that[7]

$$\Theta \to \Psi = 3/2\,(\Phi \to \Theta). \tag{7}$$

Hence one can compute $\Phi \to \Psi$ directly from (6a) and (6b) and finds:

Φ	$\Phi \to \Psi$	Φ	$\Phi \to \Psi$
♑	-15°	♋	-18°
♒ k°	$-(15+k\cdot 0;2)$	♌ k°	$-(18-k\cdot 0;2)$
♓	-16	♍	-17
♈ k°	$-(16+k\cdot 0;2)$	♎ k°	$-(17-k\cdot 0;2)$
♉	-17	♏	-16
♊ k°	$-(17+k\cdot 0;2)$	♐ k°	$-(16-k\cdot 0;2)$

Example:

$$\begin{array}{r} \Phi = ♌\,24;20 \\ -(18-24;20\cdot 0;2) = -17;11,20 \\ \hline \Psi = ♌\ 7;\ 8,40. \end{array}$$

C. Mercury

The ephemerides for Mercury consider only four phases: first (Γ) and last (Σ) visibility as morning star and first (Ξ) and last (Ω) visibility as evening star. Always two of these four are made "satellites" of the other two and accordingly we distinguish two "Systems," A_1 and A_2 (cf. Fig. 21). In A_1 the last visibilities depend on the preceding first visibilities, in A_2 the first visibilities are obtained from the preceding last visibilities. The two Systems are, however, not compatible in a strict mathematical sense. If one computes a sequence of Σ-s or Ω-s by means of System A_1 one does not find agreement with the synodic arcs as obtained by System A_2, and vice versa.

Also in the determination of the primary phases the two systems are greatly different. In A_1 three zones of synodic arcs are used for Γ and Ξ, while A_2 has four zones for Σ and Ω (cf. Fig. 22):

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6 For the other methods cf. below II A 7, 4 C and Fig. 41, p. 1332.

7 Cf. p. 1320, Fig. 18.

Syst. A_1 Γ:	$w_1 = 1{,}46°$	from	♌ 1	to	♑ 16	(1)
	$w_2 = 2{,}21;20$	from	♑ 16	to	♊ 0	
	$w_3 = 1{,}34;13{,}20$	from	♊ 0	to	♌ 1	
Ξ:	$w_1 = 2{,}40$	from	♋ 6	to	♎ 26	
	$w_2 = 1{,}46;40$	from	♎ 26	to	♓ 10	
	$w_3 = 1{,}36$	from	♓ 10	to	♋ 6	
Syst. A_2 Σ:	$w_1 = 1{,}47;46{,}40$	from	♋ 0	to	♎ 0	(2)
	$w_2 = 2,\ 9;20$	from	♎ 0	to	♑ 6	
	$w_3 = 1{,}37$	from	♑ 6	to	♈ 5	
	$w_4 = 2,\ 9;20$	from	♈ 5	to	♋ 0	
Ω:	$w_1 = 1{,}48;30$	from	♋ 0	to	♑ 0	
	$w_2 = 2,\ 0;33{,}20$	from	♑ 0	to	♓ 0	
	$w_3 = 1{,}48;30$	from	♓ 0	to	♉ 0	
	$w_4 = 2{,}15;37{,}30$	from	♉ 0	to	♋ 0	

For the periods $P = \sum \alpha_i / w_i$ computed with these parameters (α_i being the arc length of the individual zones) one finds four slightly different values:

$$P_\Gamma = \frac{44{,}33}{14{,}8} \approx 3;9{,}7{,}38,\ldots, \qquad P_\Sigma = \frac{20{,}33}{6{,}28} \approx 3;9{,}7{,}25,\ldots,$$
$$P_\Xi = \frac{25{,}13}{8{,}0} = 3;9{,}7{,}30, \qquad P_\Omega = \frac{11{,}24}{3{,}37} \approx 3;9{,}7{,}32,\ldots \tag{3}$$

all of which differ from the 46-year period

$$P = \frac{2{,}25}{46} \approx 3;9{,}7{,}49,\ldots \tag{4}$$

known from procedure texts and Goal-Year texts.[1] Above p. 390 (10b) we listed the value of the period P_Ξ as representative for Mercury because it is this value which also appears in Greek sources.[2] The differences between these values are, of course, without any practical interest; P_Ξ produces, e.g., in 46 years a deviation of only $+0;28{,}29,\ldots°$ in comparison with (4).

Before turning to the relations between different phenomena we remark that the great differences in the values for the synodic arcs shown in (1) and in (2) are far less outspoken in their final effect because they rarely remain uninfluenced by another zone. For example $w_1 = 2{,}40°$ in Ξ is greater than the whole arc $\alpha_1 = 1{,}50$ so that even for its first point the other end would reach 50° into α_2. These 50° require a reduction in the ratio $w_2/w_1 = 1{,}46;40/2{,}40 = 0;40$. Hence the greatest actual synodic arc (starting at ♋6) is only $1{,}50 + 0;40 \cdot 50 = 2{,}23;20$. For all points inside of α_1 the synodic arc decreases linearly with the distance from ♋6 until for ♎26 the value $w_2 = 1{,}46;40$ is reached which remains valid inside of α_2 until ♓$10 - 1{,}46;40 =$ ♏$23;20$ (cf. Fig. 23).

"Pushes". We have already remarked (p. 401) that in the ephemerides of Mercury two phases are treated as satellites. We call the arcs which lead from

[1] Cf. ACT No. 800 and Sachs [1948], p. 283, respectively.
[2] Cf. above p. 22, notes 7 to 9.

the independent phases to the satellites "*pushes.*" In System A_1 the pushes represent the lengths of the visible segments in the planet's motion, in System A_2 the pushes are the intervals of invisibility.

Our source material allows a complete reconstruction only for System A_1,[3] resulting in the following rules:

$\lambda(\Gamma)$ or $\lambda(\Xi)$	$\Gamma \rightarrow \Sigma$	$\Xi \rightarrow \Omega$	$\lambda(\Gamma)$ or $\lambda(\Xi)$	$\Gamma \rightarrow \Sigma$	$\Xi \rightarrow \Omega$
midpoint of ♈	12°	36°	midpoint of ♎	34°	14°
♉	14	42	♏	44	14
♊	18	46	♐	44	16
♋	22	42	♑	42	20
♌	26	36	♒	30	22
♍	30	22	♓	24	22

Fig. 24 gives the graphs for these pushes simultaneously with the synodic arcs for Γ and Ξ as function of λ.[4] A certain parallelism in the trend of $\Delta\lambda(\Gamma)$ and $\Gamma \rightarrow \Sigma$ on the one hand, and of $\Delta\lambda(\Xi)$ and $\Xi \rightarrow \Omega$ on the other hand, is quite obvious. Such a relation is not surprising since large intervals of visibility as morning star ($\Gamma \rightarrow \Sigma$) will correspond to short visibility as evening star. This is also reflected in the trend of the maximal elongation of the planet, shown here in Fig. 25 which is based on Alm. XII, 10.[5]

In System A_2 the pushes $\Sigma \rightarrow \Xi$ (invisibility at superior conjunction) are incompletely known. We are more fortunate with the interval $\Omega \rightarrow \Gamma$ which is of interest because it concerns the retrogradation of the planet at inferior conjunction. Fig. 26 shows that for the greater part of the ecliptic $\Delta\lambda$ is negative. In ♈ and ♉, however, $\Delta\lambda$ is positive which means that the planet gains in longitude slightly during invisibility. We shall see presently that this has to do with another peculiarity of Mercury's visibility. Pliny states[6] that Mercury does not become retrograde in ♉ and ♊ until ♋25°. It almost seems as if we had here a reflection of the Babylonian theory, interpreted in the form that the limits of invisibility are such that no retrogradation is observable.

"Passed by" Phenomena. Fig. 24, p. 1323 shows that the pushes $\Gamma \rightarrow \Sigma$ (visibility as morning star) have a minimum in ♈, the pushes $\Xi \rightarrow \Omega$ (visibility as evening star) in ♎ and ♏. Also the corresponding synodic arcs $\Delta\lambda(\Gamma)$ and $\Delta\lambda(\Xi)$ show in these regions comparatively small values. And in the same areas ephemerides and related texts[7] declare the "appearances" Γ and Ξ to have "passed by" – using the same term (dib = *etiq*) which is applied to non-occurring eclipses. This obviously implies that also the subsequent "disappearances" Σ and Ω do not take place. It is only for the sake of uninterrupted computation of ephemerides that longitudes and dates are assigned to events which actually do not take place.

[3] Including some variants for which cf. below p. 472.

[4] Except for the scale the latter curves are the same as in Fig. 23, p. 1322.

[5] Fig. 25 is repeated from Fig. 238, p. 1288 but changed to the scale of the present Fig. 24.

[6] Pliny, NH II, 77 (ed. Budé, Vol. 2, p. 33/34 and p. 165).

[7] For the details cf. Neugebauer [1951].

The limits for which such data indicate that a pair of phenomena has "passed by" are not accurately known to us but can be estimated from the actual texts within reasonably small errors as follows:

Γ omitted between ♈10 and ♉20, hence Σ between about ♈24 and ♊ 5

Ξ omitted between ♎ 0 and ♏ 5, hence Ω between about ♎18 and ♏30.

As we have just seen (p. 403) the pushes $\Omega \to \Gamma$ are ordinarily negative but show small positive values for Ω in ♈ and ♉ (cf. Fig. 26, p. 1324). Consequently Γ also will then be in most cases in these two signs and hence usually "pass by." This is perhaps the reason for introducing positive pushes $\Omega \to \Gamma$ although the planet was of course actually retrograde at least in a part of the period of invisibility.

In discussing the computation of the planetary phases in the Almagest (Book XIII, 8) we mentioned[8] that Ptolemy was familiar with the omission of the morning star phases of Mercury at the beginning of Taurus and of the evening star phases at the beginning of Scorpio. Ptolemy gave a correct explanation of these phenomena, caused by the great negative latitude of Mercury in these sections of the zodiac, combined with a small inclination of the ecliptic toward the horizon. This then prevents the planet reaching the visibility limit, even at maximum elongation from the sun.[9]

Exactly as in the Babylonian case the tables themselves (Alm. XIII, 10) give no indication that certain phases will be omitted. A modern user of ancient material who is not fully familiar with the theoretical background of the ancient computational techniques can easily be misled by the absence of warning signals or by their cryptic form. Schoch who unknowingly used bypassed phases of Mercury for his determination of this planet's arcus visionis[10] would have obtained more secure results had he simply accepted Ptolemy's empirical data.

2. Subdivision of the Synodic Time; Velocities

A. Summary; Jupiter

In II A 4 (above p. 395) we have seen how a mean synodic time interval $\overline{\Delta\tau}$ was derived from the mean synodic arc $\overline{\Delta\lambda}$ and how, by a light modification, the true synodic time also was associated with the true motion $\Delta\lambda$ between consecutive phases of the same kind. But the natural order of the phases leads to the problem of sectioning the synodic arcs as well as the synodic times.

From a purely formal viewpoint one might say that such a subdivision would only be needed for one single synodic arc because one complete sequence of consecutive Greek-letter phenomena would provide the initial conditions for all subsequent longitudes and dates. In other words one would need only one row in the matrix of all Greek-letter phenomena (cf., e.g., above p. 400 (5)) to compute all earlier and later positions and moments.

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[8] Above p. 241 and p. 255.

[9] Cf. Fig. 258, p. 1296.

[10] Cf. for this whole episode Neugebauer [1951].

Through procedure texts we know of slightly different methods to settle the problem of initial values. There we find simple patterns for time intervals and velocities for the transition from one phase to the next. Since the velocities are constant for each section the longitudes are always assumed to vary linearly between two consecutive phases. Occasionally, in order to make these transitions a little smoother, auxiliary "Greek-letters" were introduced, e.g. for Jupiter the points Γ' and Ω' at a distance of 30^τ from Γ and Ω, respectively. Then round time intervals were assumed for every transition from one Greek-letter to the next. In this way one has six intervals for Jupiter:

(a)	from Γ to Γ'	30^τ		
(b)	Γ' to Φ	3^m		
(c)	Φ to Ψ	4^m	retrograde	
(d)	Ψ to Ω'	3^m		(1)
(e)	Ω' to Ω	30^τ		
(f)	Ω to Γ	(30^τ).		

For the last interval (f), which is the interval of invisibility, the text gives no time, obviously because the total (a)$+\cdots+$(f) should be the synodic time $\Delta\tau$ from Γ to Γ which is variable, depending on $\lambda(\Gamma)$. Nevertheless in a simple schematic way (f) must have been taken to be 30^τ as is seen from the corresponding distance and velocity schemes.

For example for Jupiter's System A' we have for the distances[1]

Section	slow	medium	fast	
(a) (e) (f)	6;15°	7; 1,52,30°	7;30°	
(b)	10	11;15	12	
(c)	−8;20	−9;22,30	−10	(2)
(d)	9;35	10;46,52,30	11;30	
synodic arc	30°	33;45°	36°	

and for the velocities, measured in degrees per tithi

Section	slow	medium	fast	
(a) (e) (f)	$0;12,30^{\circ/\tau}$	$0;14,\ 3,45^{\circ/\tau}$	$0;15^{\circ/\tau}$	
(b)	0; 6,40	0; 7,30	0; 8	(3)
(c)	−0; 4,10	−0; 4,41,15	−0; 5	
(d)	0; 6,23,20	0; 7,11, 5	0; 7,40	

If one divides the distances in (2) by the corresponding velocities in (3) one obtains for all arcs the time intervals given in (1) and in particular for (f) the time of 30^τ.

[1] Cf. above p. 399 (4).

Obviously these simple patterns are not strictly adhered to in the ephemerides. We mentioned before (p. 399 (3)) the initial dates from ACT No. 611:

Γ	S.E. 180*	VI 13	♍10
Φ		X 16	♍26;15

for positions on the slow arc where one would expect from the rule (1) the time interval (a)+(b)$=4^m=2{,}0^\tau$. Formally the above interval could be considered to be $4^m 3^\tau$ while the actual interval was 121^d according to modern calculation.[2] Similarly the dates from p. 398 (2)

Ω	S.E. 108	IV 4	♌ 9;30
Φ		IX 4	♍ 1;45
Ψ	109	I 6	♌21;45

agree with (f)+(a)+(b)$=5^m$ but assume (c)$=4^m 2^\tau$ (or 2^d) against simply 4^m according to (1).[3] There exists also a "Normal-Star Almanac" for the same time with the following data[4]:

Ω	S.E. 108	IV 4	in ♌
Γ		V 4	in ♌
Φ		IX 5	beginning of ♍

hence again (f)$=30^\tau$ but now (a)+(b)$=4^m 1^\tau$. Since these dates were based on computations, not on observations,[5] we must admit that we do not know how in practice a sequence of initial dates for an ephemeris had been chosen.

B. Mars

It is only through the accidents of preservation that we know about some very refined theoretical considerations which concern the subdivision of the mean (not even the true!) synodic arc of Mars. As we have seen before[1] the phases Θ and Ψ are treated in the ephemerides of Mars as satellites of Φ and it is therefore not surprising that the following theory deals only with Ω, Γ and Φ but completely ignores the retrograde arc.

The subsequent discussion is based on a procedure text, published in ACT as No. 811a. Its main content can be grouped into two sections: the first concerns the total synodic time (e.g. from one Γ to the next); the second section derives the time intervals between Ω and Γ, Γ and Φ, and Φ back to Ω. From the first part we obtain a value for the synodic time of greater refinement than one could deduce from the ephemerides. The second part is of interest because it allows us to uncover the theoretical basis on which a subdivision of the synodic time was constructed, a theory much more satisfactory than one would have expected from such simple estimates described in the preceding pages.

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[2] From Parker-Dubberstein, BC.
[3] From Parker-Dubberstein: 147^d and 121^d, respectively.
[4] Pinches-Sachs, LBAT No. 1019 (obv. 2, obv. 4, rev. 4') [Sachs].
[5] Sachs [1948], p. 287.
[1] Cf. above p. 400.

Mean Synodic Time. If $\overline{\Delta\lambda}$ is the mean synodic arc then

$$\overline{\Delta\Lambda} = \overline{\Delta\lambda} + 6{,}0^\circ \tag{1}$$

is the total mean travel of the planet Mars itself between two consecutive occurrences of the same phase. Since the mean synodic time was found from[2]

$$\overline{\Delta\tau} = \overline{\Delta\lambda}^{\tau} + \left(\frac{e}{P} + 2(6{,}0+e)\right)^{\tau} = \overline{\Delta\lambda}^{\tau} + c$$

we can now write

$$\overline{\Delta\tau} = \overline{\Delta\Lambda}^{\tau} + c', \qquad c' = \frac{e}{P} + 6{,}0 + 2e. \tag{1a}$$

But

$$P = 6{,}0/\overline{\Delta\lambda}$$

and therefore

$$c' = \frac{e}{P} + 6{,}0 + 2e = \frac{e}{6{,}0}\,\overline{\Delta\lambda} + e + (6{,}0+e) = \frac{e}{6{,}0}\,\overline{\Delta\Lambda}^{\tau} + (6{,}0+e)^{\tau}.$$

Introducing a new constant

$$\gamma = \frac{e}{6{,}0}\,\overline{\Delta\Lambda}^{\tau} \tag{2}$$

we have

$$\overline{\Delta\tau} = \overline{\Delta\Lambda}^{\tau} + c', \qquad c' = \left(\gamma + (6{,}0+e)\right)^{\tau}. \tag{3}$$

These relations allow a simple geometric representation of γ. We construct a velocity diagram (cf. Fig. 27) in which both degrees and tithis are represented by the same unit of length. Then the straight line from the point $A = (-6{,}0^\circ, 0^{\tau})$ to $B = (+\overline{\Delta\Lambda}, \overline{\Delta\tau})$ is the graph for the motion of the mean sun[3]; hence $CD = (6{,}0+e)^{\tau} = 1$ year. If we draw BE under 45° toward CF we have $EF = \overline{\Delta\Lambda}^{\tau}$. Hence (3) shows that $\gamma = DE$. In other words γ^{τ} is the correction which is required because the accurate mean velocity of the sun $\bar{v}_{\odot} = 6{,}0^\circ/(6{,}0+e)^{\tau}$ is applied during one year only whereas for the whole of the remaining time too high a velocity $1^{\circ/\tau}$ is used.

The relations (2) and (3) are also valid for Saturn and Jupiter for which $\overline{\Delta\Lambda} = \overline{\Delta\lambda}$. Hence (2) gives $\gamma = e/P$ which is a much smaller correction than in the case of Mars (cf. Fig. 28).

Returning to Mars, the numerical value for $\overline{\Delta\Lambda}$ is obtainable from the basic relation[4]

$$4{,}44 \text{ years} = 2{,}13 \text{ occurrences} = 2{,}31 \text{ sid. rotations} \tag{4}$$

from which it follows that

$$\overline{\Delta\Lambda} = \frac{2{,}31 \cdot 6{,}0}{2{,}13} = \frac{15{,}6{,}0}{2{,}13} \approx 6{,}48;43{,}18{,}30^\circ. \tag{5}$$

The last number is a very good approximation since $2{,}13 \cdot 6{,}48;43{,}18{,}30 = 15{,}6{,}0 + 0;0{,}0{,}30^\circ$.

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[2] Cf. above p. 395 (3).

[3] This is astronomically evident since the sun is in A and B in the same relative position to the planet; but it can also be seen from the preceding formulae that $\bar{v}_{\odot} = 6{,}0/(6{,}0+e) = (6{,}0+\overline{\Delta\Lambda})/\overline{\Delta\tau}$.

[4] Cf. above p. 399 (1).

Using for the length of the year the canonical value[5] $12;22,8^{\mathrm{m}}$, i.e. the "epact"

$$e = 11;4^{\tau} \tag{6a}$$

we have

$$e/6,0 = 0;1,50,40 \tag{6b}$$

and hence from (2) and (5)

$$\gamma = 6,48;43,18,30 \cdot 0;1,50,40 = 12;33,51,52,47,20^{\tau}. \tag{7}$$

It is this value which is mentioned in our procedure text (Section 6) as "mean value (of time) from one appearance (Γ_n) to the next appearance (Γ_{n+1})." What is meant by this sentence is the statement that γ is the correction which applies to the complete synodic arc. We shall see presently that this γ will be split into partial corrections corresponding to the partial arcs between consecutive phases in their natural order.[6]

Substituting (5), (6a), and (7) into (3) one obtains for the mean synodic time

$$\begin{aligned}\overline{\Delta\tau} &= 6,48;43,18,30 + 12;33,51,52,47,20 + 6,11;4\\ &= 13,12;21,10,22,47,20^{\tau}.\end{aligned} \tag{8}$$

From the ephemerides one knows only the rounded value[7]

$$\overline{\Delta\tau} = 13,12;21,10,30^{\tau} \tag{8a}$$

which, however, has the same practical accuracy. Converting the tithis in (8a) to days one finds

$$\overline{\Delta\tau} \approx 12,59;57,17^{\mathrm{d}} \approx 779.955^{\mathrm{d}}$$

which compares very well with the modern value 779.936^{d}.

Subdivision of the Synodic Time. We remark first that according to (1) and (4) the following relations must hold for the synodic arc

$$\overline{\Delta\Lambda}/\overline{\Delta\lambda} = (\overline{\Delta\lambda} + 6,0)/\overline{\Delta\lambda} = 1 + P = \frac{\Pi + Z}{Z} = \frac{2,31}{18} = 8;23,20 \tag{9a}$$

i.e.

$$\overline{\Delta\Lambda} = \frac{\Pi + Z}{Z}\,\overline{\Delta\lambda} = 8;23,20\,\overline{\Delta\lambda}. \tag{9b}$$

All subsequent considerations are based on a division of the total mean synodic travel $\overline{\Delta\Lambda}$ into three sections

$$\begin{aligned}&\text{from } \Omega \text{ to } \Gamma\text{:}\quad \Delta\Lambda_1 = 1;50\,\overline{\Delta\lambda} = \frac{33}{18}\,\overline{\Delta\lambda}\\ &\text{from } \Gamma \text{ to } \Phi\text{:}\quad \Delta\Lambda_2 = 3;20\,\overline{\Delta\lambda} = \frac{1,0}{18}\,\overline{\Delta\lambda}\\ &\text{from } \Phi \text{ to } \Omega\text{:}\quad \Delta\Lambda_3 = 3;13,20\,\overline{\Delta\lambda} = \frac{58}{18}\,\overline{\Delta\lambda}\end{aligned} \tag{10}$$

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[5] Cf., e.g., above p. 396 (5b).

[6] Cf. below p. 409 (16).

[7] Cf. below II A 7, 4.

which indeed satisfy (9a) and (9b) since

$$\sum_1^3 \Delta\Lambda_i = 8;23,20\,\overline{\Delta\lambda} = \frac{2,31}{18}\,\overline{\Delta\lambda}.$$

We now call

$$\bar{\delta} = \frac{\overline{\Delta\Lambda}}{\Pi+Z} = \frac{\overline{\Delta\lambda}}{Z} = \frac{6,0}{PZ} = \frac{6,0}{\Pi} \tag{11}$$

a "*basic interval*" or a "*step*"; then we can describe (10) by saying that the mean synodic travel $\overline{\Delta\Lambda}$ is divided in such a fashion that

$$\begin{aligned} &\text{from } \Omega \text{ to } \Gamma\text{: } \Delta\Lambda_1 = 33\bar{\delta} = s_1\bar{\delta} \\ &\text{from } \Gamma \text{ to } \Phi\text{: } \Delta\Lambda_2 = 60\bar{\delta} = s_2\bar{\delta} \\ &\text{from } \Phi \text{ to } \Omega\text{: } \Delta\Lambda_3 = 58\bar{\delta} = s_3\bar{\delta} \end{aligned} \tag{12}$$

where the numbers s_i of the steps satisfy

$$\sum_1^3 s_i = \Pi + Z = 2,31. \tag{12a}$$

The fundamental importance of these "steps" for the planetary theory will become evident later.[8] At the moment we give only the numerical value of $\bar{\delta}$ as it follows from (5) and (11)

$$\bar{\delta} = 48;43,18,30/18 = 2;42,24,21,40^\circ. \tag{13}$$

Using (12), (13), and

$$\overline{\Delta\lambda} = 48;43,18,30^\circ \tag{14}$$

we obtain

$$\begin{aligned} \Delta\Lambda_1 &= 1,29;19,23,55^\circ \\ \Delta\Lambda_2 &= 2,42;24,21,40 \\ \Delta\Lambda_3 &= 2,36;59,32,56,40 \end{aligned} \tag{15}$$

and consequently

$$\overline{\Delta\Lambda} = \sum_1^3 \Delta\Lambda_i = 6,48;43,18,31,40^\circ. \tag{15a}$$

Since according to (2)

$$\gamma = \frac{e}{6,0}\,\overline{\Delta\Lambda}^{\tau}$$

it is natural to apply the same procedure to the subsections and thus to define three parameters through

$$\gamma_i = \frac{e}{6,0}\,\overline{\Delta\Lambda}_i^{\tau} \tag{16}$$

hence with (6b) and (15)

$$\begin{aligned} \gamma_1 &= 2;44,45,6,46,46,40^{\tau} \\ \gamma_2 &= 4;59,32,55,57,46,40 \\ \gamma_3 &= 4;49,33,50,5,51,6,40 \end{aligned} \tag{16a}$$

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[8] Cf. below II A 6.

and consequently

$$\gamma = \sum_1^3 \gamma_i = 12;33,51,52,50,24,26,40^\tau. \tag{16b}$$

The slight discrepancy in the last digits of (14) and (15a) and therefore also of (7) and (16b) is due to the fact that the division in (5) necessarily leads to rounding errors.

The values (16a) are indeed found in the procedure text (ACT No. 811a Section 6) but they are replaced in the subsequent sections (7 to 9) by the rounded values

$$\begin{aligned} \gamma_1 &= 2;44,45^\tau \\ \gamma_2 &= 4;59,33 \\ \gamma_3 &= 4;49,33,50 \end{aligned} \tag{17}$$

with

$$\gamma = \sum_1^3 \gamma_i = 12;33,51,50 \approx 12;33,52^\tau \tag{17a}$$

instead of (16b).

Since it is the purpose of these calculations to obtain a subdivision of the synodic time corresponding to the division of the synodic arc expressed in (10) or (12) one defines in analogy to the basic relation (above p. 407 (1a) and (3)) for the total time

$$\overline{\Delta\tau} = \overline{\Delta\Lambda}^\tau + c' = \overline{\Delta\Lambda}^\tau + (\gamma + (6,0+e))^\tau \tag{18}$$

relations for the three sections

$$\Delta\tau_i = \Delta\Lambda_i^\tau + c_i', \qquad i = 1, 2, 3 \tag{19}$$

where the c_i' contain the parameters γ_i found in (16) or (17). Just as for the whole synodic period the constant c' is represented in Fig. 27 (p. 1324) by the distance CE so do we have now (cf. Fig. 29) intervals c_i' which supplement the quantities $\Delta\Lambda_i^\tau$ in the same fashion as CE $= c'$ has to be added to EF $= \overline{\Delta\Lambda}^\tau$.

It follows from (18) and (19) that the c_i' must satisfy

$$\sum_1^3 c_i' = c' = \sum_1^3 \gamma_i + (6,0+e). \tag{20}$$

This will be the case if we make

$$c_i' = \gamma_i + r_i\left(1 + \frac{e}{6,0}\right), \qquad i = 1, 2, 3 \tag{21a}$$

with coefficients r_i such that

$$\sum_1^3 r_i = 6,0. \tag{21b}$$

These quantities r_i have a good astronomical meaning since they are multiplied with $(6,0+e)/6,0$ which is the reciprocal of the mean velocity of the sun. Hence $r_i\left(1 + \frac{e}{6,0}\right)$ is the time needed by the mean sun to travel a distance of r_i degrees.

The values assigned to the r_i in our procedure text are

$$\begin{aligned} &\text{for } \Omega\Gamma\text{: } r_1 = 30^\circ \\ &\text{for } \Gamma\Phi\text{: } r_2 = 1{,}45 = 105^\circ \\ &\text{for } \Phi\Omega\text{: } r_3 = 3{,}45 = 225^\circ. \end{aligned} \tag{22a}$$

Van der Waerden first realized[9] that these arcs are related to the elongations of the phases Ω, Γ, and Φ from the mean sun in the following fashion:

$$\begin{aligned} &\text{elongation of } \Omega\text{: } && -15^\circ \\ &\text{elongation of } \Gamma\text{: } && +15^\circ \\ &\text{elongation of } \Phi\text{: } && +120^\circ. \end{aligned} \tag{22b}$$

The values r_i in (22a) are simply the increments of the elongations as the planet moves from Ω to Γ and to Φ. The total is, of course, 360° as required by (21b).

The γ_i in (21a) are, as before, the corrections required if one first converts the planetary travel $\Delta\Lambda_i^\circ$ in each section to $\Delta\Lambda_i^\tau$. Hence (19) and (21a) combine to the correct time intervals between the phases under the assumptions expressed in (12) and (22b).

The numerical values for the c_i' are found in the Section 7 to 9 of our procedure text, following (21a) with (17) and (22a), (6b):

$$\begin{aligned} c_1' &= 2;44,45 &&+ \ 30;55,20 = && 33;40,5^{10} \\ c_2' &= 4;59,33 &&+ 1,48;13,40 = && 1,53;13,13 \\ c_3' &= 4;49,33,50 &&+ 3,51;55 = && 3,56;44,33,50 \end{aligned} \tag{23}$$

the last digit in c_3' being rounded to 34. The total agrees, of course, exactly with (17a) when one adds $6,11;4^\tau$ as required by (21a) and (21b).

As final result we can write for the time intervals between the phases Ω, Γ, Φ, and again Ω:

$$\Delta\tau_i = \left(s_i\bar{\delta} + r_i\frac{6{,}0+e}{6{,}0}\right)^\tau \qquad i = 1, 2, 3 \tag{24}$$

with (12) and (22a) defining the steps $s_i\bar{\delta}$ and the elongations r_i. The norm of 120° for the elongation of the first station (Φ) of Mars and probably in general for all outer planets left many traces in Hellenistic-Roman literature.[11]

If one compares the theory outlined here for the subdivision of the synodic time for Mars with the simple patterns known, e.g., for Jupiter (above p. 405) one finds a great difference of approach. Of the three parameters, longitudinal distance between consecutive phases, time to traverse it, and velocity, two are always chosen independently in the simple patterns. The only restriction consists in maintaining the proper totals for the different zones of the synodic arcs. In the refined theory of Mars, however, none of the basic parameters represents a time interval. The longitudinal divisions are based on integer multiples of

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[9] Van der Waerden [1957], p. 52; also Anf. d. Astr., p. 190.

[10] Scribal error in the text: omission of the final 5.

[11] E.g. in Vitruvius, Archit. IX, 1, 11; Pliny NH II, XII 59 (Jan-Mayhoff, p. 145); Paulus Alex., Apot. 15 (Boer, p. 31 f.); "Heliodorus", Comm. 12 (Boer, p. 19f.) etc.; still in Copernicus, Comment. (Rosen TCT, p. 78; Swerdlow [1973], p. 480).

"steps" of fixed length $\bar{\delta}=6{,}0/\Pi$ each. The corresponding time intervals are then computed by means of the requirement of given elongations from the mean sun. The extant procedure texts concern only the mean situation but it is clear that true time intervals also could be found in the same fashion if one only replaces the mean steps $\bar{\delta}$ by the same number of "true" ones while one probably leaves the characteristic elongations unchanged. This reconstruction is confirmed by our sources only for the longitudes [12]; for the time intervals and hence for the elongation evidence is still lacking.

3. Daily Motion

We now turn to the last step in the Babylonian planetary theory, the construction of an "ephemeris" in the strict sense of the term, i.e. a sequence of day-by-day planetary longitudes. Such an ephemeris is obtained by interpolation between positions and dates of the Greek-letter phenomena, taken in their natural order.

The simplest device consists, of course, in linear interpolation. In actual practice modifications in the initial data may become necessary in order to obtain convenient differences. It is also possible that not only one constant velocity is assumed for the transition from one phase to the next but that the interval is broken up into two or three shorter sections which are better adapted to the factual variations of the daily motion. We have described such devices before [1] and we shall come back to these linear interpolation methods in our discussions of the individual planets.[2] Here we shall only describe more refined methods which are based on arithmetical progressions of the second and third order.

The purpose of these ephemerides is not quite clear. It could be that the astrologically important question of a planet's crossing from one zodiacal sign into the next provided the initial stimulus. This seems a plausible enough explanation but it must be admitted that the procedure texts do not contain any reference to this problem nor to any other astrologically important configuration of planetary positions. One may even think of a purely mathematical interest in the construction of such sophisticated procedures as we shall describe in the following, reminiscent of a tendency which is undoubtedly extant in the Old-Babylonian mathematical texts.

Quite independent of the question of motivation the reference to the mathematical tradition is of interest in the present context. Mathematical problems involving arithmetical progressions occur repeatedly in Old-Babylonian mathematical texts.[3] It is, however, a particularly fortunate accident that we also have from the late period mathematical texts from Uruk, written or owned by the same scribes whom we know from the astronomical texts.[4] One problem in a tablet written by a "scribe of Enūma-Anu-Enlil" (the great astrological series)

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[12] Cf. below p. 424.
[1] Above p. 405.
[2] Cf. also the excellent discussion of these methods by P. Huber [1957].
[3] Cf. Neugebauer, MKT III, p. 83 s.v. Reihen; Neugebauer-Sachs, MCT, p. 100.
[4] Cf. ACT I, p. 14f.

concerns the sum of consecutive squares.[5] In free translation the text runs as follows:

"Squares, from 1 times 1, i.e. 1, to 10 times 10, i.e. 1,40. Find the total. $1 \cdot 0;20 = 0;20$ $(0;20 = 1/3)$. $10 \cdot 0;40 = 6;40$ $(0;40 = 2/3)$. $0;20 + 6;40 = 7$. $7 \cdot 55 = 6,25$. This is the total."

In modern formulation: find $S = \sum_1^n i^2$ for $n = 10$. Answer:

$$S = (1/3 + 2/3\,n) \sum_1^n i \qquad \left(\sum_1^n i = 1/2\,n(n+1)\right) \tag{1}$$

hence for $n = 10$ $S = (0;20 + 0;40 \cdot 10) \cdot 55 = 6,25$.

[Remark. Fig. 30 suggests the following argument leading to (1): one counts first all points in each line but then subtracts the points which do not belong to a square. Hence

$$\sum_1^n i^2 = n \sum_1^n i - \left(1 + (1+2) + \cdots + \sum_1^{n-1} i\right) = n \sum_1^n i - A$$

where

$$A = \sum_1^n 1/2\,i(i-1) = 1/2 \sum_1^n i^2 - 1/2 \sum_1^n i.$$

Thus

$$3/2 \sum_1^n i^2 = (n + 1/2) \sum_1^n i$$

q.e.d.]

The idea of using difference sequences of higher order is not restricted to interpolation problems. Many applications will be found in the lunar theory with the purpose of obtaining smoothly changing functions[6] or periodic functions of a sinusoidal type.[7]

A. Jupiter

The most interesting text for the study of Babylonian methods of interpolation is an ephemeris for Jupiter, covering, on a single tablet, one whole synodic period from Γ to Γ, i.e. entries for about 400 days.

The text, written in Babylon, is now in a badly damaged condition (cf. Pl. VI). Four columns constitute a unit (dates, longitudes, first and second differences) and each side held three such quadruple columns, continued from the obverse over the lower edge to the reverse, as is the common practice in Babylon texts.[1] One fragment of the first group of columns was known to Kugler, through a copy by Strassmaier, and correctly dated to S.E. 147/148 (−164/163); together with a new fragment it was published in ACT as No. 654. A badly damaged piece of similar type did not accurately fit the other fragments and was thus published in ACT as No. 655. But finally P. Huber [1957] succeeded in bridging

[5] Cf. Neugebauer, MKT I, p. 103; first explained by Waschow [1932], p. 302f.

[6] Cf., e.g., the relations between Φ and G in the lunar System A (below II B 3, 2 B, p. 485).

[7] Cf., e.g., column J in System B, obtained by summation of the linear zigzag function H (below II B 3, 5 B, p. 493).

[1] Cf. ACT I, p. 2.

Table 8

k	t	λ	v	$\dot{v}$	$\ddot{v}$	
1	147 IX 1	♏ 29	+0;12,40	0		Γ
2	2	29;12,39,54	+0;12,31,54	−0;0, 0, 6	−0;0,0, 6	
3	3	29;25,19,36	+0;12,39,42	−0;0, 0,12	−0;0,0, 6	
4	4	29;37,59	+0;12,39,24	−0;0, 0,18	−0;0,0, 6	
121	148 I 3	♐ 16; 7,56	+0; 0,34	−0;0,12	−0;0,0, 6	
122	4	16; 8,17,54	+0; 0,21,54	−0;0,12, 6	−0;0,0, 6	
123	5	16; 8,27,36	+0; 0, 9,42	−0;0,12,12	−0;0,0, 6	Φ
124	6	16; 8, 5,52	−0; 0,21,44(!)	−0;0,12, 2	+0;0,0,10	
125	7	16; 7,32,16	−0; 0,33,36	−0;0,11,52	+0;0,0,10	
126	8	16; 6,46,58	−0; 0,45,18	−0;0,11,42	+0;0,0,10	
180	148 III 3	♐ 11;53,26,16	−0; 7, 9,36	−0;0, 2,42	+0;0,0,10	
181	4	11;46,14, 8	−0; 7,12, 8	−0;0, 2,32	+0;0,0,10	
182	5	11;38,59,38	−0; 7,14,30	−0;0, 2,22	+0;0,0,10	Θ
183	6	11;31,47,40	−0; 7,11,58	+0;0, 2,32	(+0;0,4,54)	
184	7	11;24,38,24	−0; 7, 9,16	+0;0, 2,42	+0;0,0,10	
239	148 V 2	♐ 7;21,43,24	−0; 0,24, 6	+0;0,11,52	+0;0,0,10	
240	3	7;21,31,20	−0; 0,12, 4	+0;0,12, 2	+0;0,0,10	Ψ
241	4	7;21,55,36	+0; 0,24,16(!)	+0;0,12,12	+0;0,0,10	
242	5	7;22,31,58,30	+0; 0,36,22,30	+0;0,12, 6,30	−0;0,0, 5,30	
243	6	7;23,20,22	+0; 0,48,23,30	+0; 0,12, 1	−0;0,0, 5,30	
244	7	7;24,20,41	+0; 1, 0,19	+0;0,11,55,30	−0;0,0, 5,30	

the gap and in restoring completely the two first quadruple columns which cover the most interesting part in the planet's motion, from Γ to Ψ. The following is based on Huber's publication which should be consulted for all details. Table 8 gives some (slightly modified) excerpts from the text.[2] The continuous line numbering is the same as in Huber [1957], p. 298 to 303 but the columns for the velocity v and the acceleration $\dot{v}$ in the text precede the longitudes λ; the third differences are modern additions. Fig. 31 gives a graphic representation of the whole text from Γ to Ψ, using greatly different scales for the successive differences. The last curve gives a comparison with the actual motion[3] without correcting the difference in the zero points.[4] It is quite evident that the Babylonian text assumes a retrograde motion that is slightly too small (about 8;45° instead of 9;40°) but it is possible that this difference is the result of adjustments made because of the interpolation procedures. Almost all numbers shown in Table 8 belong to restored areas but there hardly exists any other possibility for filling the gaps although some errors occur in the text.

One must, of course, assume that the data for the phases were found in advance but it is also evident that one could not expect to obtain from these given numbers convenient or at least manageable differences. Hence it is not surprising that the

[2] To facilitate the reading I have given all numbers with their complete sexagesimal order. The text omits all initial zeros as well as signs.

[3] Using the Tuckerman Tables.

[4] Cf. above p. 369.

numbers as they stand do not fit any known computing pattern for the phases.[5] Only the initial position, $\Gamma = ♏ 29$, seems genuine:

Γ: $k=1$	$t=$ S.E. 147	IX 1	$\lambda = ♏ 29$	
Φ: 123	148	I 5	♐ 16; 8,27,36	(1)
Θ: 182		III 5	♐ 11;38,59,38	
Ψ: 240		V 3	♐ 7;21,31,20.	

The dates as given in the text are undoubtedly meant to be days in the real lunar calendar since some months are hollow. But that the original dates for the phases were given in tithis is practically certain (since all Greek-letter ephemerides operate with tithis in their determination of the dates) and it seems very unlikely that results obtained for tithis were carefully converted to calendar dates – a conversion which would require additional information about the zero points of the tithis in relation to the solar days, information of which no trace has been found anywhere in the cuneiform material. Hence an interval from IX 1 in line 1 to I 5 in line 123 could just as well represent 125–1 steps (tithis) as 123–1 steps (lines = calendar days).

There are certain inconsistencies or errors visible in the text, of course becoming most easily apparent at the points where the characteristic phases require changes in the differences. In (1) and in Table 8 the phases are assigned to those lines where either λ has an extremal value (at Φ or Ψ) or where the velocity reaches its greatest negative value (at Θ). In the case of Φ there occurred an error in changing from positive to negative values. One should have computed

$$+0;0,9,42 - 0;0,12,2 = -0;0,2,20$$

instead of changing the sign of v one line too early and thus finding (in line 124)

$$-0;0,9,42 - 0;0,12,2 = -0;0,21,44.$$

Only recently P. Huber found that the corresponding error also occurred at Ψ at the change from negative to positive velocities. In going from line 240 to 241 one should have used

$$-0;0,12,4 + 0;0,12,12 = +0;0,0,8$$

but the text seems to have operated with

$$+0;0,12,4 + 0;0,12,12 = +0;0,24,16.$$

This explains the numbers still seen by Pinches.[6] The errors committed in both cases influence all subsequent lines but they seem to be at least consistent.

[5] The nearest attested positions in Aaboe-Sachs [1966], Text D (System A) are

Γ:	♏ 29;20 → * → ♏ 28;30	
Φ:	♐ 18;42	♐ 17;42 → *
Θ:	♐ 12;42	♐ 11;42 → *
Ψ:	* → ♐ 6;42	♐ 5;42

where * indicates the relative positions in our text (cf. Table 8).

[6] Pinches-Sachs, LBAT 133, rev. I (better than ACT No. 655, rev. I, 2–5) corresponding to the ends of the lines 258 to 261 for λ.

Near Θ the third difference should remain constant (+0;0,0,10), hence the second difference ($\dot{v}$) could not suddenly change from −0;0,2,22 to +0;0,2,32. But $\dot{v}$ should go through zero at Θ and this was achieved in this clumsy fashion, formally using the difference 0;0,0,10 instead of at least preserving symmetry and thus jumping only to +0;0,2,22.

Also at Ψ exist small inconsistencies. The second and third differences would place the stationary point at line 241 but λ and v require line 240. These discrepancies seem to suggest as original dates (in tithis) I, 4 III, 4 V, 5 for Φ, Θ, and Ψ, respectively and it may only be due to the difficulty of fitting the differences exactly that the lines appear to be 123, 182, 240 (or 241), respectively. For the final function $\lambda(t)$ these small inaccuracies are of course of no significance whatever.

We now can investigate the theoretical determination of the difference sequences which should lead from one phase to the next. Operating with constant third differences we should have (a, b, c, d being constants, i and n integers) for n steps

$$\begin{aligned} \ddot{v} &= d \\ \dot{v} &= c + dn \\ v &= b + cn + \frac{d}{2} n(n+1) = b + \left(c + \frac{d}{2}\right) n + \frac{d}{2} n^2 \\ \lambda &= a + bn + \left(c + \frac{d}{2}\right) \sum_1^n i + \frac{d}{2} \sum_1^n i^2 \end{aligned} \tag{2}$$

hence, using (1), p. 413

$$\begin{aligned} \lambda &= a + bn + \left(c + \frac{d}{2} + \frac{d}{2} \frac{1+2n}{3}\right) \sum_1^n i \\ &= a + bn + c \sum_1^n i + \frac{d}{3}(n+2) \sum_1^n i. \end{aligned} \tag{2a}$$

We apply these relations first to the section from Γ to Φ. We consider to be given the number n of steps (days or tithis) and the increment

$$s = \lambda(\Phi) - \lambda(\Gamma) \tag{3}$$

for the longitudes. At Γ we norm

$$\lambda(0) = 0 \quad \text{thus } a = 0 \tag{4a}$$

$$\dot{v}(0) = 0 \quad \text{thus } c = 0. \tag{4b}$$

The second condition is the consequence of the plausible assumption that the motion at Γ is the same as during the interval of invisibility from Ω to Γ. At Φ one has with (3)

$$\lambda(n) = s \qquad v(n) = 0$$

because the planet is stationary. Consequently

$$s = bn + \frac{d}{6} n(n+1)(n+2) \tag{5a}$$

and, using (2):

$$b + \frac{d}{2} n(n+1) = 0 \tag{5b}$$

hence

$$s=b\left(n-\frac{n+2}{3}\right)=\frac{b}{3}(2n-2)$$

or

$$b=\frac{3s}{2(n-1)}. \tag{6}$$

Finally from (6) and (5 b)

$$d=-\frac{2b}{n(n+1)}=-\frac{3s}{(n-1)\,n(n+1)}. \tag{7}$$

If we take the data from (1), p. 415 we have

$$s\approx 17;8,30, \qquad n=123-1=2,2. \tag{8}$$

Substituting these values into (6) and (7) gives for the initial velocity

$$b=v(0)\approx 0;12,45^{\circ/\mathrm{d}} \qquad \text{instead of } 0;12,40^{\circ/\mathrm{d}}$$

used in the text and for the constant third difference

$$d\approx -0;0,0,6,7 \qquad \text{instead of } d=-0;0,0,6.$$

The values of the text are almost exactly obtainable for $n=2,3=123$:

$$b\approx 0;12,39, \qquad d\approx -0;0,0,5,58.$$

This speaks in favor of the suggestion made above to consider S.E. 148 I 4 as the date of Φ. The value $n=2,3$ is also supported by the following consideration. It seems most likely that in fact b was not computed from (6) but given as velocity from Ω to Γ.[7] Hence one can find n from s and b and obtains $n\approx 2,2;48\approx 2,3$. Furthermore $n=2,2$ gives $s\approx 17;2$ while $n=2,3$ leads to $s\approx 17;10$ to be compared with $s\approx 17;8,30$ of the text. Whatever the case may be, the parameters used in the text for the transition from Γ to Φ agree very well indeed with the theoretically required values.

One cannot expect the same for the retrograde section since we know of the existence of an irregular jump at Θ. We shall nevertheless ask which parameters would lead from Φ to Ψ with Θ being in the midpoint which is under all circumstances very nearly the case in the text.

As before we find

$$v=b+c\,n+\frac{d}{2}\,n(n+1)$$

$$\lambda=a+b\,n+\left(\frac{c}{2}+\frac{d}{4}\right)n(n+1)+\frac{d}{2}\sum_{1}^{n} i^2.$$

At Φ we have $\lambda(0)=0$ and $v(0)=0$ hence

$$a=b=0.$$

[7] It must be admitted, however, that the known pattern would suggest for Ω to Γ $b=0;12,30^{\circ/\tau}$ and not $0;12,40^{\circ/\tau}$. Cf. ACT II, p. 312.

Also at Ψ one finds that $v(n)=0$ but $\lambda(n)=s$, thus

$$c=-\frac{d}{2}(n+1)$$

and with (1), p. 413

$$s=\left(\frac{c}{2}-\frac{c}{2(n+1)}\right)n(n+1)-\frac{c}{n+1}\cdot\frac{1+2n}{3}\cdot\frac{n(n+1)}{2}=\frac{c}{6}(n^2-n).$$

Thus

$$c=\frac{6s}{n(n-1)},\quad d=-\frac{2c}{n+1}=-\frac{12s}{(n-1)\,n(n+1)}. \tag{9}$$

As suggested before we interpret $\Delta t \approx$ V 4 − I 4 as $n=2{,}0$ steps. The retrograde arc is now $s\approx -8;47°$ (probably too small[8]). Hence

$c\approx -0;0,13,17$ against $-0;0,12,12$ (or $-0;0,12,6$) in the text

and

$d\approx 0;0,0,13,10$ against $0;0,0,10$.

Here the jump of almost $30d$ in line 183 reflects the fact that $d=0;0,0,13$ (or perhaps $0;0,0,15$) would have been a much better third difference than $0;0,0,10$.

B. Mercury

Beside the Jupiter ephemeris discussed in the preceding section we have only one more planetary text for day-by-day positions based on nonlinear interpolation. This text is a Mercury ephemeris from Uruk, covering one calendar year, probably S.E. 122 (i.e. −189/188).[1] Actually preserved is only the major part of the ephemeris for the months IV to IX but for Mercury this is just enough to give us about one synodic period, here from one Ξ to the next.

The data for the phases as they stand in the text[2] (i.e. probably modified to a certain extent for the sake of convenient interpolation) are as follows

obv.	II,1	[S.E. 122 V 5]	♍[12;37	Ξ]	
	II,24	[V] 28	♎ 2;12,6	[Ω]	
	III,21	[VI] 25	♍ 29;30	Γ	(1)
rev.	I,21	[VII] 27	♎ 28;11	Σ	
	III,5	[IX 11	♑ 13;42	Ξ].	

None of the known computing schemes agrees with this sequence.[3] That the scribe who computed this text was not very skillful becomes evident when one looks at his attempts to interpolate with constant second differences (cf. Fig. 32). This problem in itself would not give a new insight since the Jupiter ephemeris operates even with third differences. But the interest of the text lies in its errors, or inaccuracies, which show that the Greek-letter phenomena were not too sharply

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[8] Cf., e.g., above p. 398 (1) and p. 399 (4).

[1] This date was suggested by Huber [1957], p. 276. The text is published in ACT No. 310.

[2] The restorations are quite secure.

[3] For details cf. ACT II, p. 326f.

defined since Ω and Γ were apparently identified with the stationary points (which otherwise do not count among the tabulated phases of Mercury). This can be seen from the numerical data in the text (plotted in Fig. 33[4]) which suggest considering the points Ω and Γ as extrema with $v=0$. This is furthermore supported by the boundary conditions which determine the interpolation scheme.

For the section from Ξ to Ω we assume[5]

$$\dot{v}=c$$

$$v=b+cn$$

$$s=nb+\frac{c}{2}n(n+1)$$

where, according to (1)

$$s\approx 19;35°, \quad n=23. \tag{2}$$

If we consider Ω as a stationary point then we have $v(n)=0$, hence

$$b=-nc, \quad c=\frac{2s}{n-n^2} \tag{3}$$

or with (2)

$$c=-\frac{39;10}{8,26}\approx -0;4,39, \quad b\approx 1;47 \tag{4a}$$

as compared with

$$c=-0;4,12, \quad b=1;45 \tag{4b}$$

used in the text.

One can go even farther. It seems again a plausible assumption[6] that b was given as the constant mean velocity for the interval of invisibility from the preceding Σ to Ξ. Then one finds that the values (4b) correspond exactly to

$$n=-\frac{b}{c}=\frac{1;45}{0;4,12}=25$$

and again exactly to

$$s=\frac{c}{2}(n-n^2)=0;2,6\cdot 10,0=21°.$$

The fact that we obtain integer values for n and s strongly suggests that these are the data which were actually used, of course in combination with $v(n)=0$, i.e. identifying Ω with a stationary point (or nearly so, if one takes $n=25$ as an intentional deviation from $n=23$ found in (1)).

The remainder of the text shows so many irregularities that it makes no sense to determine the correct parameters. The second differences for the section $\Gamma\rightarrow\Sigma$ (cf. Fig. 32) could suggest an additional subdivision of this part – similar to a division of the synodic arc of Jupiter[7] – but our text is not reliable enough to take this shift of differences seriously.

Fig. 35 shows the actual motion of Mercury in longitude and latitude for the time in question. Our ephemeris tries to give the longitudinal velocity components

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[4] The modern data are taken from the Tuckerman Tables. The curve of the text is not corrected for the difference of the zero points (cf. above p. 369), otherwise Γ would come near to the level of point 10.

[5] Using the same notation as with Jupiter.

[6] Above p. 416.

[7] Cf. above p. 405; also ACT, p. 312.

on the basis of the longitudes for the phases. It is perhaps not too surprising when the result, shown in Fig. 33, represents the retrograde section (from point 5 to point 11) only qualitatively. To the modern historian or astronomer this is a renewed warning to use ephemerides of Mercury only with great caution in all problems which concern retrogradations or visibility conditions.

§ 6. The Fundamental Patterns of Planetary Theory

All planetary theory must take its point of departure from the experience of periodicity of the observable phenomena. This periodicity will not be very exact for short intervals of time but the larger stretches one has at one's disposal the more one will be able to detect periodic sequences which form a reliable basis for predicting future events. We have also seen that not quite accurate periodic returns were utilized to determine on purely arithmetical grounds much larger periodic intervals for which accurate returns may be expected.[1] (The number of events of the same kind within such a presumably accurate period we called "*number period*" Π.) Experience must also have shown that such a period corresponds to an integer number of sidereal rotations of the planet and therefore also to an integer number of years since all planetary phenomena of the same kind occur in the same relation to the sun, a fact easily established by observing the relative motion of the planet and of the sun against the background of the fixed stars. These empirical relations are: Π occurrences of the same planetary phase correspond to[2]

$$\begin{array}{rl} \Pi+Z & \text{years or } Z \text{ sidereal rotations of Saturn or Jupiter} \\ 2\Pi+Z & \text{years or } \Pi+Z \text{ sidereal rotations of Mars} \\ \Pi+Z & \text{years or sidereal rotations of Venus} \\ Z & \text{years or sidereal rotations of Mercury} \end{array} \tag{2}$$

(where we may always assume that Π and Z are relatively prime numbers).

Having once established a number period Π one can construct an ideal "mean" situation in which all anomalies are ignored and hence all events of the type under consideration follow one another at equidistant intervals. If the planet once was found at λ_0 it will for the first time after Π events again be at $\lambda_0=\lambda_\Pi$. All intermediary events will be located at equidistant points of the ecliptic which form a "*regular Π-gon*" with an arc of length

$$\bar{\delta}=6,0/\Pi \tag{3}$$

between them (cf. the example of Jupiter, Fig. 35). We call $\bar{\delta}$ a mean "*basic interval*" or a mean "*step.*" This arc $\bar{\delta}$, however, is not the distance between consecutive events in their actual order which is given by the "*mean synodic arc*"[3]

$$\overline{\Delta\lambda}=6,0/P \tag{4}$$

[1] Cf., e.g., above p. 391 (12).
[2] Cf. above p. 389 (4) and (6).
[3] Cf. above p. 382.

where P is the "*period*"

$$P = \Pi / Z. \tag{5}$$

From (3), (4), and (5) it follows that

$$\overline{\Delta\lambda} = Z\bar{\delta} \tag{6}$$

which shows that the mean synodic arc can also be defined as the length of Z steps.

Experience shows that in fact the phenomena in question are not spaced with exactly equal intervals. Fortunately, however, the distribution of the points in question remains always the same in the same area of the ecliptic.[4] Hence one can introduce "true steps" of variable lengths δ which depend only on λ. Then true synodic arcs $\Delta\lambda$ can be defined by means of

$$\Delta\lambda = Z\delta \tag{7}$$

with the same factor Z as in (6), known from (2). All further considerations depend on purely arithmetical assumptions concerning the values of the arcs δ and their distribution. If δ is assumed to be stretchwise constant on each of k arcs of the ecliptic then we call this arrangement of type "*System A*." If the δ form an arithmetic progression we speak of a "*System B*" model. In the following we have to show that all the computational rules which we deduced from the ephemerides of the two systems are consequences of the assumptions made for the steps in combination with (7).

The distribution of the δ as function of λ must be ultimately based on empirical data. For us, however, nothing is left beyond the final arithmetical idealization which leads to the models known from the ephemerides and procedure texts. It is futile to speculate about data and preliminary models which may have preceded the still extant types.

1. System A

We assume the ecliptic divided in k arcs α_i (the largest attested value for k is 6) such that each arc α_i contains a number π_i of steps of length δ_i; hence

$$\pi_i \delta_i = \alpha_i \tag{1}$$

and, of course,

$$\sum_1^k \pi_i = \Pi. \tag{2}$$

With only a few exceptions[1] the π_i in our sources are integers but this is not essential for the following. The numerical values for the parameters in (1) and (2) are listed in Table 9, p. 423 and can be considered from now on to be given.

We define "true synodic arcs" $\Delta\lambda$ as the sum of Z consecutive steps. As long as all Z steps belong to the same arc α_i the resulting $\Delta\lambda$ has the length

$$w_i = Z\delta_i. \tag{3}$$

[4] The slow rotation of all apsidal lines escaped notice in early astronomy.

[1] Cf. below p. 423, Table 9.

If, however, such an aggregate of Z steps contains only $m<Z$ steps which belong to the arc α_i while $Z-m$ steps belong to α_{i+1} then we call their total length

$$w'=r+s$$

where

$$r=m\delta_i$$
$$s=(Z-m)\delta_{i+1}=w_{i+1}-r\frac{\delta_{i+1}}{\delta_i}=w_{i+1}-r\frac{w_{i+1}}{w_i}.$$

Hence

$$w'=w_{i+1}+r\left(1-\frac{w_{i+1}}{w_i}\right)=w_{i+1}-c_{i,i+1}\cdot r \tag{4}$$

which is exactly the transition rule for System A found above p. 376 (3a) and (3b). Consequently, as before,

$$\frac{r}{w_i}+\frac{s}{w_{i+1}}=1. \tag{5}$$

Furthermore with (2) and (1)

$$\Pi=\sum_1^k \pi_i=\sum_1^k \alpha_i/\delta_i$$

hence with (3)

$$P=\frac{\Pi}{Z}=\sum_1^k \frac{\alpha_i}{Z\delta_i}=\sum_1^k \frac{\alpha_i}{w_i}. \tag{6}$$

As we have seen all computing rules for System A flow from the relations (5) and (6).

A. Numerical Data

Our sources concerning Venus are so incomplete[2] that we exclude the theory for this planet from the following discussion. The data for all the remaining models which follow System A are given in Table 9.

All basic intervals δ_i are represented by regular numbers. This does not hold for the w_i because the wave numbers Z do not need to be regular (cf. Mercury). All transition coefficients, however,

$$c_{i,i+1}=\frac{w_{i+1}}{w_i}-1 \tag{7}$$

are again regular numbers, a fact of great importance for the convenience of computation. Indeed, with one minor exception,[3] all ratios w_{i+1}/w_i are of the form

$$w_{i+1}/w_i=1\pm\frac{1}{n} \tag{8}$$

with small regular integers n.[4] This gives to the coefficients (7) simply the form $\pm 1/n$. These values are shown in the last column of Table 9.

B. Subdivision of the Synodic Arc

The problem of spacing the Greek-letter phenomena in their natural order has been discussed in its general aspect above section II A 5 (p. 397 ff.). But really

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[2] Cf. below II A 7, 5.

[3] Mercury Ξ has $2{,}40/1{,}36=1+2/3$; cf. Table 9.

[4] This has been first observed by Aaboe [1965], p. 224.

Table 9

System		$\Pi = \sum v_i$	Z	$P = \Pi/Z = \sum \alpha_i/w_i$	$\bar{\delta} = 6{,}0/\Pi$	i	α_i $\sum\alpha_i = 6{,}0$	v_i $\sum v_i = \Pi$	$\delta_i = \alpha_i/v_i$	$w_i = Z\delta_i$	$c_{i,i+1}$
☉	A	46,23	3,45	12;22,8	0;7,45,41,4,...°	1	3,14°	24,15	0; 8°	30°	−0; 3,45
						2	2,46	22, 8	0; 7,30	28; 7,30	+0; 4
☿ A₁	Γ	44,33	14, 8	3;9,7,38,...	0;8,4,50,...	1	2,45	22, 0	0; 7,30	1,46	+0;20
						2	2,14	13,24	0;10	2,21;20	−0;20
						3	1, 1	9, 9	0; 6,40	1,34;13,20	+0; 7,30
	Ξ	25,13	8, 0	3;9,7,30	0;14,16,34,11,...	1	1,50	5,30	0;20	2,40	−0;20
						2	2,14	10, 3	0;13,20	1,46;40	−0; 6
						3	1,56	9,40	0;12	1,36	+0;40
☿ A₂	Σ	20,23	6,28	3;9,7,25,...	0;17,31,41,...	1	1,30	5,24	0;16,40	1,47;46,40	+0;12
						2	1,36	4,48	0;20	2, 9;20	−0;15
						3	1,29	5,56	0;15	1,37	+0;20
						4	1,25	4,15	0;20	2, 9;20	−0;10
	Ω	11,24	3,37	3;9,7,32,...	0;31,34,44,...	1	3, 0	6, 0	0;30	1,48;30	+0; 6,40
						2	1, 0	1,48	0;33,20	2, 0;33,20	−0; 6
						3	1, 0	2, 0	0;30	1,48;30	+0;15
						4	1, 0	1,36	0;37,30	2,15;37,30	−0;12
♂	A	2,13	18	7;23,20	2;42,24,21,...	1	1, 0	24	2;30	45	−0;20
						2	1, 0	36	1;40	30	+0;20
						3	1, 0	27	2;13,20	40	+0;30
						4	1, 0	18	3;20	1, 0	+0;30
						5	1, 0	12	5	1,30	−0;15
						6	1, 0	16	3;45	1, 7;30	−0;20
♃	A	6,31	36	10;51,40	0;55,14,34,...	1	3,25	3,25	1	36	−0;10
						2	2,35	3, 6	0;50	30	+0;12
	A₁	16,19	1,30	10;52,40	0;22,3,47,59,...	1	3,22	8,25	0;24	36	
						2	2,38	7,54	0;20	30	
	A′	6,31	36	10;51,40	0;55,14,34,...	1	2, 0	2,24	0;50	30	+0; 7,30
						2	53	56;32	0;56,15	33;45	+0; 4
						3	2,15	2,15	1	36	−0; 3,45
						4	52	55;28	0;56,15	33;45	−0; 6,40
	A″	6,31	36	10;51,40	0;55,14,34,...	1	2, 0	2,24	0;50	30	
						2	56;15	1, 0	0;56,15	33;45	
						3	2,15	2,15	1	36	
						4	48;45	52	0;56,15	33;45	
	A‴	4,53	27	10;51,6,40	1;15,46,4,...	1	2, 0	1,48	1; 6,40	30	
						2	48	38;24	1;15	33;45	
						3	2,20	1,45	1;20	36	
						4	52	41;36	1;15	33;45	
♄	A	4,16	9	28;26,40	1;24,22,30	1	3,20	2,33;36	1;18, 7,30	11;43,7,30	+0;12
						2	2,40	1,42;24	1;33,45	14; 3,45	−0;10

detailed information about the theoretical background is available only for Mars. In II A 5, 2 B (p. 406 ff.) we have seen from a procedure text (ACT No. 811 a) how the mean synodic arc was divided into sections $\Omega \rightarrow \Gamma$ and $\Gamma \rightarrow \Phi$ and how the corresponding time intervals were determined. For the true synodic arcs (but not for the dates) we have other texts[5] which give for a whole number period of Mars (therefore $134 = \Pi + 1$ lines) in 5 columns longitudes for the consecutive phases Ω, Γ, Φ, Θ, Ψ, beginning with

$$\begin{array}{lll} \text{line 1:} & \Omega & ♏ 22 \\ & \Gamma & ♈ 6 \\ & \Phi & ♌ 24;20 \\ & \Theta & ♌ 17;27,28 \\ & \Psi & ♌ 7;8,40 \\ \text{line 2:} & \Omega & ♒ 3 \\ & \Gamma & ♉ 29 \\ & & \text{etc.} \end{array} \tag{1}$$

All longitudes for Ω, Γ and Φ are computed with System A (cf. above p. 400 (3) and (4)) as is seen in the following example: $\Omega =$ ♏ 22 belongs to the sector α_4 in which $w_4 = 1{,}0$. Thus for the next Ω (line 2):

$$♏ 22 + 1{,}0 = ♑ 22 \quad \text{and} \quad ♑ 22 + 0;30 \cdot 22 = ♒ 3.$$

The retrogradations are computed with "scheme S" (cf. above p. 401 (6 b) and (7)):

$$\begin{aligned} \Phi \rightarrow \Theta \text{ from } \Phi = ♌ 24;20\text{:} \quad & -7;12 + 24;20 \cdot 0;0,48 = -6;52,32 \\ \text{thus } \Theta = & ♌ 24;20 - 6;52,32 = ♌ 17;27,28 \\ \text{and } \Theta \rightarrow \Psi = & 3/2(\Phi \rightarrow \Theta) = -3/2 \cdot 6;52,32 = -10;18,48 \\ \text{thus } \Psi = & ♌ 17;27,28 - 10;18,48 = ♌ 7;8,40. \end{aligned}$$

This shows that we can restrict our discussion to the direct motions $\Omega \rightarrow \Gamma$ and $\Gamma \rightarrow \Phi$.

Following the general principles formulated above p. 421, we consider now the "true steps" δ_i defined by

$$\delta_i = w_i / Z \tag{2}$$

which gives with the known w_i (above p. 400 (3)) and $Z = 18$ the values

$$\begin{array}{lll} \delta_1 = 2;30^\circ & \delta_2 = 1;40 & \delta_3 = 2;13,20 \\ \delta_4 = 3;20 & \delta_5 = 5 & \delta_6 = 3;45 \end{array} \tag{3}$$

as listed in Table 9. For the sake of comparison we recall the value for the mean step (cf. above p. 409 (13))

$$\bar{\delta} \approx 2;42,24^\circ. \tag{3a}$$

In order to determine the number of steps between Ω and Γ it is best to find in the text a pair Ω, Γ which belongs to the same zone in order to avoid transition coefficients. Hence one looks into arcs where w_i and thus δ_i is smallest, i.e. in $\alpha_2 =$ ♋ + ♌. This situation is realized in line 12 of our text with

$$\Omega \rightarrow \Gamma = ♌ 27;40 - ♋ 2;40 = 55^\circ = 33\,\delta_2.$$

[5] Published by Aaboe-Sachs [1966], p. 9 f. and p. 24 f. (Texts G to J).

Since, according to (2), the δ_i will transform exactly as the w_i one will find for all zones α_i that

$$\Omega \to \Gamma = 33\,\delta_i. \tag{4a}$$

For $\Gamma \to \Phi$ we cannot avoid crossing boundaries. In line 40 we find, e.g.,

$$\Gamma \to \Phi = ♏22 - ♋11.$$

Scorpio belongs to α_4 which requires a coefficient 0;30 with respect to α_3. Hence with respect to the beginning of α_3 (♍0°) we only have an arc of $1,0+0;40 \cdot 22 = 1,14;40°$. But w_3 is enlarged by 0;20 when starting in α_2, hence with respect to ♋0° we only have an arc of $1,0+0;45 \cdot 1,14;40 = 1,56°$ which gives for the original progress $1,56-11 = 1,45° = 1,3 \cdot \delta_2$.

This shows that our text operates with

$$\Gamma \to \Phi = 1,3\,\delta_i \tag{4b}$$

and because $6,0+\Delta\lambda$ must contain $\Pi + Z = 2,31$ steps[6] we must have

$$\Phi \to \Omega = 55\,\delta_i. \tag{4c}$$

The subdivision of the true synodic arc is therefore based on

$$\Omega \to \Gamma = s_1\,\delta_i, \quad \Gamma \to \Phi = s_2\,\delta_i, \quad \Phi \to \Omega = s_3\,\delta_i \tag{5a}$$

with

$$s_1 = 33, \quad s_2 = 1,3, \quad s_3 = 55. \tag{5b}$$

The resulting values for all six zones are given in the following table

i	$\Omega \to \Gamma$ $33\delta_i$	$\Gamma \to \Phi$ $1,3\delta_i$	$\Phi \to \Omega$ $55\delta_i$	α_i
1	1,22;30°	2,37;30°	2,17;30°	♉, ♊
2	55	1,45	1,31;40	♋, ♌
3	1,13;20	2,20	2, 2;13,20	♍, ♎
4	1,50	3,30	3, 3;20	♏, ♐
5	2,45	5,15	4,35	♑, ♒
6	2, 3;45	3,56;15	3,26;15	♓, ♈

(6)

For the transitions (4), p. 400 remains valid.

In discussing the mean situation for Mars (above p. 409) we found that the same phases are separated by

$$s_1 = 33, \quad s_2 = 1,0, \quad s_3 = 58 \tag{7}$$

mean steps, respectively, in contrast to (5b). A difference of 3 steps in the position of Φ can correspond to a longitudinal difference between $3\delta_2 = 5°$ (in ♋ and ♌) and $3\delta_5 = 15°$ (in ♑ and ♒). Whether such a change in the position of Φ relative to Γ and Ω is also reflected in a change of the elongation of Φ[7] cannot be established unless the theory for the corresponding time intervals also becomes known.

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[6] Cf. also above p. 409 (12) and (12a).

[7] Cf. above p. 411 (22b).

C. Approximate Periods

From the GADEx material two periods for Mars are known:

$$\begin{array}{l}\text{47 years, for the prediction of synodic (i.e. Greek-letter) phenomena} \\ \text{79 years, as sidereal period, i.e. in relation to Normal-Stars.}^{8}\end{array} \qquad (1)$$

The "accurate" period of 284 years then reveals itself as a linear combination of these two smaller periods:

$$284 = 47 + 3 \cdot 79. \qquad (2)$$

Since this is a parallel to similar combinations of periods of Jupiter[9] it is a plausible conjecture that (2) was based on deviations from exact returns with opposite signs and different amounts after 47 and 79 years, respectively.

It is indeed possible to find explicit support for this assumption. Since

$$\begin{array}{l}\text{47 years contain about 22 occurrences} \\ \text{79 years contain about 37 occurrences}\end{array} \qquad (3)$$

deviations in the ratio -3 to $+1$ would suggest not only (2) but also

$$22 + 3 \cdot 37 = 133 \text{ occurrences in 284 years.} \qquad (4)$$

Hence $\Pi = 133$ and $2\Pi + Z = 284$, thus $Z = 18$ and

$$P = 2{,}13/18 = 7;23{,}20 \text{ mean syn. arcs.}$$

Consequently $3P$ agrees nearly with 22 occurrences because

$$3P = 22;10\,\overline{\Delta\lambda} = 22\,\overline{\Delta\lambda} + 3/18\,\overline{\Delta\lambda}.$$

Because a mean step $\bar{\delta}$ is given by

$$\bar{\delta} = \frac{6{,}0}{\Pi} = \frac{\overline{\Delta\lambda}}{Z}$$

one can say (with $Z = 18$) that

$$3P = 22\,\overline{\Delta\lambda} + 3\bar{\delta}.$$

Similarly

$$5P = 36;56{,}40\,\overline{\Delta\lambda} = 37\,\overline{\Delta\lambda} - 1/18\,\overline{\Delta\lambda} = 37\,\overline{\Delta\lambda} - \bar{\delta}.$$

Applying again the principle that a true situation is obtainable from the mean one by using the same number of true steps we find with (3), p. 424 the following corrections[10]

i	47 years $-3\delta_i$	79 years $+\delta_i$	
1	− 7;30°	2;30°	
2	− 5	1;40	(5)
3	− 6;40	2;13,20	
4	−10	3;20	
5	−15	5	
6	−11;15	3;45	

[8] This distinction was established by Sachs (cf. Pinches-Sachs, LBAT, p. [XXV]).

[9] Cf. above p. 391; also note 4 there.

[10] Cf. also below II A 7, 4 A, p. 456.

These corrections are indeed given in a procedure text[11] with the exception of one rounding (2;15 for δ_3) and probably one scribal error (3;40 for δ_6). This is the most explicit use of the "steps" we have in our sources.

2. System B

In the preceding section we have shown that the rules of "System A" are the consequence of arranging all "steps" in such a fashion that their lengths remain constant on k arcs of the ecliptic. We will now demonstrate that the rules of "System B" result from a symmetric distribution on two semicircles of steps such that their lengths form an arithmetic progression. In other words we will operate with the following model: we assume that a certain diameter of the ecliptic connects the region of greatest density of events with the region of lowest density such that each half contains $1/2\,\Pi$ events, symmetrically arranged to the said diameter.[1] We then assume that the distances between neighboring points increase on each semicircle in arithmetic progression. Hence, taking the minimum distance as unit, these distances, or "*steps*," will be on each semicircle of length

$$1, 2, 3, \ldots, 1/2\,\Pi - 1, 1/2\,\Pi. \tag{1}$$

Any "*true synodic arc*" is then the sum of Z consecutive steps, e.g.

$$\delta_\nu = k + (k+1) + (k+2) + \cdots + (k+Z-1) \tag{2a}$$

followed by

$$\delta_{\nu+1} = (k+Z) + (k+Z+1) + \cdots + (k+2Z-1) \tag{2b}$$

where we assume for the moment that both δ_ν and $\delta_{\nu+1}$ completely belong to the same semicircle. Obviously for any such pair of consecutive synodic arcs holds that

$$\delta_{\nu+1} - \delta_\nu = Z^2 = d. \tag{3}$$

Hence we have shown that the δ_ν also form an arithmetic progression, with $d = Z^2$ as difference, as long as a sequence of Z consecutive steps does not transgress the boundary between the two semicircles. If we plot these synodic arcs δ_ν as function of ν their endpoints lie on increasing and decreasing line-segments with slope $\pm d$, respectively.

The smallest of these synodic arcs is

$$\delta_m = 1 + 2 + \cdots + Z = 1/2\,Z(Z+1) = 1/2\,d + 1/2\,Z. \tag{4}$$

By reason of symmetry such an arc must occur twice, one from an upgoing, the other from a downgoing branch, in adjacent positions in our graph. Hence the corresponding branches intersect at

$$\delta_m - 1/2\,d = 1/2\,Z = m. \tag{5}$$

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[11] ACT No. 811, Sect. 3 (p. 381).

[1] We assume here and in the following that Π is even (hence Z odd). This is convenient for our formulations but factually irrelevant. Nor is it essential for the following that the diameter in question be sidereally fixed.

Similarly there exists a pair of greatest synodic arcs of length

$$\begin{aligned}\delta_M &= 1/2\,\Pi + (1/2\,\Pi - 1) + \cdots + (1/2\,\Pi - Z + 1)\\ &= 1/2\,\Pi Z - (1 + 2 + \cdots + Z - 1) = 1/2(\Pi + 1)\,Z - 1/2\,d\end{aligned} \tag{6}$$

which define an intersection of the two branches at

$$\delta_M + 1/2\,d = 1/2(\Pi + 1)\,Z = M. \tag{7}$$

Hence we have found an amplitude

$$\Delta = M - m = 1/2\,\Pi Z \tag{8}$$

and therefore a value

$$P = \frac{2\Delta}{d} = \frac{\Pi}{Z} \tag{9}$$

as it should be for a linear zigzag function of number period Π and wave number Z.

We have not yet shown, however, that all linear segments intersect at m and M, respectively. If we begin on the increasing semicircle with the shortest Z steps (which add up to δ_m) and proceed with consecutive arcs of Z steps each we will not find δ_M before having to cross over to the decreasing semicircle.

Let us, e.g., assume that the last true synodic arc on the increasing semicircle, say δ_n, leaves just one step free on the increasing branch. In other words the step of length $1/2\,\Pi$ is not included in δ_n. Instead, the first step of δ_n has the length $1/2\,\Pi - Z$, hence

$$\delta_n = \delta_M - 1/2\,\Pi + (1/2\,\Pi - Z) = \delta_M - Z. \tag{10a}$$

On the decreasing semicircle the arc δ_{n+2} ends now $Z - 1$ steps before the upper boundary, hence its length is given by

$$\delta_{n+2} = (\delta_M - d) + (1/2\,\Pi - Z + 1) - (1/2\,\Pi - 2Z + 1) = \delta_M - d + Z. \tag{10b}$$

In general: if δ_n leaves k intervals ($k < 1/2\,Z$) free before the end of the increasing branch we find that

$$\delta_n = \delta_M - kZ \tag{11a}$$

and

$$\delta_{n+2} = \delta_M - d + kZ. \tag{11b}$$

Hence we have always

$$\delta_n + \delta_{n+2} = 2\,\delta_M - d \tag{12a}$$

independent of k, i.e. independent of the asymmetry of the position of the synodic arcs with respect to the maximum. Using (7) we can replace (12a) by

$$\delta_n + \delta_{n+2} = 2M - 2d. \tag{12b}$$

If we now call δ_{n+1} the synodic arc which corresponds to the transition from the ascending to the descending branch and define

$$\delta_{n+1} = \delta_n + d \qquad \text{for } k < 1/2\,Z \tag{13a}$$

or

$$\delta_{n+1} = \delta_{n+2} + d \qquad \text{for } k > 1/2\,Z \tag{13b}$$

then we always have, because of (12b)

$$\delta_n + \delta_{n+1} = 2M - d. \tag{14}$$

Similarly one finds for the arcs δ_n and δ_{n+1} which straddle a minimum that always

$$\delta_n+\delta_{n+1}=2m+d. \tag{15}$$

In other words: all synodic arcs satisfy the reflection rules of linear zigzag functions at the same fixed extrema m and M which are defined by (5) and (7), respectively.

What we have shown so far is that steps arranged in arithmetic progression on two semicircles lead to synodic arcs which define a linear zigzag function of period $P=\Pi/Z$ (cf. (9)). Having assumed for the lengths of the steps the special sequence 1, 2, 3, ... we have obtained for the amplitude the value

$$\Delta_0=1/2\,\Pi Z \tag{16}$$

(cf. (8)). If we are now given any arbitrary amplitude Δ we make the steps of length

$$c, 2c, 3c, \ldots$$

where

$$c=\Delta/\Delta_0=2\Delta/\Pi Z. \tag{17}$$

Consequently two consecutive synodic arcs will have the lengths

$$ck+c(k+1)+\cdots+c(k+Z-1)=c\delta_\nu$$

and

$$c(k+Z)+c(k+Z+1)+\cdots+c(k+2Z-1)=c\delta_{\nu+1}$$

(cf. (2a) and (2b)). Hence one obtains for the constant difference the value

$$d=cZ^2=\frac{2\Delta}{\Pi Z}\cdot Z^2=\frac{2\Delta}{P} \tag{18}$$

which is indeed the proper difference for the given amplitude Δ.

According to (5) the minimum would still have the value

$$m_0=1/2\,cZ=\Delta/\Pi. \tag{19}$$

In order to obtain a given minimum m we have only to add to each step the constant correction

$$m-m_0=m-\frac{\Delta}{\Pi}. \tag{20}$$

The maximum will automatically be correct since Δ has been already transformed by (17).

Example. Jupiter System B.

We are given the basic period relation (cf. Table 9, p. 423)

$$\Pi=6{,}31 \qquad Z=36. \tag{21}$$

Thus

$$P=\frac{6{,}31}{36}=10;51{,}40 \qquad \overline{\Delta\lambda}=\frac{6{,}0}{P}=33;8{,}44{,}48,\ldots\approx 33;8{,}45°. \tag{22}$$

Let us assume that observations established that the true synodic arcs may deviate as much as $\pm 5°$ from the mean. Thus we estimate

$$M\approx 38° \qquad m\approx 28° \qquad \Delta\approx 10°. \tag{23}$$

Now with (16) and (17):

$$\Delta_0 = 1/2\,\Pi Z = 6{,}31 \cdot 18 = 1{,}57{,}18 \tag{24}$$

thus

$$c = \Delta : \Delta_0 \approx 10 : 1{,}57{,}18 = 0;0{,}5{,}6, \ldots \approx 0;0{,}5 \tag{25}$$

and from (18)

$$d = c \cdot Z^2 = 0;0{,}5 \cdot 21{,}36 = 1;48^\circ. \tag{26}$$

We now must adjust the estimates in (23) and find with (24) the final values

$$\Delta = c \cdot \Delta_0 = 0;0{,}5 \cdot 1{,}57{,}18 = 9;46{,}30^\circ \tag{27}$$

and therefore with (22):

$$\begin{aligned} M &= 33;8{,}45 + 4;53{,}15 = 38;\ 2^\circ \\ m &= 33;8{,}45 - 4;53{,}15 = 28;15{,}30^\circ. \end{aligned} \tag{28}$$

Thus all parameters of a zigzag function are exactly determined[2], without making use of any new empirical information beyond (21) and (23).

The consecutive intervals themselves are given by the sequence

$$m_0 + c, \qquad m_0 + 2c, \qquad m_0 + 3c, \ldots$$

where with (19)

$$m_0 = 1/2\,cZ = 0;0{,}5 \cdot 18 = 0;1{,}30$$

hence by

$$0;1{,}35 \qquad 0;1{,}40 \qquad 0;1{,}45 \quad \text{etc.} \tag{29}$$

The corresponding synodic arcs are obtained by adding $m - m_0 = 28;14$ to each term in the sequence (29).

Having shown that the Systems A and B can be derived from basically identical concepts we can now look again at the historical relation between the two systems. It seems to me that in several respects B can be considered to be more "primitive" than A. The use of linear zigzag functions for the description of periodic phenomena is certainly mathematically the most obvious approach and it is not surprising to find this method applied in the earliest phase of Babylonian astronomy to problems like the variable length of daylight.[3] When the far more complex problems of planetary theory come into view, where a planetary phase returns to its initial position only after several sidereal rotations, it is again a mathematically rather obvious idea to account for the observable differences in the spacing of events by an arithmetical progression for their variable distribution, a model which leads, as we have seen, directly to the computing rules of System B.

In fact, the simplicity of this method also entails serious disadvantages. As soon as the most fundamental empirical parameters, Π, Z, and μ, Δ are chosen no freedom is left for taking any observed asymmetries into account.

From this viewpoint System A appears as a very clever modification of the density distribution assumed in System B. Instead of being forced into one rigid pattern one can now introduce as many free parameters as one needs. To do this in a practical fashion requires, however, a great deal of arithmetical skill, again in contrast to System B where one has only to adjust Δ if one wishes to obtain a convenient difference d.

[2] They agree exactly with the parameters found in the ephemerides; cf. ACT II, p. 310f.

[3] Cf. below IV D 1,1.

The advantages of System A over B are not restricted to the planetary theory. The greater complexity of the lunar theory repeatedly requires summations over preceding sequences, a process which increases the degree of the resulting sequence over the preceding one. Hence it is preferable to have stretchwise constant primitive functions in contrast to the linear sequences in System B. Again one could well consider A as a methodological improvement of B.

Since Kugler's days it has been customary to think of B as the more recent theory, mainly because certain parameters in the lunar theory are more accurate (by modern standards) in B than in A. This may be so – although the two systems also have important parameters in common – but any association with final parameters need not be chronologically equivalent with the origin of methods. And in the planetary theory there is no evidence whatsoever for a precedence of A over B.[4]

These considerations should not be understood as an argument for declaring System B the predecessor of A. On the contrary it is my impression that it makes little sense to assume any significant chronological gap between the two methods. It seems to me perfectly possible that within the same group of men both systems were developed practically simultaneously as alternate possibilities which would have to prove their relative merits in the time to come. Indeed both systems remained in contemporary use for the whole period at our control.

As far as we can see it is mainly the System B version which exercised the main influence on Greek astronomy. This in itself is again not indicative for the relative chronology of the two methods. As soon as the Greeks had adopted the use of geometrical methods for the description of celestial motions the computational methods of both systems lost all interest. Only the period relations and the extrema, i.e. exactly the parameters Π, Z, and μ, Δ basic for System B remained of importance. Hence it is not surprising that we have no evidence in Greek astronomy for data which are characteristic for System A.

3. Historical Reminiscences

During the academic year 1933/1934 I lectured at the Mathematics Institute of the University of Copenhagen[1] on pre-Greek mathematics. These lectures were published[2] in a volume which was supposed to be the first in a sequence of three, the second being reserved for Greek mathematics, the third for ancient mathematical astronomy. As a preparation for the first volume I had compiled an edition of all the mathematical cuneiform texts I could find.[3] Being thus preoccupied with cuneiform sources I considered it economical to prepare next the section on Babylonian astronomy for the third volume. Little did I foresee that this would mean the end of the whole plan.

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[4] Repeated attempts to establish accurate dates for the origin of the two systems require too many explicit and implicit assumptions to be taken seriously.

[1] Then under the directorship of Harald Bohr.

[2] Neugebauer, Vorl. (1934).

[3] Published as "Mathematische Keilschrift Texte" in QS A 3 in three volumes (1935, 1937).

When I started work on the astronomical texts, I had, of course, a general knowledge of Kugler's great work. I considered it as my main, and comparatively simple, task to extract from Kugler's material the mathematically oriented texts, i.e. the ephemerides and the procedure texts, supplemented by texts from Uruk which had recently become accessible by the publications of Thureau-Dangin (texts in the Louvre)[4] and Schnabel (Berlin Museum).[5]

It was evident from the outset that any summary of Kugler's work had to present his brilliant results in a form mathematically more concise than in the original publication which preserved many of the often involved, however ingenious, ways of actual discovery. Also the practical task of restoring damaged passages and sections of texts made it imperative to operate as systematically as possible. Consequently I developed checking methods for all numerical columns in ephemerides, based on the simple idea of representing periodic sequences of fixed amplitude by monotone sequences in infinitely many strips.[6] This method provided, at the same time, information about the size of the gap between related fragments of ephemerides. Combining in a linear diophantine equation the number n of intermediary lines with the number m of traversed strips (i.e. the number of periods of the original function) one obtains the time difference between the lines in question because each line in an ephemeris represents a known unit of time, e.g., in the lunar ephemerides a mean synodic month. This "*diophant*" became the main tool in dating and classifying scattered fragments.

The consistent application of this method completely upset my original program. Even in Kugler's published material new joins could be established and similar situations occurred with the texts from Uruk. Obviously nothing short of a new edition of all available texts could be considered as the basis for any adequate presentation of Babylonian mathematical astronomy. Meantime I had obtained access to unpublished texts in Berlin, Chicago, Istanbul, and – through Strassmaier's notebooks[7] – the British Museum, a repository whose wealth only became fully evident through the direct exploration of A. Sachs and by his obtaining access to Pinches' volumes of masterful copies which had been kept secret for over 50 years.[8]

In the present context we have to return to the method of diophantine dating. Originally constructed for linear zigzag functions,[9] it can easily be extended to step functions.

Let us consider a System A model with k arcs α_i and Π and Z as number period and wave number, respectively. We then know that $P = \Pi/Z$ counts the number of occurrences contained in 360° of longitude, i.e., after P synodic arcs the longitude of the phenomenon in question should have gained 360°, using the proper synodic arcs on each of the k sections (cf. Fig. 36). Actually how this progress was reached is without interest. The same result can be obtained by moving with constant velocity from O to A, dropping discontinuously at the end ($\lambda = 6{,}0$) down to B.

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[4] Thureau-Dangin, TU (1922).

[5] Schnabel, Ber. (1923), a rather chaotic publication.

[6] Cf. above p. 374.

[7] Cf. above p. 348.

[8] The majority of Pinches' copies are now available in Pinches-Sachs LBAT (1955).

[9] Neugebauer [1936, 1].

We make use of this procedure, beginning at λ_0 and ending at λ_n after m circuits through the ecliptic (cf. Fig. 37). The only assumption made in this construction is that both γ_0 and γ_n belong to the same arc α_i for which the synodic arc has the value w_i. The drop after each complete rotation is then given by $Pw_i - 6{,}0$. The total progress during n lines is given by

$$\lambda_n + m \cdot 6{,}0 - \lambda_0$$

and must be the same as nw_i diminished by m reductions in the amount $Pw_i - 6{,}0$ each. Thus

$$\lambda_n + m \cdot 6{,}0 - \lambda_0 = nw_i - m(Pw_i - 6{,}0)$$

or

$$nw_i - mPw_i = \lambda_n - \lambda_0. \tag{1}$$

This is a diophantine equation for n and m which gives, if solutions exist, all possible distances between the given longitudes λ_n and λ_0, counted in numbers of occurrences.

This is the method which I used throughout for the dating and restoration of texts by joins between fragments, eventually published in the ACT.[10] In the above formulation of the procedure we have adopted the attitude that we were dealing with continuous motions whereas in fact we only have single events at discrete longitudes. Since the period P is not an integer the points A and B in Fig. 35 cannot represent actual events. Indeed, only after Π lines will an occurrence again return to $\lambda = 0$ (as assumed in Fig. 35) after Z crossings of the strips of 6,0° width. Apparently the diophantine method seems to introduce such a meaningless concept as fractions of an event.

This deficiency of formulation was repaired when van der Waerden reduced[11] all Z strips into one by dividing each synodic arc w_i into Z "*steps*" of length $\delta_i = w_i/Z$. Then one obtains the previous formulae by saying that consecutive occurrences are always separated by Z steps, regardless of the arcs on which the endpoints of a sequence of Z steps are located. That this is so we have shown above p. 421 f.

Returning to the diophantine equation (1) with the new definition $w_i = Z\delta_i = \frac{\Pi}{P}\delta_i$ we have now

$$nZ\delta_i - m\Pi\delta_i = \lambda_n - \lambda_0$$

or

$$nZ - m\Pi = \frac{1}{\delta_i}(\lambda_n - \lambda_0). \tag{2}$$

Since Π and Z are relatively prime one can say: two longitudes λ_n and λ_0 of events on the same arc α_i are connectible if and only if $\lambda_n - \lambda_0$ is an integer multiple N of the step δ_i which is associated with the arc α_i. In other words N is a solution of the congruence

$$nZ \equiv N \mod \Pi \quad \text{with} \quad N = \frac{1}{\delta_i}(\lambda_n - \lambda_0). \tag{3}$$

Van der Waerden made effective use of this concept of "steps" in particular for the theory of Mars which we have considered above p. 406 ff. and p. 424 ff. He

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[10] London 1955. The omission of the date on the title page was overlooked by all concerned.

[11] Van der Waerden [1957], p. 47.

also realized that the Babylonian astronomers were aware of these short intervals δ_i which appear explicitly in a procedure text.[12]

The full significance of the arrangement of all Π occurrences on the ecliptic at distances δ_i, depending only on α_i, was finally recognized by A. Aaboe. Working with A. Sachs on a group of fragmentary texts[13] which give longitudes only for all Greek-letter phenomena of the outer planets he saw that a uniform pattern dominates the distribution of the Π points where the phases occur, resulting in a density distribution characteristic for a System A model. And, as we have seen above p. 424f., also the subdivision of the synodic arcs is ultimately based on these "steps" or "basic intervals." There can be little doubt that this was the foundation on which the Babylonian planetary theory was erected. The fact that System B also can be derived in a similar fashion[14] supports this conclusion.

It is a curious coincidence that the idea of starting from the set of all occurrences of a certain phase for the total number period is not restricted to Babylonian astronomy. Newcomb, in his study of cycles for solar eclipses[15] took his point of departure from the same idea. He investigated the behavior of the "conjunction points" of two celestial bodies, rotating with mean motions of conveniently chosen integer ratio. If this ratio were exact the "conjunction points" would form the regular polygon of our mean "steps." But in nature such exact arithmetical conditions are not satisfied and the problem arises of modifying the mean situation to fit the empirical data. Here, of course, the parallelism of ancient and modern methods ends,[16] and, needless to say, the existing parallelism did not become apparent until after the background of the Babylonian theory had been clarified. But it seems to me not a small tribute to the mathematical ability of the Babylonian astronomers that they operated with concepts which proved fruitful in the hands of a Newcomb.

§ 7. The Single Planets

1. Introduction

We now turn to a description of the details of Babylonian planetary theory. It was a fortunate accident that the material published in ACT was comparatively uniform in character, thus making it possible to obtain a more or less consistent picture of the underlying methods. Texts that have come to light in the meantime, however, provided evidence for a much greater complexity than had been contemplated around 1955, making it only too clear that we are far removed from real insight into the origin and development of astronomical traditions in Mesopotamia.

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[12] Cf. above p. 427; van der Waerden. Anf. d. Astr., p. 187ff.
[13] Cf. Aaboe [1965] and Aaboe-Sachs [1966].
[14] Above p. 427f.
[15] Newcomb [1897].
[16] It may be mentioned, however, that in Newcomb's discussion also a linear diophantine equation stands in the center of the problem.

We shall summarize in the following, planet by planet, what we know about the different computational systems, their parameters and basic assumptions. Beyond cuneiform sources Indian material, from Varāhamihira's Pañca-Siddhāntikā chapter XVII[1] (sixth century A.D.), will also be utilized.

That Indian astronomy was greatly influenced by Babylonian methods is well known.[2] It may suffice to mention the use of the ratio 3:2 for the extrema of the length of daylight,[3] of the 19-year cycle,[4] or of the characteristic parameters of the anomalistic lunar motion,[5] well attested in cuneiform sources as well as in Greek papyri.[6] The planetary data found in Pc.-Sk. chapter XVII, however, are frequently not available in cuneiform texts although they are undoubtedly of ultimately Babylonian origin. It is therefore clear that this Indian material should be utilized in conjunction with Babylonian and Greek sources, even if we are left ignorant of chronological relations and other influences.

It is obvious that many problems will remain unsolved. For example the order of the planets in Pc.-Sk. XVII is

♀ ♃ ♄ ♂ ☿

attested nowhere else.[7] Chap. XVI, 23 gives visibility limits for the moon (12°) and for the planets:

♀	♃	☿	♄	♂	
9°	11°	13°	15°	17°	(1)

reckoned in time degrees between planet-rise and sun-rise.[8] The arithmetical pattern is, at least in spirit, Babylonian in character.

How close some Indian procedures come to the basic concepts of Babylonian planetary theory is shown by rules for obtaining the mean position of a planet, starting from a given position in the past. One rule[9] tells us to multiply the number N of occurrences since epoch of a certain planetary phase (e.g. Γ) by a number Z and divide the result by Π, where Z and Π are the characteristic Babylonian parameters which express the fundamental period relations.[10] Since

$$\Pi/Z = P \quad \text{and} \quad 6{,}0/P = \overline{\Delta\lambda}$$

we see that

$$N \cdot Z/\Pi = N/P = N \cdot \overline{\Delta\lambda}/6{,}0.$$

In other words the Indian rule simply says: reduce the total mean motion since epoch modulo 360°.

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[1] I quote from the commentary to the edition of the Pañca-Siddhāntikā by Neugebauer-Pingree, Vol. II.

[2] We need not to discuss here the question as to whether this influence is due to hellenistic intermediaries or to Iranian contacts; cf. Pingree [1963, 1].

[3] Pc.-Sk. I, 15.

[4] Pc.-Sk. II, 8 and XII, 5.

[5] Cf. below p. 481 (Pc.-Sk. II, 2–6; III, 4–9); also V A 2, 1 D 2.

[6] Cf. below V A 2, 1.

[7] Cf. Neugebauer-Pingree, Pc.-Sk. II, p. 109.

[8] Pc.-Sk. II, p. 108; Chap. XVII, 58 (p. 125) gives a different sequence. The order chosen here in (1) is numerical; in this part of the text the order is the ordinary Indian one, i.e. the order of the weekdays.

[9] Pc.-Sk. II, p. 115.

[10] Cf. above p. 420 and p. 390f., as compared with Pc.-Sk. II, p. 112, Table 24.

The second rule[11] tells us to multiply the longitudinal increment since epoch, $\lambda_0 - \lambda$, by a number b and divide the result by another number a where the comparison with the Babylonian data shows that

$$a/b = \overline{\Delta\lambda}. \tag{2}$$

Hence: reduce the total progress in longitude modulo the mean synodic arc. In both cases the residues will tell us how far the mean planet has moved beyond the initial position.[12]

The numbers a and b in (2) provide us with a good example for the way in which Babylonian parameters were assimilated in Indian astronomy.[13] We write the Babylonian values for the mean synodic arcs[14] (including complete rotations) as decimal integers plus sexagesimal fractions, assuming convenient roundings for the latter:

$$\begin{array}{lll} \text{Saturn:} & 6{,}12;39{,}22{,}30^\circ & \approx 372;40^\circ \\ \text{Jupiter:} & 6{,}33;\ 8{,}44{,}48,\ldots & \approx 393;\ 8{,}34{,}17,\ldots \\ \text{Mars:} & 12{,}48;43{,}18{,}29,\ldots & \approx 768;45 \\ \text{Venus:} & 3{,}35;30 & = 215;30 \\ \text{Mercury:} & 1{,}54;12{,}36{,}38,\ldots & \approx 114;12{,}36. \end{array} \tag{3}$$

The fractional parts are expressible by small rational numbers:

$$\begin{array}{l} 0;40 = 2/3 \\ 0;8{,}34{,}17,\ldots \approx 1/7 \\ 0;45 = 3/4 \\ 0;30 = 1/2 \\ 0;12{,}36 = 1/5 + 1/100 = 21/100 \approx 6/29. \end{array} \tag{4}$$

The denominators of these fractions are the Indian numbers b. To produce the integers a we multiply the numbers in (3) by b and find[15]

$$\begin{array}{ll} \text{Saturn:} & 372 \cdot 3 + 2 = 1118 \\ \text{Jupiter:} & 393 \cdot 7 + 1 = 2752 \\ \text{Mars:} & 768 \cdot 4 + 3 = 3075 \\ \text{Venus:} & 575 \cdot 2 + 1 = 1151 \\ \text{Mercury:} & 114 \cdot 29 + 6 = 3212. \end{array} \tag{5}$$

Consequently the quotients a/b will be close approximations of the $\overline{\Delta\lambda}$ in (3).

2. Saturn

The fundamental period relations for this planet are[1]

$$4{,}16 (=256) \text{ occurrences} = 9 \text{ sidereal revolutions} = 4{,}25 (=265) \text{ years} \tag{1a}$$

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[11] Pc.-Sk. XVII, 64–80 (Vol. II, p. 126, Table 33).

[12] Concerning the subsequent rules for obtaining the true from the mean positions cf. the discussion in Pc.-Sk. II, p. 115 ff. and p. 126 ff.

[13] The following is based on a suggestion made by A. Sachs.

[14] Listed above p. 391 (11).

[15] Pc.-Sk. II, p. 126, Table 33 (where 53,12 for Mercury is a misprint for 55,12). For Venus 575 = 215 + 360.

[1] Cf. above p. 390 (10a) and (11).

and consequently

$$\Pi = 4{,}16, \quad Z = 9, \quad P = 28;26{,}40 \tag{1b}$$

with

$$\overline{\Delta\lambda} = 12;39{,}22{,}30°, \quad \bar{\delta} = 1;24{,}22{,}30° \tag{1c}$$

for the mean synodic arc and the mean "step," respectively.

The goal-year texts operate with a period of 59 years. From astrological texts we know of a period of 560 years,[2] in excess of the ACT period but probably related to it by

$$560 = 265 + 5 \cdot 59 \ (= 29 + 9 \cdot 59).$$

We know of two Babylonian versions of System A (using two zones) and one from Indian sources (three zones). System B is attested in ACT ephemerides and in procedure texts.

Aaboe [1958] made a systematic comparison between Babylonian planetary models (both of System A and B) and close approximations to a Kepler motion. The case of Saturn is discussed on p. 240–242 of his publication.

A. System A

What seems to be the main model of System A with two zones is only known from the procedure texts ACT Nos. 801 and 802 and from "template" texts[1] which cover a whole number period. The characteristic parameters are

$$\begin{array}{llll} \alpha_1 = 3{,}20° & \text{from } ♌ 10 \text{ to } ♊\ 0 & w_1 = 11;43{,}7{,}30° \\ \alpha_2 = 2{,}40° & \text{from } ♊ 0° \text{ to } ♌ 10 & w_2 = 14;3{,}45. \end{array} \tag{2a}$$

Thus $w_1 : w_2 = 5:6$ and

$$\delta_1 = 1;18{,}7{,}30°, \quad \delta_2 = 1;33{,}45° \tag{2b}$$

for the steps.

The template texts differ in several respects from the ordinary ephemerides. First they give only longitudes and no dates. Secondly they do not arrange the different phases in different columns but they list the phases in their natural order, e.g.[2]

$$\begin{array}{lcl} 9;50{,}37{,}30 & \approx & \Gamma \\ 17;20{,}37{,}30 & \approx & \Phi \\ 10;40{,}37{,}30 & \approx & \Psi \\ 18;13{,}45 & \approx & \Omega \\ 21;33{,}45 & \approx & \Gamma \\ & \text{etc.} & \end{array} \tag{3}$$

Furthermore one misses the positions of the oppositions (Θ), which are listed in ephemerides of System B.[3] Finally it should be mentioned that in the retrograde motion from Φ to Ψ a recrossing of the discontinuity (at ♌ 10 or ♊ 0) is ignored by making the total retrogradation always $-6;40°$ or $-8°$, depending on whether

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[2] Van der Waerden (AA, p. 110 ≈ BA, p. 111) discusses a 589-year period which is, however, only the result of an incorrect restoration by Kugler of a broken passage; cf. Neugebauer-Sachs [1967], p. 206, n. 32.

[1] Aaboe-Sachs [1966], p. 3f., Texts A and B.

[2] Aaboe-Sachs [1966], p. 13 obv. column I, lines 1 to 5.

[3] ACT Nos. 704 and 704a.

Φ belongs to the slow or to the fast zone.[4] For an ephemeris, however, the column Ψ would be computed according to the general rules, independent of Φ.

Another dateless list of longitudes for only one (unknown) phase of Saturn[5] is computed on the basis of a variant of System A. Its parameters are

$$\begin{array}{llll} \alpha_1 = 2{,}50° & \text{from } ♌20 \text{ to } ♒10, & w_1 = 11;43,7,30° \\ \alpha_2 = 3{,}10 & \text{from } ♒10 \text{ to } ♌20, & w_2 = 14;3,45 \end{array} \tag{4a}$$

thus $w_1 : w_2 = 5:6$ as before, but operating with much shorter steps

$$\delta_1 = 0;3,7,30°, \qquad \delta_2 = 0;3,45° \tag{4b}$$

because Π and Z are now much larger

$$\Pi = 1,45,4, \qquad Z = 3,45, \qquad P = 28;1,4 \tag{4c}$$

but less accurate than in (1b). It is impossible to say whether we have here a recognized variant of System A or only some isolated exercise of planetary computations.

From the Pañca-Siddhāntikā[6] we know of the existence of a System A model for Saturn with three zones:

$$\begin{array}{llll} \alpha_1 = & 45;51° \text{ containing} & \pi_1 = 30 \text{ occurrences} \\ \alpha_2 = & 2,57;34 & \pi_2 = 2,7 \\ \alpha_3 = & 2,16;35 & \pi_3 = 1,39 \\ \text{total:} & 6,0° & \text{total: } \Pi = 4,16. \end{array} \tag{5}$$

From (5) one obtains for the corresponding steps

$$\begin{array}{ll} \delta_1 = 45;51/30 & = 1;31,42° \\ \delta_2 = 2,57;34/2,7 & = 1;23,53,23,\ldots \\ \delta_3 = 2,16;35/1,39 & = 1;22,46,40. \end{array} \tag{6}$$

Multiplying these numbers by $Z = 9$ one finds

$$\begin{array}{ll} w_1 = 13;45,18° & \text{hence } \alpha_1/w_1 = 3;20 \\ w_2 = 12;35,0,28,\ldots & \alpha_2/w_2 = 14;6,40,0,23,\ldots \\ w_3 = 12;25 & \alpha_3/w_3 = 11. \end{array} \tag{7}$$

Since α_2/w_2 does not end in a finite sexagesimal fraction special conditions for the location of the discontinuities must be satisfied. We shall find analogous situations in Babylonian material.[7]

A very similar model is also known for Jupiter.[8] In both cases one arc (α_2) is almost, or exactly, 180°, while a very short arc (45;51° and 20;30°, respectively) contains the longest steps, i.e. the greatest synodic arcs, while the synodic arcs on the remaining two sections deviate very little from the mean value. This concentration of almost the whole anomaly into a comparatively small area

[4] Aaboe-Sachs [1966], p. 4.
[5] Aaboe-Sachs [1966], p. 5f., Text C, beginning and end broken away.
[6] Pc.-Sk. XVII, 16–19 (Neugebauer-Pingree II, p. 112).
[7] Cf. below p. 532.
[8] Cf. below p. 446.

near the "perigee" is a strange feature, nowhere else attested in a System A-type model of planetary motion, excepting, perhaps, System A_3 for Mercury (Ω).[9]

B. System B

System B is attested in twelve fragments of ephemerides, as well as in procedure texts.[1] The sequences of longitudes form zigzag functions which are based on

$$\begin{aligned} m &= 11;14,\ 2,30^\circ & \mu &= 12;39,22,30^\circ \\ M &= 14;\ 4,42,30 & d &= \ 0;12. \end{aligned} \tag{8a}$$

The mean value μ is exactly the mean synodic arc $\overline{\Delta\lambda}$ derived from the basic period relations (above p. 437). For the dates we have both accurate (8 b) and approximate (8 c) parameters:

$$\begin{aligned} m &= 22;41,23,7,30^\mathrm{t} & \mu &= 24;\ 6,43,7,30 \\ M &= 25;32,\ 3,7,30 & d &= \ 0;12 \end{aligned} \tag{8b}$$

and

$$\begin{aligned} m &= 22;41,25^\mathrm{t} & \mu &= 24;\ 6,45 \\ M &= 25;32,\ 5 & d &= \ 0;12. \end{aligned} \tag{8c}$$

Since for all these lists $\Delta = M - m = 2;50,40$ and $d = 0;12$ they have the same period $P = 28;26,40$ as required by (1 b), p. 437. For some evidence of abbreviated parameters for longitudes cf. ACT No. 707.

As we have seen above p. 396 (7) there should hold the nearly exact relation

$$\overline{\Delta\tau} - \overline{\Delta\lambda}^\mathrm{t} = \frac{e}{P} + (6,0 + e)^\mathrm{t} \tag{9}$$

between the mean synodic time $\overline{\Delta\tau}$ and the mean synodic arc $\overline{\Delta\lambda}$, counted in tithis, e being the epact. From (8 a) and (8 b) it follows that

$$\overline{\Delta\tau} - \overline{\Delta\lambda}^\mathrm{t} = 6,24;6,43,7,30 - 12;39,22,30 = 6,11;27,20,37,30^\mathrm{t}.$$

From it we find with $P = 28;26,40$ exactly

$$e = (6,11;27,20,37,30 - 6,0) \cdot 28;26,40/29;26,40 = 11;4^\mathrm{t}. \tag{10}$$

Hence System B for Saturn uses exactly the same epact as System A of the solar theory since (10) implies that one year contains $6,0 + 11;4^\mathrm{t} = 12;22,8$ mean synodic months.[2]

C. Subdivision of the Synodic Arc; Daily Motion

From the dateless "template" texts, described above p. 437, one derives the following intervals between consecutive phases:[3]

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[9] Cf. below p. 469.

[1] ACT Nos. 700 to 709 and Nos. 801 and 802. For an example cf. above p. 381, Table 6 (ACT No. 702).

[2] Cf. above p. 378.

[3] Aaboe-Sachs [1966], p. 4, Table 3; there it was remarked that the procedure texts ACT Nos. 801 and 802 erroneously describe a motion as retrograde which is actually the direct motion $\Psi \to \Omega$. This correction was overlooked by van der Waerden (AA, p. 185, BA, p. 263) and amplified by an error of his own: arbitrarily increasing the numbers for the velocities in (11 b) for $\Phi \to \Theta$ by 0;10.

	slow arc	fast arc	
$\Gamma \to \Phi$	7;30°	9°	
$\Phi \to \Psi$	−6;40	−8	(11 a)
$\Psi \to \Omega$	7;33,7,30	9;3,45	
$\Omega \to \Gamma$	3;20	4	
total $\Gamma \to \Gamma$	$w_1 = 11;43,7,30$	$w_2 = 14;3,45$.	

Procedure texts[4] give further details:

	duration	slow arc	fast arc	
$\Omega' \to \Omega$	30^{τ}			
$\Omega \to \Gamma$	?	$0;5^{\circ/\tau}$	$0;6^{\circ/\tau}$	
$\Gamma \to \Gamma'$	30			
$\Gamma' \to \Phi$	90	0;3,20	0;4	(11 b)
$\Phi \to \Theta$	52;30	−0;4,[13,20]	−0;5,4,24	
$\Theta \to \Psi$	60	−0;3,20	−0;4	
$\Psi \to \Omega'$	90	0;3,35,30	0;4,18,40.	

Combining these two lists one would find 40 tithis for the time of invisibility and thus a synodic period of $392;30^{\tau}$ which is a little too long. That no specific time for $\Omega \to \Gamma$ is given in the text is perhaps explicable by the use of this interval to absorb discrepancies between simple schematic patterns and accurately computed ephemerides.

In the Pañca-Siddhāntikā[5] we again find a parallel, although without reference to a System A model:

$\Gamma \to \Gamma'$	1;20°	16^{τ}	thus	$0;5^{\circ/\tau}$	
$\Gamma' \to \Phi$	3;53	56		0;4, 8, ...	
$\Phi \to \Theta$	−3	55		−0;3,16, ...	
$\Theta \to \Psi$	−4	60		−0;4	(12)
$\Psi \to \Omega'$	8	112		0;4,17, ...	
$\Omega' \to \Omega$	3	36		0;5	
total $\Gamma \to \Omega$	9;12°	335^{τ}			

The Babylonian scheme (11 b) would lead to $352;30^{\tau}$ for $\Gamma \to \Omega$.

Finally we have a Greek papyrus which gives a dateless template for the daily motion of Saturn, based on difference sequences.[6] Well preserved are the sections

$\Psi \to \Omega$	117^{d}	8; 7,2,42°	velocity increasing	(13)
$\Omega \to \Gamma$	31	4;24,9,36	to $0;8,15,18^{\circ/d}$.	

The whole synodic period amounts to 378^{d}, a number which is also given by Cleomedes.[7] If one uses the parameters of System B one obtains from the rela-

[4] ACT Nos. 801 and 802; cf. the preceding note.

[5] Pc.-Sk. XVII, 19–20 (Neugebauer-Pingree II, p. 118).

[6] Cf. below p. 791, Table 1.

[7] Cf. below p. 964.

tion (9), p. 439 a mean synodic time of

$$\overline{\Delta\tau}=384;6,43,7,30^{\tau}\approx 378;6^{d}. \tag{14}$$

This value can be used to obtain an estimate for the length of the sidereal year. Assuming a solar mean motion of about $0;59,8^{\circ/d}$ it takes the sun about $12;50^{d}$ to travel the distance $\overline{\Delta\lambda}=12;39,22,30°$. Therefore the sidereal year would be $378;6-12;50=365;16^{d}$ which is at least the right order of magnitude.

3. Jupiter

Among the ephemerides assembled in ACT Jupiter is by far the best represented planet.[1] A commonly used period relation is

$$6,31\ (=391)\ \text{occurrences}=36\ \text{sidereal revolutions}=7,7\ (=427)\ \text{years}. \tag{1a}$$

Consequently

$$\Pi=6,31, \qquad Z=36, \qquad P=10;51,40 \tag{1b}$$

and

$$\overline{\Delta\lambda}=33;8,44,48,\ldots°, \qquad \bar{\delta}=0;55,14,34,\ldots°. \tag{1c}$$

The rounding

$$\overline{\Delta\lambda}=33;8,45° \tag{1d}$$

is used as mean value in zigzag functions describing the synodic progress of the planet (System B and System B′).[2]

As we have remarked before[3] the period of 427 years is the combination of two smaller periods, the universally known Jupiter period of $a=12$ years and the goal-year period $b=71$ years. Their sum, $a+b=83$ years is a second goal-year period. Beyond these four periods we know from procedure texts[4] and from an astrological text[5] four additional periods which progress in a simple arithmetical pattern to the final period:

$$\begin{array}{ll} 2a+b\ =\ 95 & a=12\ \text{years}, \quad b=71\ \text{years} \\ 3a+2b=178^{6} & \\ 4a+3b=261 & \\ 5a+4b=344 & \\ 6a+5b=427. & \end{array} \tag{2}$$

The constant difference is, of course, $a+b=83$ years. The increasing accuracy of the periods listed in (2) can be demonstrated from data provided by the procedure texts. Let α be the correction[7] of the period a, β of b. Then two sets

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[1] Cf. the diagram ACT II, p. XII.

[2] Cf. below p. 446. For a comparison of the results with the actual facts cf. Aaboe [1958], p. 242–245.

[3] Above p. 391 (12).

[4] ACT No. 812, Sect. 10 (p. 395 f.) and No. 813, Sect. 20 (p. 414).

[5] Kugler SSB I, p. 48, a passage which is part of Sachs LBAT No. 1593 (rev. 12 ff.).

[6] ACT No. 813, Sect. 20 gives the meaningless number 2,46,40 (instead of 2,58); the parallel passage in No. 814 is damaged. An emendation to 2,46 makes no sense because $2,46=2\cdot 83$ is not a new period.

[7] That is to say: the sidereal longitude of a given phase will be $\lambda^*+\alpha$ after 12 years and $\lambda^*+\beta$ after 71 years.

of such corrections are mentioned in procedure texts[8]:

$$\alpha = +4;10°, \quad \beta = -5° \quad \text{and} \quad \alpha = +5°, \quad \beta = -6°. \tag{3}$$

Consequently we find for the sequence (2) the following corrections:

$2\alpha + \beta$	+3;20°	+4°
$3\alpha + 2\beta$	+2;30	+3
$4\alpha + 3\beta$	+1;40	+2
$5\alpha + 4\beta$	+0;50	+1
$6\alpha + 5\beta$	0	0.

The differences $\alpha + \beta$ are the corrections $-0;50°$ and $-1°$ for the 83 year period. The ratios between the two sets are 5:6 because of (3).

A. Sachs noted that the ratio 5:6 in (3) can be derived from the basic model of ephemerides of the System A type. In these ephemerides we have two zones of synodic arcs,[9] one with $w_1 = 30°$, the other with $w_2 = 36°$. We can describe this arrangement also by saying that the points in the ecliptic at which a certain phase (e.g. Γ) can occur are distributed in two different densities: a greater density on the "slow arc" (hence shorter synodic progress), a lesser density on the "fast arc", such that their ratio is 6:5.

Let us now assume that there exists an exact period which brings the planet after Π occurrences of the phase Γ, and thus after Z complete revolutions, back to exactly the same sidereal longitude. (We need not know the value of Π, it suffices to assume the existence of such a large number, and of a corresponding number Z, greater than 6 in our case). We then consider a smaller and approximate period, in our case of 12 years corresponding to one revolution, or of 71 years, corresponding to 6 revolutions. Starting from one point on the zodiac we know that these revolutions do not bring us exactly back to the same place but to another point in the set of the Π possible points; in other words Γ will occur an integer number k of steps beyond of, or short of, a multiple of Π. Wherever we start with a phase Γ, the next Γ will be k steps distant from it. Thus, if we begin within a slow arc the next Γ will be k short intervals away, if we start on a fast arc we have an increment of k long steps. Hence the deviations from exact return will have the same ratio as the synodic arcs $w_1 : w_2$, q.e.d.

These considerations are not limited to a strict System A arrangement. Since we are dealing with approximate periods we will very nearly return to the same point, i.e. it is only a small number of steps which will be gained or lost from an exact return.[10] If we operate with any smooth density distribution, depending only on sidereal longitudes and with extrema in diametrically opposite areas the corrections for approximate periods will also vary in the same ratio as the extrema of the synodic arcs. Hence the corrections given in (3) only indicate that the 12-year period results in positions between 4;10° and 5° ahead of the starting point whereas the 71-year period produces phases between $-5°$ and $-6°$ from the initial position.

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[8] ACT No. 813, Sect. 1 and No. 814, Sect. 1; No. 812, Sect. 10 and No. 813, Sect. 20.

[9] Cf. for details below II A 7, 3 A.

[10] In our case 5 for the 12-year period, 6 for the 71-year period, among 391.

Sachs has also shown, by modern computation of the phases of Jupiter for many cycles, that the limits (3) correspond excellently to the facts. Fig. 38 (p. 1331) shows for 13 consecutive years[11] the corrections of the sidereal longitudes of Γ both 12 years and 71 years later. In the first case the longitudes increase by at most 5.2° in ♓ and decrease by −6.1° in the second case. The smallest increase is about 4.2° and −5.1°, respectively, in ♍. The midpoints of the two arcs of System A are ♓ 12;30 and ♍ 12;30.[12]

It is easy to see that the parameters α in (3) are in fact embedded in the model of System A. The 12-year period corresponds to one rotation through the zodiac and the computing rules of the ephemerides produce exactly the increment of +4;10° if one moves from a point in the slow arc to the corresponding point one revolution later.[13] Since $w_1 : w_2 = 5:6$ the correction in the fast arc is necessarily 5°.

Turning to the synodic times we may expect to find them based on an epact value $e = 11;4^{\tau}$ since this is the norm expressly mentioned in the procedure texts.[14] Nevertheless we will have to tolerate some approximations since the term $1/\Pi$ appears in the relations between $\Delta\lambda$ and $\Delta\tau$ and $\Pi = 6,31$ for Jupiter does not result in a finite sexagesimal expression for $1/\Pi$ (in contrast to the case for Saturn[15] with $\Pi = 4,16$).

First we determine the mean synodic time $\overline{\Delta\tau}$ on the basis of[16]

$$\overline{\Delta\tau} = \overline{\Delta\lambda}^{\tau} + \frac{e}{P} + (6,0 + e)^{\tau}. \tag{5a}$$

Since $\overline{\Delta\lambda} = 6,0/P$ and $P = \Pi/Z$ we have

$$\overline{\Delta\tau} = (6,0 + e)\left(\frac{1}{P} + 1\right) = (6,0 + e)\,\frac{\Pi + Z}{\Pi}. \tag{5b}$$

With $\Pi = 6,31$ and $Z = 36$ this gives

$$(\Pi + Z)/\Pi \approx 1;5,31,27,28 \tag{6}$$

thus

$$\overline{\Delta\tau} = 6,11;4 \cdot 1;5,31,27,28 \approx 6,45;13,52,56^{\tau}. \tag{7a}$$

Indeed we find in the ephemerides of System B

$$\overline{\Delta\tau} = 12^{m} + 45;14^{\tau} \tag{7b}$$

and in System B′ even

$$\overline{\Delta\tau} = 12^{m} + 45;13,53^{\tau}. \tag{7c}$$

In other aspects, however, System B′ seems to be less accurate than System B.[17]

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[11] Starting with year 24 of Darius I, i.e. −497.

[12] Cf. below p. 447.

[13] Cf. the "checking rules" ACT, p. 307 and p. 309.

[14] ACT No. 812, Sect. 2 and No. 813, Sect. 13.

[15] Above p. 439 and p. 395.

[16] Above p. 395 (3).

[17] Cf. below p. 446.

Secondly one uses a relation of the type (5a) also for synodic arcs and times in general, i.e. not only for the mean values. Thus:

$$\Delta\tau=\Delta\lambda+6{,}0+e\left(\frac{1}{P}+1\right)=\Delta\lambda+6{,}0+e\cdot\frac{\Pi+Z}{\Pi} \tag{8a}$$

and with (6)

$$\begin{aligned}\Delta\tau-6{,}0&\approx\Delta\lambda+11;4\cdot 1;5{,}31{,}27{,}30\\&=\Delta\lambda+12;5{,}8{,}8{,}20^{\mathrm{r}}\approx\Delta\lambda+12;5{,}10^{\mathrm{r}}.\end{aligned} \tag{8b}$$

The increment $12;5{,}8{,}8{,}20^{\mathrm{r}}$ is mentioned in procedure texts[18] whereas the ephemerides of System A use for $\Delta\lambda=30°$ and $\Delta\lambda=36°$, respectively

$$\begin{aligned}d&=12^{\mathrm{m}}+42;5{,}10^{\mathrm{r}}\\D&=12^{\mathrm{m}}+48;5{,}10^{\mathrm{r}}\end{aligned} \tag{8c}$$

to transfer synodic arcs into time intervals.[19] In System A′ the intermediate arc with $\Delta\lambda=33;45°$ leads to the medium difference

$$d'=12^{\mathrm{m}}+45;50{,}10^{\mathrm{r}}. \tag{8d}$$

Similar relations hold for System B.[20]

A. System A

Ephemerides of this type[1] are represented in two versions: one with two zones (the "System A" proper):

$w_1=30°$	from ♊ 25 to ♐ 0	thus $\alpha_1=2{,}35°$	(9)
$w_2=36$	from ♐ 0 to ♊ 25	$\alpha_2=3{,}25$	

and one with four zones (System A′), hence with smoother transitions:

$w_1=30°$	from ♋ 9 to ♏ 9	thus $\alpha_1=2{,}0°$	(10)
$w_2=33;45$	from ♏ 9 to ♑ 2	$\alpha_2=53$	
$w_3=36$	from ♑ 2 to ♉ 17	$\alpha_3=2{,}15$	
$w_4=33;45$	from ♉ 17 to ♋ 9	$\alpha_4=52.$	

In both cases the periods are

$$\Pi=6{,}31,\qquad Z=36,\qquad P=10;51{,}40. \tag{11}$$

In System A exists a diameter of symmetry through the points ♍ 12;30 and ♓ 12;30.[2] In System A′ the midpoint of the slow arc is ♍ 9, of the fast arc ♓ 9;30.

The rules of computation resulting from these schemes we have described before (p. 392 to 394). Fragments of 12 columns are preserved in a text which gives the longitudes (but no dates) for the consecutive phases Γ, Φ, Θ, Ψ, Ω. The restoration shows that two tablets of 18 columns (of 55 lines each) would cover

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[18] ACT No. 812, Sect. 2.

[19] In the procedure text ACT No. 813, Sect. 1 (p. 403/404) the value of d is abbreviated to $6{,}42^{\mathrm{r}}$.

[20] Cf. below p. 446.

[1] ACT Nos. 600 to 608 and Nos. 609 to 614.

[2] ACT No. 814, Sect. 2 (p. 424) seems to mention instead ♍ 12 and ♓ 12.

a complete number period,[3] beginning and ending with Ω at ♊25.[4] The computation strictly follows System A for all phases, thus avoiding inconsistencies noted in similar templates for longitudes of Saturn.[5]

The dates in an ephemeris are found directly from the longitudes by adding to $\Delta\lambda$ a constant which is appropriate to the respective arc according to (8c) and (8d), p. 444. Arcs which contain a discontinuity, hence a synodic arc $\Delta\lambda$ that differs from one of the values w_i, lead to a $\Delta\tau = \Delta\lambda + 12;5,10^{\mathrm{r}}$ as required by (8b).[6]

A slightly different method for going from the synodic arcs of System A′ to the corresponding synodic times is described in a procedure text.[7] A coefficient $1;1,50^{\mathrm{r}}$ represents the time (in tithis) required by the sun to travel 1°. Indeed, the solar mean motion is about 0;59,8° during $1^{\mathrm{d}} \approx 1;0,57^{\mathrm{r}}$, and thus it takes the sun $1;0,57/0;59,8 = 1;1,50,49 \approx 1;1,50^{\mathrm{r}}$ to proceed 1°. Then $e = 11;3,20^{\mathrm{r}}$ is assumed as epact such that

$$11;3,20 + 1;1,50 \cdot w_i = \Delta\tau \tag{12}$$

gives the time in excess of $12^{\mathrm{m}} = 6,0^{\mathrm{r}}$ for the sun to traverse one synodic arc. In this way one finds

$$d = 41;48,20^{\mathrm{r}}, \quad d' = 45;50,12,30^{\mathrm{r}}, \quad D = 48;9,20^{\mathrm{r}} \tag{13}$$

instead of the increments given in (8c) and (8d), p. 444. The effect will be a small, but accumulative error which can be estimated by applying the same procedure to the mean synodic arc. This leads to a mean synodic time of about $45;12,40^{\mathrm{r}}$, instead of $45;13,53^{\mathrm{r}}$ in (7c). Hence the predicted dates will fall behind each year by about $0;1,10^{\mathrm{r}} \approx 0;1^{\mathrm{d}}$ in relation to the dates obtained in an ordinary ephemeris. The main cause for this discrepancy lies, of course, in the use of $11;3,20^{\mathrm{r}}$ as epact, a value very near to the epact $11;3,10^{\mathrm{r}}$ of the 19-year cycle,[8] but not attested to anywhere else.

Also for the computation of longitudes exist variants of the two main systems A and A′, using the same values for the w_i as given in (9) and (10) but changing the locations of the discontinuities and the lengths of the arcs. The parameters for these variant systems (called A_1, A″, and A‴) are conveniently tabulated in Aaboe [1965], p. 221.[9] A_1 and A‴ are known only from procedure texts but A″ is attested also in a template for longitudes of one phase,[10] covering a whole number period. Aaboe has pointed out[11] that this model is the only one among all System A types (for all planets) in which the arcs α_i do not consist of an integer number of degrees – in A″ one has $\alpha_2 = 56;15°$, $\alpha_4 = 48;45°$. In return all four arcs in A″ contain an integer number of steps, in contrast to A′ and A‴ where fractions of steps exceed the lengths of the intermediate arcs.

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[3] Aaboe-Sachs [1966], p. 16–21.
[4] Thus beginning at the boundary of the slow arc (cf. above (9)). The next line gives Γ at ♋1 and could also motivate the beginning and end for a number period.
[5] Cf. above p. 437f.
[6] For an example cf. ACT, p. 308.
[7] ACT No. 813, Sect. 14 to 16.
[8] Cf. above p. 359.
[9] Cf. ACT II, p. 310, No. 813, Sect. 7 and 8, No. 813b, Sect. 3. Still another (six-zone) variant seems to be mentioned in the fragmentary Section 1 of No. 811.
[10] Aaboe-Sachs [1966], p. 8 and p. 22f. (Text E).
[11] Aaboe [1965], p. 223, p. 221.

Finally we know from the Pañca-Siddhāntikā of a three-zone model for Jupiter,[12] similar to the one for Saturn.[13] The characteristic parameters are:

$$\begin{array}{lll} \alpha_1 = 2{,}39;30^\circ & \text{containing } \pi_1 = 3{,}0 \text{ occurrences} & \\ \alpha_2 = 3{,}0 & \pi_2 = 3{,}15 & \\ \alpha_3 = 20;30 & \pi_3 = 16 & \\ \text{total: } 6{,}0^\circ & \text{total: } \Pi = 6{,}31. & \end{array} \qquad (14)$$

It is clear that the period relations are exactly the same as in (1), p. 441. For the lengths of the steps, however, one finds from (14):

$$\delta_1 = 0;53{,}10^\circ, \quad \delta_2 = 0;55{,}23{,}4,\ldots^\circ, \quad \delta_3 = 1;16{,}52{,}30. \qquad (15)$$

As in the case of Saturn the fastest motion is concentrated on a very short arc.

B. System B

We know of two models for the synodic motion of Jupiter that operate with linear zigzag functions for the variable lengths of the synodic arcs. We have shown before (above p. 429f.) that the parameters of one of them ("System B" proper) can be derived from the fundamental periods $\Pi = 6{,}31$, $Z = 36$, and the assumption that the true synodic arcs deviate at most about $\pm 5^\circ$ from the mean.

On the other hand we have derived from the same mean synodic arc and the epact $e = 11;4^\tau$ the mean synodic time and found (above p. 443) that the result agrees excellently with $\overline{\Delta\tau}$ in System B′ while System B operates with a rounded value. If one compares all parameters in the two systems

System B	System B′	System B	System B′
$m = 28;15{,}30^\circ$	$m = 28;19{,}10^\circ$	$\Pi = 6{,}31$	$\Pi = 2{,}24{,}49$
$\mu = 33;\ 8{,}45$	$\mu = 33;\ 8{,}45$	$Z = 36$	$Z = 13{,}20$
$M = 38;\ 2$	$M = 37;58{,}20$	$P = 10;51{,}40$	$P = 10;51{,}40{,}30$
$d = 1;48$	$d = 1;46{,}40$		(15)
$m = 40;20{,}45^\tau$	$m = 40{,}20{,}15^\tau$	$\overline{\Delta\lambda} = 33;\ 8{,}44{,}48{,}29,\ldots$	$\overline{\Delta\lambda} = 33;8{,}43{,}16{,}55,\ldots$
$\mu = 45;14$	$\mu = 45;13{,}53$	$\bar{\delta} = 0;55{,}14{,}34{,}40$	$\bar{\delta} = 0;2{,}29{,}9{,}14,\ldots$
$M = 50;\ 7{,}15$	$M = 50;\ 3{,}31$		
$d = 1;48$	$d = 1;46{,}40$		

one finds that B has the same periods as A and A′ and therefore also the same mean step $\bar{\delta} = 6{,}0/\Pi$. The parameters of System B′, however, by reducing $\Delta = M - m$ for the synodic arcs slightly in comparison with B (but maintaining μ) produce a vastly greater number period without gaining any practical advantage. Nevertheless there exists an ephemeris (ACT No. 640) which is based on B′. The majority of extant ephemerides (15 texts, ACT Nos. 620 to 629) operate with System B. Both systems are represented in procedure texts side by side, e.g. in No. 813, Sect. 21 and 22.

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[12] Pc.-Sk XVII 9–11 (Neugebauer-Pingree II, p. 113).

[13] Cf. above p. 438.

The minima in both systems are supposedly located in ♍ 15, the maxima in ♓ 15.[1] I do not know the exact implications of such a statement for the numerical arrangement of an ephemeris. In System A the corresponding "apsidal line" would be the diameter ♍ 12;30/♓ 12;30.[2]

C. Subdivision of the Synodic Arc

As remarked before[1] an ephemeris could be described as a matrix in which all elements are known as soon as any one pair λ, τ is given (e.g. by observation). Such a structure requires, however, not only rigid computational rules for each column, i.e. for each individual phase Γ, Φ, etc., but also definite rules for the relations between consecutive phases within a synodic arc, i.e. rules for the subdivision of such an arc. In fact we do have a text[2] which could serve (at least for System A) as norm for the whole matrix in question since it gives the longitudes for all five phases of Jupiter over one whole number period. Thus one might well expect to find the same subdivision of the synodic arc in all texts of the same System. We shall see, however, that this expectation is not born out by the extant ephemerides and still less by the procedure texts which mention a bewildering variety of rules for the intervals between consecutive phases. It is difficult to find a reason for so great a number of variants, none of which is much different from the next one. In any case, to construct a satisfactory division of the synodic arc should have presented many fewer problems than the development of a precise computing scheme for the longitudes and times of an individual phase.

We begin by repeating the pattern abstracted from the just mentioned template for longitudes computed with System A:[3]

Γ to Φ	16;15°	19;30°	
Φ to Θ	−5	−6	
Θ to Ψ	−5	−6	(1)
Ψ to Ω	17;45	21;18	
Ω to Γ	6	7;12	
total	$w_1 = 30°$	$w_2 = 36°$.	

For System A′ we have correspondingly:[4]

Γ to Φ	16;15°	18;16,52,30°	19;30°	
Φ to Ψ	−8;20	−9;22,30	−10	
Ψ to Ω	15;50	17;48,45	19	(2)
Ω to Γ	6;15	7; 1,52,30	7;30	
total	$w_1 = 30°$	$w_2 = w_4 = 33;45°$	$w_3 = 36°$.	

In both cases the partial arcs have the same ratios as the total synodic arcs but an essential difference between (1) and (2) lies in the assumption for the length

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[1] ACT No. 805, Sect. 1; No. 812, Sect 1; No. 813, Sect. 12, 21 and 22.

[2] Cf. above p. 444. Ptolemy found in the second century A.D. for Jupiter the apogee ♍ 11; cf. above p. 179 (6).

[1] Above p. 397.

[2] Above p. 398 from Aaboe-Sachs [1966], Text D.

[3] Above p. 398 (1).

[4] Procedure text ACT No. 813, Sect. 1–2; also above p. 399 (4) and p. 405 (2) and ACT Nos. 610 and 611.

of the retrograde arcs. We shall come back to this feature in the next section.[5] Another variant for the slow arc is mentioned in ACT No. 813, Sect. 9:

$$\begin{array}{ll} \Gamma \text{ to } \Phi & 15;37,30^\circ \\ \Phi \text{ to } \Psi & -10 \\ \Psi \text{ to } \Omega & 18;\ 7,30 \\ \Omega \text{ to } \Gamma & 6;15 \\ \text{total:} & w_1 = 30^\circ \end{array} \qquad (3)$$

whereas a "second method" in the same section gives the values for the slow arc as in (2).[6]

From ephemerides we have for the distances from Ψ to Ω the following values:

ACT No.	slow arc	medium arc	fast arc
612	16;10°	18;11,15°	19;24°
613a	16	18	19;12
602, 605	13;15	–	15;54.

(4)

A group of four (originally five) related ephemerides[7] of System A is based on the following arcs $\delta\lambda$ between consecutive phases on the slow arc[8]

ACT No.	for	λ	$\delta\lambda$
606	Ω	♌ 9;30	
–	[Γ]	[♌ 15;30]	6°
600	Φ	♍ 1;45	16;15
604	Θ	♌ 27;20	–4;25
601	Ψ	♌ 21;45	–5;35
606	Ω	♍ 9;30	17;45

(5)

The distances $\delta\lambda$ agree with (1), excepting the retrogradations, though they show the same total of $-10°$.

Retrogradations. Several procedure texts mention only the total retrograde arc (from Φ to Ψ) whereas another group and the ephemerides also take the opposition (Θ) into consideration. To the first group belongs the pattern (2) for System A′:

$$\text{slow arc: } -8;20^\circ \quad \text{medium: } -9;22,30^\circ \quad \text{fast: } -10^\circ \qquad (6)$$

while ACT No. 813 Sect. 9 gives both $-8;20°$ and $-10°$ as possibilities for the slow arc. Indeed we have ample evidence that $-10°$ belongs also to the slow

[5] Below p. 449.

[6] ACT No. 813, Sect. 23 is probably only a garbled version of (2) for the slow arc. Also the Sect. 24 and 31 are marred by errors, Sect. 11 is incomplete.

[7] Cf. above p. 398 (2).

[8] The intervals on the fast arc are, of course, obtainable by using the factor 6/5. For the corresponding dates cf. below p. 450.

arc,[9] hence $-12°$ to the fast arc. Examples are the arcs listed above in (1) and (5):

Φ to Θ	slow arc: $-5°$	or	$-4;25°$	fast: $-6°$	or	$-5;18°$	(7)
Θ to Ψ	-5		$-5;35$	-6		$-6;42.$	

The asymmetry in the position of Θ is still more pronounced in ACT No. 813 Sect. 2 = No. 814 Sect. 2 and the ephemeris No. 611[10]

Φ to Θ	slow arc: $-4°$	fast arc: $-4;48°$	(8)
Θ to Ψ	-6	$-7;12$	

The same ratio is at the basis of a refined velocity scheme for the retrograde motion on the slow arc[11] (cf. Fig. 39):

Φ to Φ'	$-0;2^{\circ/\tau}$	during 30^τ	thus $\delta\lambda = -1°$	
Φ' to Θ	$-0;6$	30	-3	(9)
Θ to Ψ'	$-0;8$	30	-4	
Ψ' to Ψ	$-0;4$	30	$-1.$	

The ephemeris ACT No. 612 shows for $\Theta \to \Psi$ the retrogradations

$$\Theta \text{ to } \Psi \quad \text{slow arc: } -4;30° \quad \text{medium: } -5;3,45° \quad \text{fast: } -5;24° \tag{10}$$

in the correct ratios.[12]

A duration of 4 months (as in (9)) for the retrogradation is well attested[13] though we know also about slightly different intervals, as in

Φ to Θ	58^τ	(11)
Θ to Ψ	$2^m 4^\tau.$	

Time Intervals. The mean synodic time $\overline{\Delta\tau}$ for Jupiter should be about $12^m + 45;14^\tau$, i.e. about 13 1/2 months.[14] This total is indeed found in ACT No. 813 Sect. 23,2 and 30:

Γ to Φ	$4^m\ 4^\tau$	
Φ to Θ	58^τ	
Θ to Ψ	$2^m\ 4^\tau$	(12)
Ψ to Ω	$4^m\ 10^\tau$	
Ω to Γ	29^τ	
total:	$13^m\ 15^\tau.$	

The ephemerides which we utilized before in (5) to give us a subdivision of the synodic arc provide us also with the following time intervals

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9 Cf. above (3)

10 Cf. above p. 399 (3). An incorrectly computed ephemeris (No. 603) has only $-5;55°$ and $-7;10°$ for $\Theta \to \Psi$.

11 ACT No. 819b, Sect. 2.

12 Some insecure values of $-4°\ -4°$, $-4°\ -5°$, $-3°\ -5;10°$ for $\Phi \to \Theta \to \Psi$ are mentioned in ACT No. 813, Sect. 31, 24, 23, respectively. The fragmentary ephemeris No. 612 seems to operate with $-3;45°$ for $\Phi \to \Theta$ on the slow arc.

13 Cf., e.g., above p. 405 (1).

14 Cf. above p. 443 (7b).

ACT No.	for	τ	$\delta\tau$	
606	Ω	S.E. 108 IV 4		
–	[Γ]	[V 4]	1^m	
600	Φ	IX 4	4^m	(13a)
604	Θ	XI 5	2^m 1^τ	
601	Ψ	S.E. 109 I 6	2^m 1^τ	
606	Ω	V 16;5,10	4^m 10;5,10^τ	

The total is 13^m 12;5,10^τ for the slow arc; for the fast arc one finds similarly

$$\begin{array}{lll} \Omega \text{ to } \Phi & 5^m \; 4;27^\tau & \\ \Phi \text{ to } \Theta & 2^m+0;\;7^\tau & \\ \Theta \text{ to } \Psi & 2^m-0;\;7^\tau & \\ \Psi \text{ to } \Omega & 4^m \; 13;38,10^\tau & \\ \text{total:} & 13^m \; 18;\;5,10^\tau. & \end{array} \qquad (13\text{b})$$

A fragment (ACT No. 813, Sect. 11) gives

$$\begin{array}{lll} \Theta \text{ to } \Psi & 54^\tau & \text{i.e. } 1^m 24^\tau \\ \Psi \text{ to } \Omega & 135^\tau & \text{i.e. } 4^m 15^\tau \\ \Omega \text{ to } [\Gamma] & 32^\tau & \text{i.e. } 1^m\;2^\tau. \end{array} \qquad (14)$$

Rounded values only are given in ACT No. 813, Sect. 9,2 and No. 818, Sections 1 and 2 for the slow arc

$$\begin{array}{ll} \Gamma \text{ to } \Gamma' & 30^\tau \\ \Gamma' \text{ to } \Phi & 3^m \\ \Phi \text{ to } \Psi & 4^m \\ \Psi \text{ to } \Omega' & 3^m \\ \Omega' \text{ to } \Omega & 30^\tau \end{array} \qquad (15)$$

probably to be completed with 30^τ for Ω to Γ,[15] hence totalling only 13^m.

From actual ephemerides one can also obtain time intervals which, however, are presumably to be understood as calendar days. Furthermore day numbers are normally given only as integers, omitting the fractions which would be the result of following the exact computational rules. In using ephemerides one must therefore allow at least $\pm 1^d$ as margin of incertitude in relation to tithis. The following data are thus obtained:

Syst. A	slow	fast	ACT No.	
Φ to Θ	$2^m 14^d$	[?]	607	
Θ to Ψ	2^m	2^m	603	(16)
Ψ to Ω	$4^m 10^d$	[?]	605	
	4^m 9^d	$4^m 12^d$	602	

[15] Cf. above p. 405 (1).

and

Syst. A′	slow	medium	fast	ACT No.
Γ to Φ	$4^m\ 4^d$	$4^m\ 7^d$	$4^m\ 8^d$	611
Φ to Θ	$2^m\ 2^d$	$2^m\ 2^d$	$2^m\ 1^d$	611
Θ to Ψ	$1^m 23^d$	$1^m 23^d$	[?]	612
	2^m	[?]	[?]	609
Ψ to Ω	$4^m\ 9^d$	$4^m 10^d$	$4^m 10^d$	612
	$4^m 11^d$	$4^m 13^d$	$4^m 14^d$	610

(17)

The values for $\Theta \to \Psi$ in ACT No. 612 probably should be emended to $2^m 3^d$. A simple computing error in $\Delta\tau$ for Θ by $+10^\tau$ would explain the low value $1^m 23^d$. Under all circumstances, however, the ephemerides show that the time intervals are practically independent of the longitudinal increments $\delta\lambda$ on the respective arcs. The small deviations are explicable by the above-mentioned roundings and transformations to calendar days. Consequently the velocities should show the same ratios as the $\delta\lambda$, i.e. in System A′ the ratios 9/8 and 6/5 with respect to the slow arc. This is indeed the case as we know from the procedure texts. Hence the $\delta\tau$ are theoretically independent of the arcs of different synodic velocity.

Velocities. System A′ is best fitted to illustrate the fact that the ratios of the velocities are identical with the ratios of the lengths of the synodic arcs. Procedure texts give the following parameters[16]:

	slow arc	medium	fast
Γ to Γ'	$0;12,30^{\circ/\tau}$	$0;14,\ 3,45^{\circ/\tau}$	$0;15^{\circ/\tau}$
Γ' to Φ	0; 6,40	0; 7,30	0; 8
Φ to Ψ	−0; 4,10	−0; 4,41,15	−0; 5
Ψ to Ω'	0; 6,23,20	0; 7,11,15	0; 7,40
Ω' to Ω	0;12,30	0;14, 3,45	0;15
Ω to Γ	0;12,30	0;14, 3,45	0;15

(18)

The numbers for the medium arc are obtainable from the slow arc by the factor 33;45/30 = 1;7,30, the numbers in the fast arc by the factor 36/30 = 1;12.

According to (15) p. 450 the retrograde motion lasts for $4^m = 2,0^\tau$. Consequently (18) assigns 8;20° to the retrogradation on the slow arc, a parameter also known from (6) p. 448. But we also know of 10° retrogradations on the slow arc (e.g. from (7) p. 449). This is confirmed by ACT No. 813 Sect. 9 which gives us velocities which we can relate to longitudinal distances $\delta\lambda$ by using the $\delta\tau$ from (15) p. 450:

Γ to Γ'	$0;12,30^{\circ/\tau}$	$\delta\tau = 30^\tau$	thus $\delta\lambda =$ 6;15°
Γ' to Φ	0; 6,15	1,30	9;22,30
Φ to Ψ	−0; 5	2, 0	−10
Ψ to Ω'	0; 7,55	1,30	11;52,30
Ω' to Ω	0;12,30	30	6;15
Ω to Γ	0;12,30	[30	6;15 total: $w_1 = 30°$]

(19)

[16] ACT No. 813, Sect. 9, No. 810, Sect. 3, No. 812, Sect. 4, No. 818, Sect. 1 for the slow arc; No. 810, Sect. 4 and 6 for the medium arc. No. 810, Sect. 5 committed an error for the fast arc by using the ratio 6/5 with respect to the medium arc instead of the slow arc (cf. ACT, p. 378/9). Cf. also above p. 405 (3).

Several passages in the procedure text remain obscure. ACT No. 813a, Sect. 2, e.g., mentions a velocity of $0;12^{\circ/\tau}$ during 30^τ. Perhaps this refers to the arc of invisibility Ω to Γ in (1), p. 447.

ACT No. 813, Sect. 4 and 814, Sect. 2 describes a motion of Jupiter until last visibility (Ω), giving a sequence of arcs which depend on the zodiacal signs according to a linear zigzag function:

♋		$6;10^\circ = m$	$d = 0;1,40^\circ$	
♊	♌	6;11,40		
♉	♍	6;13,20		
♈	♎	$6;15 = \mu$		(20)
♓	♏	6;16,40		
♒	♐	6;18,20		
♑		$6;20 = M.$		

The mean value $6;15^\circ$ could be, according to (19), the arc Ω' to Ω, traversed in 30 days, but it remains a mystery why this arc should depend on the longitudes as indicated in (20) instead of following the very different arrangement of the synodic arcs.[17] Also a variation within $\pm 0;5^\circ$ seems hardly significant for the phases of Jupiter.

It is not surprising to see the Babylonian subdivisions of the synodic motion assimilated in India, together with other features of Babylonian planetary theory. The details of the Indian version, however, differ slightly from their prototype and hardly provide us with some new element belonging to the latter. For example the Pañca-Siddhāntikā not only mentions auxiliary phases Γ' and Ω' (cf. above (18) and (19)) but inserts even a Γ'' between Γ' and Φ[18] which reduces the velocity at the approach of the first station. The retrograde arc is divided in -6° and -6° as in (1), p. 447 for the fast arc, but the total synodic arc is here rounded to 34°. The synodic time is 399^d i.e. $\approx 6,45;20^\tau$, nearly $\overline{\Delta\tau}$ in (7), p. .

In Greek astronomy we have not yet any traces of this approach to planetary theory.

D. Daily Motion

Jupiter and Mercury are the only planets for which we have "ephemerides" in the strict sense of the word, i.e. lists of day-by-day positions of the planet. Two of these texts we already described in the introductory chapter.[1] For Jupiter there remains a group of fragments from Uruk which were published in ACT and subsequently properly arranged and dated by P. Huber.[2] He was able to show that these splinters were parts of three large tablets inscribed with ephemerides of Jupiter for the years S.E. 116, 117, and 119, respectively (i.e. for the period from -195 to -191). Each tablet had six (or, if necessary, seven) columns on each side for the two halves of the year, each column being 29 or 30 lines long depending on the character of the calendaric lunar month. This, incidentally,

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[17] Cf., e.g., the position of the "apsidal line" (above p. 447).
[18] Pc.-Sk. XVII 12–13, Neugebauer-Pingree II, p. 118, Table 26.
[1] Jupiter, above II A 5, 3 A, Mercury II A 5, 3 B.
[2] Huber [1957], p. 269–276. He combined ACT Nos. 652, 1015, 1016 in "Text A", Nos. 650, 651, 653 in "Text B", Nos. 1014, 1021, 1032 in "Text D".

shows that an ephemeris of this type, computed for a year ahead, presupposes the existence of lunar ephemerides which determined beforehand whether a month was full or hollow.

The layout of these Jupiter ephemerides is as follows. In each column we have first the day numbers from 1 to 29 or 30; then comes a sequence of velocities of stretchwise constant amounts which we know from System A′:

$$-0;4,10^{\circ/\tau} \text{ (from } \Phi \text{ to } \Psi) \tag{1a}$$

on the slow arc,

$$0;7,30^{\circ/\tau} \text{ (from } \Gamma' \text{ to } \Phi), \qquad -0;4,41,15^{\circ/\tau} \text{ (from } \Phi \text{ to } \Psi) \tag{1b}$$

on the arc of medium velocity, and

$$0;15^{\circ/\tau} \text{ (from } \Omega' \text{ to } \Gamma'), \qquad 0;8^{\circ/\tau} \text{ (from } \Gamma' \text{ to } \Phi) \tag{1c}$$

on the fast arc.[3] A third sequence in each column provides the longitudes of the planet such that the preceding numbers serve as differences. For example:[4]

18	15	17, 2
19	15	17,17
20	15	17,32
21	8	17,40
22	8	17,48
23	8	17,56

As Huber has shown, the dates belong to month I of S.E. 116, the longitudes are degrees and minutes in Aries. It follows from (1c) that the above given excerpt indicates that the phase Γ' took place on I 20 at ♈ 17;32. The restored continuation of the text shows that Φ falls on IV 26 at ♈ 30;4 and Ψ on IX 5 at ♈ 19;34. Hence we find that on the fast arc (of which Aries is a part) the synodic arc is divided such that

$$\begin{aligned} \Gamma' \text{ to } \Phi&: \quad 12;32^\circ \text{ in } 3^m\, 4^\tau \\ \Phi \text{ to } \Psi&: \; -10;30^\circ \text{ in } 4^m\, 6^\tau. \end{aligned} \tag{2a}$$

Assuming for $\Gamma \to \Gamma'$ a motion of 7;30° during 1^m we would have

$$\Gamma \text{ to } \Phi: \approx 20^\circ \text{ in } 4^m\, 4^\tau. \tag{2b}$$

This compares reasonably well with the data 19;30° and 4^m obtained before from the procedure texts.[5] Similarly we know of retrogradations[6] of -10° during 4^m.

Texts of the present type also make it clear why auxiliary phases like Γ' and Ω' were introduced. Obviously such points do not correspond to any real planetary phase but they are useful to make the transition smooth from the arc of greatest velocity near conjunction (i.e. $\Omega \to \Gamma$) to the preceding or following station.[7] On the other hand ordinary "ephemerides", dealing only with single specific phases, will never tabulate Γ' or Ω'.

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[3] Cf. above p. 451 (18).
[4] ACT No. 652, Obv. I, 10 to 15 = Huber, Text A, p. 292.
[5] Cf. above p. 447 (2) and p. 450 (15).
[6] Above p. 448 (6) and p. 450 (15).
[7] Cf. the analogous situation for the retrograde arc. above p. 449 (9).

Some remarks are needed for the relation between "days" and tithis. The texts know only one term, *ūmu*, i.e. "day" in ordinary usage. On the other hand it is clear that the whole theory of obtaining the synodic times $\Delta\tau$ from the synodic arcs is based on tithis and not on calendar days. The same holds also for many of the rules in the procedure texts concerning the subdivisions of the synodic arc.[8] All these operations would be impossible should one be forced to take into consideration the irregular, and a priori unknown, alternations between full and hollow months. Consequently there remains in practice only one procedure: one simply identifies calendar dates with the numerical results obtained for tithis. The computation with tithis guarantees that one remains in the mean in agreement with the actual civil lunar calendar. But in the transformation from tithis to days there is always inherent some uncontrollable element. Not only can certain phases only fall at certain parts of the day (Γ, Ω, and Θ) but one also does not know when a tithi begins or ends within a given civil day. Thus one is bound to make some arbitrary identifications and it would be pointless to introduce calendar days into the computation of ephemerides. Only as the very last step, months are given their proper length (computed in advance for one or two years, as we know from the lunar ephemerides) such that results which were originally meant in tithis are identified with days in the calendar scheme.[9]

The ephemerides from Uruk, discussed here, represent a direct application of the pattern for synodic arcs characteristic for System A′. The progress of the planet is determined by stretchwise constant velocities, such that the resulting synodic arc for each specific phase obtains one of the three values 30°, 33;45°, and 36°, depending on the area of the ecliptic to which the arc in question belongs. Unfortunately we do not have ephemerides which could show us how one proceeded when some phases began to cross over the boundary between adjacent zones of velocity.

Of an entirely different character is the ephemeris from Babylon for S.E. 147/148 (−164/−163).[10] Its intricate method of interpolation has no relation to the simple arithmetical schemes of type A or B. I do not think, however, that one can conclude anything about the methodology or the level of "schools" from three accidentally preserved ephemerides from Uruk and one from Babylon.

4. Mars

A. Periods; System A

We list again some of the relations which underlie the ephemerides of Mars[1]:

4,44 (=284) years = 2,13 (=133) occurrences = 2,31 (=151) sidereal rotations (1)

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[8] Cf., e.g., above II A 4 or II A 7, 3 C the proportionalitiy of (2), p. 447 and (18), p. 451.

[9] Huber [1957], p. 274 says that the procedure texts give velocities in degrees per day, not in degrees per tithi. In the same paragraph, however, he declares that "months" in these texts are schematically reckoned as 30 days. I do not see how these two statements can be reconciled.

[10] Huber [1957], p. 279–291, p. 298–303. Cf. above II A 5, 3 A.

[1] Cf. above p. 399.

hence
$$\Pi = 2{,}13 \qquad Z = 18 \qquad P = 7;23{,}20 \qquad \overline{\Delta\lambda} = 48;43{,}18{,}30°. \tag{2}$$

"System A" divides the ecliptic into six zones which coincide exactly with two zodiacal signs each. Combining the corresponding synodic arcs w_i with the lengths $\alpha_i = 1{,}0°$ of these sections one obtains the following table for the respective steps $\delta_i = w_i/Z$ and their number $\pi_i = \alpha_i/\delta_i$ in each section:

i	α_i	w_i	δ_i	π_i
1	♉ ♊	45°	2;30° = 5/2	24
2	♋ ♌	30	1;40 = 5/3	36
3	♍ ♎	40	2;13,20 = 20/9	27
4	♏ ♐	1,0	3;20 = 10/3	18
5	♑ ♒	1,30	5	12
6	♓ ♈	1,7;30	3;45 = 15/4	16

(3)

The total number of steps is Π, as it should be [2]:
$$\sum \pi_i = \sum \alpha_i/\delta_i = Z \sum \alpha_i/w_i = Z \cdot P = \Pi, \tag{4}$$
exactly covering the ecliptic.

The progress w_i of the planet in each synodic period is by definition $w_i = Z\delta_i$. Hence one can say that the planet gains in each synodic period Z steps (or "basic intervals") beyond one complete rotation.[3]

The shortness of the zones α_i brings the true synodic arcs under the influence of consecutive zones. Fig. 40 shows the resulting variation of the synodic arcs as function of the initial longitude λ. The minimum is located in Cancer where Ptolemy also found the apogee of Mars.[4]

System A for Mars is well suited to illustrate the effect of fixed velocity zones on the actual synodic arcs $\Delta\lambda$. Suppose one wishes to assign $\Delta\lambda$ in Cancer the minimum value of 30°. Indeed, for every λ between ♋0° and ♌0° our computing schemes give $\Delta\lambda = w_2 = 30°$. But since $Z \cdot \delta_2 = 18 \cdot 5/3 = 30°$ any λ which lies k steps inside of Leo deducts $k \cdot \delta_2$ from $\Delta\lambda$ and adds instead $k \cdot \delta_3$ to it. Consequently $\Delta\lambda$ increases linearly from ♌0 to ♍0 where it reaches the value $w_3 = 40°$. Now w_3 remains valid until ♏0 $- Z \cdot \delta_3 =$ ♏0 $- 40° =$ ♍20 where a new linear ascent sets in, etc. Similarly the descent of $\Delta\lambda$ shows the influence of the next zone always $Z \cdot \delta_i$ ahead of the next discontinuity. Hence a more or less symmetric arrangement of the w_i, as shown in Fig. 40, has a skewing effect on the distribution of the true arcs $\Delta\lambda$. It is clear the Babylonian computers were aware of this property of a System A scheme; patterns as found, e.g., for Mercury in Fig. 23 (p. 1322) leave no doubt that the location of the zones was chosen with such an effect in mind. It is tempting to see in the planetary theory support for the view, suggested above p. 430f., that System A represents a development beyond, rather than prior to, the methodology of System B.

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[2] Making use of p. 422 (6).

[3] This convenient formulation was first given by van der Waerden [1957], p. 47.

[4] Cf. above p. 179 (6): ♋25;30.

We also discussed[5] the combination of two goal-year periods, of 47 and 79 years, to produce the final period (1):

$$284 = 47 + 3 \cdot 79 \text{ years.} \tag{5}$$

We have from Uruk[6] an ephemeris for the first stationary point (Φ) that covers exactly 79 years in 38 lines. Its first line gives as longitude ♉ 17;30 while the last line is ♉ 20. This shows that the first zone (♉ and ♊) produces an increment of 2;30° in longitude and we see from (3) that this is exactly the amount of one basic interval δ_1. But if the 79-year period exceeds the return to the same sidereal longitude by one step $\delta_i = w_i/18$ then the identity (5) for the exact period requires that the 47-year period falls short by $3\delta_i = w_i/6$ in order to cancel the deviations caused by three 79-year periods. This leads to the table which we have derived in a slightly different way above p. 426(5) and which is attested in a procedure text (ACT No. 811, Sect. 3). It is explicit evidence for the use of the principle of proportionality between synodic arcs w_i and corrections for approximate periods, similarly established for the periods of Jupiter.[7]

The appearance of the same six-zone arrangement for the synodic motion of Mars in the Pañca-Siddhāntikā and in demotic texts illustrates the spread of Babylonian astronomical methods to India and the Hellenistic-Roman world. We shall describe some facets of the Indian version in connection with the Babylonian schemes for the retrogradation of Mars and the subdivision of the synodic arcs.[8] At the moment it will suffice to mention that the period $P = 133/18 = 7;23,20$ is also attested in the Pañca-Siddhāntikā.[9]

The evidence from the demotic material is much less straightforward. Not only is the arithmetical accuracy far inferior to a Babylonian ephemeris but the basic problem underwent a drastic transformation. A Babylonian ephemeris progresses in steps of synodic arcs, i.e. from one planetary phase (e.g. Γ) to its next occurrence. The demotic texts, however, are designed to find the dates of the planet's entry into the consecutive zodiacal signs. This change of approach is undoubtedly due to the requirements of Hellenistic-Roman astrology: to cast a horoscope one needs the sign in which the planet is located, not the longitudes of the phases.[10] Hence one no longer deals with the synodic arcs of one specific phase but one must reconstruct the distribution of the velocity zones from the velocity scheme of the total planetary motion. Nevertheless van der Waerden succeeded in showing[11] that at least the "Stobart Tables"[12] are based on the Babylonian six-zone pattern.

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[5] Above p. 426; also Sachs [1948], p. 283, Table IV and Kugler SSB I, p. 44.
[6] ACT No. 501.
[7] Cf. above p. 442.
[8] Cf. below p. 458.
[9] Pc.-Sk. II, p. 112, Table 24.
[10] The determination of the entry of a planet into consecutive signs (in direct or retrograde motion) is not unknown from Babylonian sources. The "Almanacs" and some of the "Normal Star Almanacs" regularly predict these dates; cf. Sachs [1948], pp. 277–282, p. 287. We also have a list of dates of entry for Mars during the year Philip Arrhidaeus 5 (−318/317), published Neugebauer-Sachs [1969], p. 94 Text H.
[11] Van der Waerden [1972], p. 77–87 and BA, p. 316–320.
[12] Cf. below p. 785, p. 788.

The Babylonian procedure for finding the synodic times from the synodic arcs has been described before in II A 4. The parameters characteristic for Mars are

$$\Delta\tau = \Delta\lambda^{\mathrm{r}} + 12,23;37,52^{\mathrm{r}}. \tag{6}$$

For the mean synodic time we found

$$\overline{\Delta\tau} = 13,12;21,10,30^{\mathrm{r}} \approx 12,59;57,17^{\mathrm{d}} \approx 780^{\mathrm{d}} \tag{7}$$

in very near agreement with the actual facts.[13]

B. System B

Among the small fragments of tablets from Uruk, shown on Pl. V, was found a splinter[1] from an ephemeris of Mars with longitudes derived, as P. Huber recognized,[2] from differences which form a linear zigzag function (cf. Table 10).

Table 10

	[0]	I
0.	31;36,40	[3;43,57 ♍]
	48;36,40	[22;20,37] ♎
	1, 5;36,40	[27;57,]17 ♐
	1,17;38,17	[15;3]5,34 ♓
	1, 0;38,17	[1]6;13,51 ♉
5.	43;38,17	29;52, 8 ♊
	26;38,17	[2]6;30,25 ♋
	25	[21;]30,25 ♌
	42	[3;]30,25 ♎
	59	[2;]30,25 ♐
10.	1,16	[18;30,25] ♒
	1, 7;14,57	[25;45,22 ♈]

The parameters are

$$\begin{aligned} M &= 1,20;\ 7,28,30^\circ & 2M-d &= 2,23;14,57 \\ m &= 17;19,\ 8,30 & 2m+d &= 51;38,17 \\ \mu &= 48;43,18,30 & \Delta &= 1,\ 2;48,20 \\ d &= 17 & P &= 7;23,20. \end{aligned} \tag{8}$$

The minimum is located in ♋/♌, as to be expected (cf. above p. 455, and below p. 1332, Fig. 40). The mean value μ agrees exactly with the mean synodic arc $\overline{\Delta\lambda}$ derived from the fundamental periods.[3] Also P is the same because

$$P = \frac{\Pi}{Z} = \frac{2,13}{18} = \frac{2\Delta}{d} = \frac{2,5;36,40}{17} = 7;23,20. \tag{9}$$

[13] Above p. 408.

[1] Cf. below p. 1451, right upper corner on Pl. V, second row, second fragment from the right; published in ACT No. 510.

[2] Centaurus 5 (1958), p. 246; van der Waerden BA, p. 274.

[3] Cf. above p. 391 (11) or p. 407 (5) and p. 455 (2).

This numerical identity also explains a passage in a procedure text (ACT No. 811 a, Sect. 11) in which the product $\Pi \cdot d = 2{,}13 \cdot 17 = 37{,}41$ is divided by $Z = 18$. The result is $2{,}5;36{,}40 = 2\Delta$. Since in the preceding step of the text $\mu = 48;43{,}18{,}30$ had been found one can now compute $M = \mu + 1/2\Delta$ and $m = \mu - 1/2\Delta$. Thus we have here an example for the determination of the extrema of a zigzag function from given periods Π and Z and a conveniently chosen difference d. The result is, of course, not adaptable to asymmetries in the distribution of the synodic arcs as assumed in System A.

C. Subdivision of the Synodic Arc; Retrogradation

As we have seen, the retrograde motion of Mars ($\Phi \rightarrow \Theta \rightarrow \Psi$) is treated as "satellite" to the positions of the first station (Φ).[1] Subdivision of the synodic arc therefore primarily concerns the relative positions of Ω, Γ, and Φ. For these we have found two different versions: one abstracted from dateless template texts[2] which give, measured in steps, the following distances between the consecutive phases:

$$\Omega \rightarrow \Gamma = 33\,\delta_i, \qquad \Gamma \rightarrow \Phi = 63\,\delta_i, \qquad \Phi \rightarrow \Omega = 55\,\delta_i, \tag{10}$$

the second connected with a refined scheme that is represented in a procedure text[3]

$$\Omega \rightarrow \Gamma = 33\,\bar{\delta}, \qquad \Gamma \rightarrow \Phi = 60\bar{\delta}, \qquad \Phi \rightarrow \Omega = 58\,\bar{\delta}. \tag{11}$$

In both cases the number of steps is correctly $\Pi + Z = 2{,}31$. The second scheme deals only with the mean situation. With it are associated solar motions r_i as follows:[4]

$$\Omega \rightarrow \Gamma\colon\ r_1 = 30°, \qquad \Gamma \rightarrow \Phi\colon\ r_2 = 105°, \qquad \Phi \rightarrow \Omega\colon\ r_3 = 225° \tag{12}$$

which represent the change of elongation of the mean sun from the planet. Thus the interval of invisibility extends from the elongation $-15°$ to $+15°$ whereas the elongation of Φ is $15 + 105 = 120°$. These are round values well attested in Greek and mediaeval astronomy.[5] The same scheme of elongations and phases is also found in the Pañca-Siddhāntikā.[6]

Turning to the theory of retrogradation we first establish what we may expect as obtainable from observational data. For this an epicyclic model is most convenient:[7] if we decrease the direct component of the planetary motion, caused by the motion of the center of the epicycle on the deferent, then the stationary point must move nearer to the point of tangency to the epicycle of the line of sight. This will increase the arc between the stations and hence we should expect a maximum retrogradation where the synodic progress is smallest. This is indeed what we find expressed in the Babylonian schemes which all have a maximum retrograde arc in ♋/♌ where the synodic arcs are smallest. One of

[1] Above p. 400.
[2] Above p. 425 (6).
[3] Above p. 409 (12).
[4] Above p. 411 (22).
[5] Cf. above p. 411, n. 11 and below IV D 3, 4 or V A 3, 2.
[6] Pc.-Sk. II, p. 127, Table 34.
[7] Cf. e.g., below Fig. 18 (p. 1320).

these schemes, called "S", is attested in actual use in the extant ephemerides and template texts.[8] Closely related to S is the scheme R which avoids the linear increases in every second sign but agrees with S in the constant levels (cf. Fig. 41). The existence of R is not only attested in procedure texts (ACT No. 803) but also in the Pañca-Siddhāntikā.[9] Similar to R is the scheme T (known from ACT No. 811 a, Sect. 1) with the only difference that its maximum is 7;30° instead of 7;12° in R and S. Finally we know (from ACT No. 804) of a strictly linear pattern U, also with 7;30° as maximum. All four schemes have the same minimum of 6°, as shown in the subsequent table.

	m	M	μ	Interpol.	
R	6°	7;12°	6;36°		
S	6	7;12	6;36	±0;0,48	(13)
T	6	7;30	6;45		
U	6	7;30	6;45	±0;0,30	

All four schemes only concern the arcs from Φ to Θ. We are less well informed about the second part of the retrograde motion. Since "opposition" is rather to be understood as acronychal rising than as 180° of elongation,[10] Θ should be nearer to Φ than to Ψ. This is confirmed by the rule which is attested[11] for the scheme S:

$$\Theta \to \Psi = 3/2\,(\Phi \to \Theta) \tag{14a}$$

or the equivalent rule

$$\Phi \to \Psi = 5/2\,(\Phi \to \Theta). \tag{14b}$$

Since in S the extrema for $\Phi \to \Theta$ are 6° and 7;12° we have for the total retrogradation

$$15° \leqq (\Phi \to \Psi) \leqq 18° \quad \text{thus } \mu = 16;30°. \tag{15a}$$

Under the assumption that (14b) also applies to T and U we would have for these schemes

$$15° \leqq (\Phi \to \Psi) \leqq 18;45° \quad \text{thus } \mu = 16;52,30°. \tag{15b}$$

It may be noted that Ptolemy obtained from his cinematic model the following retrogradations[12]:

at minimum distance:	11;12,14°	$\Delta t = 64;30^d$	
at maximum distance:	19;53,32	80	(16)
at mean distance:	16;18,44	73.	

From the ephemeris ACT No. 500 we see that all dates for Θ are obtained from the date of the preceding Φ by adding the fixed interval of $47;55,4^{\tau}$. If (14b) also applied to the dates the total time of retrogradation would be almost 120^{τ}.

[8] Described in detail above p. 401 and Fig. 20 (p. 1320).
[9] Pc.-Sk. II, p. 120, Table 29.
[10] Cf. above p. 399 and Fig. 18, p. 1320.
[11] Cf., e.g., Aaboe-Sachs [1966], p. 10, Table 9.
[12] Cf. above p. 196; p. 193.

This amount considerably exceeds the theoretical interval between the stationary points but could be explained by the extremely unsharp observational situation near the stationary points.

Excepting the scheme S we are not on very secure grounds for the determination of longitudinal and time intervals between stations. ACT No. 501a can be interpreted as indicating the existence of retrogradations greater than permitted by (15a) or (15b). ACT 811a, Sect. 2 gives values for $\Theta \to \Psi$ in scheme T but they do not agree exactly with (14a).

We remarked before that the scheme R for $\Phi \to \Theta$ was known in India[13] and so was the total retrogradation of 18° in agreement with (15a).[14] For the time intervals we have values between 51^d and 84^d but the details are not secure.[15]

5. Venus

A. Periods

The great regularity of the motion of Venus[1] is probably the reason why all data concerning periodic returns of this planet are the consequence of one single short-time relation[2]

$$5 \text{ synodic periods} = 8 \text{ years.} \tag{1a}$$

It was known, however, that this relation is not exact but leaves a small deficit in the sidereal returns:

$$5 \text{ synodic periods} = 8 \text{ sidereal revolutions } -2;30°. \tag{1b}$$

One can eliminate this correction by multiplying (1b) by 360/2;30 = 144 and thus obtain

$$720\,(=12,0) \text{ synod. per.} = 1152 - 1 = 1151 (=19,11) \text{ sid. years.} \tag{2a}$$

Hence

$$\Pi = 12,0 \qquad Z = 7,11 \qquad P = 12,0/7,11 = 1;40,13,\ldots \tag{2b}$$

and therefore for the mean synodic arc, exactly

$$\overline{\Delta\lambda} = 6,0/P = 6,0 \cdot 7,11/12,0 = 3,35;30°. \tag{2c}$$

The latter relation is, of course, the equivalent of (1b) since

$$5\overline{\Delta\lambda} = 17,57;30° \equiv -2;30° \mod 6,0°. \tag{2d}$$

There is no trace of approximate periods which could be used in linear combinations to eliminate deviations from exact returns.[3] Even in sources which probably

[13] Above p. 459.
[14] Pc.-Sk. II, p. 119 and p. 127.
[15] Pc.-Sk. II, p. 120, Tables 27 to 29.
[1] For the smallness of the eccentricity of the orbit of Venus cf. below p. 1443, Fig. 34.
[2] This is also the goal-year period for Venus; cf. below p. 554 (1).
[3] In contrast, e.g., to Jupiter, above p. 442. I think I was mistaken to seek evidence for approximate periods in the fragment ACT No. 815.

are earlier than the ACT material we find[4] only periods which are multiples of (1 a): 8 years, 16 years, 48 years, 64 years. We must admit, however, that we do not know why such trivial multiples were considered.

From $\overline{\Delta\lambda}$ we can derive the mean synodic time by using the epact $e = 11;4^{\mathrm{r}}$ and find[5]

$$\overline{\Delta\tau} = \overline{\Delta\lambda}^{\mathrm{r}} + 6,17;41,28,40^{\mathrm{r}} = 9,53;11,28,40^{\mathrm{r}}. \tag{3a}$$

This value is rounded in the ephemeris ACT No. 400 to

$$\overline{\Delta\tau} = 9,53;10^{\mathrm{r}}. \tag{3b}$$

On the basis of this latter value one obtains for five synodic periods

$$\begin{aligned} 5\overline{\Delta\tau} &= 49,25;50^{\mathrm{r}} = 49,30^{\mathrm{r}} - 4;10^{\mathrm{r}} \\ &= 1,39^{\mathrm{m}} - 4;10^{\mathrm{r}} = 8 \cdot 12^{\mathrm{m}} + 3^{\mathrm{m}} - 4;10^{\mathrm{r}}. \end{aligned} \tag{4}$$

Since 8 calendar years contain either 2 or 3 intercalary months we can say that after 8 years (including perhaps the adjustment of one month name) the planet will return to the same phase $4;10^{\mathrm{r}}$ earlier.

If one uses the unabbreviated value (3 a) for $\overline{\Delta\tau}$ one obtains

$$5\overline{\Delta\tau} = 49,30 - 4;2,36,40^{\mathrm{r}} \approx 1,39^{\mathrm{m}} - 4^{\mathrm{d}} \tag{4a}$$

which explains the use of this decrement for the calendar dates in the ephemerides.[6] Of course, day-(or tithi-)fractions are in any case meaningless in intervals between phases which by definition must take place in a definite part of the day (morning or evening). It is therefore in many cases irrelevant whether we denote such increments as days or as tithis.

B. Ephemerides

We have one fragment (ACT No. 401, from Uruk) which gives dates and positions of Venus at last visibility as morning star (Σ) in steps of 8 years. Perhaps this is only an auxiliary table, either for checking ephemerides or for providing initial values. The column for the longitudes is almost completely destroyed but the remnants would allow the assumption of a constant difference of $-2;30°$. The dates (between S.E. 175 and 303, i.e. $-136/5$ to $-8/7$) show the increment

$$\Delta t = 99^{\mathrm{m}} - 4;5^{\mathrm{d}} \tag{5}$$

from line to line, in agreement with (4), considering $4;5^{\mathrm{d}}$ the equivalent of $4;10^{\mathrm{r}}$.

Based strictly on mean motions is an ephemeris from Uruk (ACT No. 400) for the reappearances of the evening star (Ξ) in the years S.E. 111 to 135 ($-200/199$ to $-176/5$), i.e. for three 8-year periods. The constant differences are

$$\Delta\tau = \overline{\Delta\tau} = 20^{\mathrm{m}} - 6;50^{\mathrm{r}}, \qquad \Delta\lambda = \overline{\Delta\lambda} = 7^{\mathrm{s}} + 5;30° \tag{6}$$

in agreement with (3 b) and (2 c), respectively.

An ephemeris of this type (we call it "System A_0") is completely determined by one single pair of longitudes and dates since the differences (6) are the equivalent

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[4] Neugebauer-Sachs [1967], p. 207.

[5] The method is described in II A 4.

[6] Cf., e.g., below p. 462 (from ACT No. 410).

of the basic period relations. We also have, however, ephemerides where five consecutive occurrences of the same phase are not spaced at constant intervals but located according to special rules which depend on the phases and their positions in the zodiac.

For example in ACT No. 410 (cols. III/IV) the first six lines for dates and longitudes for the stations of the evening star (Ψ) run as follows

(S.E.) 3,56	X	18	26	♒
3,58	V	9	9;30	♎
3,59	XII	25	9	♉
4,1	VII	26	16;30	♐
4,3	III	15	15	♋
4,4	X	14	23;30	♒

This shows that after 8 years the day numbers decrease by 4, the longitudes by 2;30° as required by (4a) and (1b). The intervals, however, between consecutive dates and positions are no longer the mean values (6) but

in ♒	$20^m - 9^d$	and	$7^s + 13;30°$ (+1 rot.)	
♎	-14		$-\ 0;30$	
♉	$+\ 1$		$+\ 7;30$	(7)
♐	-11		$-\ 1;30$	
♋	$-\ 1$		$+\ 8;30$	
total	$99^m - 4^d$		$36^s - 2;30°$ (+5 rot.)	

In this way one can extract from the ephemerides rules for the spacing of five consecutive lines for each phase and for each zodiacal sign, though one is much handicapped by the extremely fragmentary character of the few available texts. One ephemeris (ACT No. 420) contained at the end of the reverse the complete list of the 5+5 increments of the type (7) for all six phases; even in its incomplete preservation this list supplements what we know from the ephemerides.[7] As an example I give here the table for Ψ

in	$\Delta\lambda$		Δt	
♈ ♉ ♊	$3,37;30° = 7^s +$	$7;30°$	$20^m + 1^d$	
♋ ♌	3,38;30	$+\ 8;30$	$-\ 1$	
♍ ♎ ♏	3,29;30	$-\ 0;30$	-14	(8)
♐ ♑	3,28;30	$-\ 1;30$	-11	
♒ ♓	3,43;30	$+13;30$	$-\ 9$	

Fig. 42, p. 1333 is a graphical representation of the additions to 7^s and 20^m, respectively, as listed in (8).

It should be noted, however, that these rules do not guarantee recessions of 4^d and 2;30° after any sequence of five consecutive occurrences of the same phase. This difference will only appear when each of the five zones is used

[7] Cf. the tabulation in ACT, p. 301/302 and No. 821 b, p. 441 f.

exactly once. But since the synodic arcs always amount to nearly 7^s it is possible that two consecutive synodic intervals begin and end in one of the zones which are 3^s in length. For example one could have for Ψ according to (8) the following sequence

♈ 10 → ♏ 17;30 → ♊ 17 → ♑ 24;30 → ♌ 23 → ♈ 1;30

i.e. a recession of 8;30° instead of 2;30° (and correspondingly 24^d instead of 4^d) because ♈ and ♊ belong to the same zone. The preserved material is much too fragmentary[8] to show us how such discrepancies were avoided.

We call the set of rules to which (8) belongs "System A_1". This terminology is misleading insofar as the rules considered here are not exactly of the same type as in a "System A" for the outer planets. The synodic arcs for a given phase of Venus depend only on the initial zone without being influenced by the effect of other zones. Consequently we have no coefficients here for the crossing of boundaries and the computation of an ephemeris is extremely simple. On the other hand the above-mentioned serious discrepancies are possible which would be excluded in a real "System A".

Another unexplained feature of the ephemerides for Venus is the fact that there exists a modification of System A_1 (denoted as "System A_2") which assumes $-2;40°$ for the change of longitudes after the completion of one 8-year period, while -4^d remains as change for the dates. The majority of the rules for the consecutive synodic arcs seem to be the same in A_1 and A_2; for Ψ, e.g., (8) remains valid except for the arc ♍, ♎, ♏ where A_2 prescribes $\Delta\lambda = 3,29;20° = 7^s - 0;40°$ instead of $-0;30°$ in A_1. A decrement of 2;40° for the 8-year period is, of course, contradictory to the fundamental period relations and must lead comparatively soon to appreciable errors. In view of this fact it is still more surprising to see the two "Systems" applied in the same ephemeris and in its set of rules.[9] There we find A_2 applied to the phases Ψ, Ω, and Φ (because they belong to the retrograde arc?) while Γ and Σ are computed with A_1 (unfortunately the rules for Ξ are not preserved). It is obvious that this mixture of contradictory parameters must rapidly lead to impossible relative positions of consecutive phases.

A_1 and A_2 do not seem to be the only incompatible procedures applied in the same ephemeris. We have a fragment (ACT No. 411) in which $\Delta\tau$ for Γ follows $A_1 = A_2$ while both $\Delta\lambda$ for Γ and $\Delta\tau$ for Φ do not agree with the rules of either system. Another ephemeris (ACT No. 421a which in part duplicates No. 420 and its list of coefficients No. 821b) gives longitudes for Ξ which do not follow A_1 and A_2, valid in the remaining columns. Finally ACT No. 421 is a fragment from a duplicate of No. 420 in a bad enough state of preservation so as not to exclude longitudes for Ψ that agree with System A_2.

To summarize this situation we can say that we have three ephemerides that follow System A_1 which is in agreement with the basic period relation. The evidence for the variant procedure, called A_2, concerning longitudes only, is actually restricted to three phases in one text and two fragmentary duplicates such that one may doubt whether A_2 is really a recognized system for the computation of ephemerides and not only some isolated computational exper-

[8] We have only fragments of three ephemerides computed with System A_1: ACT Nos. 410, 412, and 430 (for the restoration of No. 430 cf. van der Waerden [1957], p. 59).

[9] This is the above mentioned text ACT No. 420+821b (cf. p. 462).

iment. In short the material for Venus is by far the least reliable in all our sources for the investigation of ephemerides.[10]

There still remain two problems for discussion, both connected with empirical data. One is the question, why one constructed complicated patterns for the synodic arcs, as shown, e.g., in Fig. 42; the other concerns the intervals between different phases taken in their actual sequence.

Concerning the first question it is easy to see that the distance between consecutive phases of the same kind must depend on their position in the zodiac. We know (cf., e.g., (6)) that this distance is of the order of magnitude of 7 signs; hence consecutive phases occur in opposite parts of the ecliptic and therefore take place under very different inclinations of the ecliptic toward the horizon while the planet is also in an opposite position with respect to the nodes of its orbit. Both these elements are of decisive influence on the elongation of the planet from the sun, required for first or last visibility. In the case of the moon the Babylonian astronomers obtained a far reaching insight into these visibility conditions[11] and there is no reason that could have prevented them from doing the same for the planets. Our sources, however, reveal no trace of such a theoretical approach and it seems therefore that the schemes at hand are only the result of some schematization of empirical data, not subjected to a theoretical analysis of the underlying causes. The absence of any theory of planetary latitudes[12] supports this impression.

This does not prevent us from trying to get at least a general idea of what one may expect for the variations of the synodic arcs of a given phase. To this end we make use of Ptolemy's tables for the elongations of the planetary phases in the Handy Tables. Let $\delta_1 = \lambda_\odot - \lambda_1$ be the elongation of Venus from the sun at a moment when a certain phase (e.g. Σ) takes place. The next time the same phase will occur 7 signs later, i.e. at an elongation $\delta_2 = \lambda_\odot + 7^s - \lambda_2$. These values δ_1 and δ_2 are tabulated in the Handy Tables as function of the longitudes at the beginning of each sign and for each of the seven climata.[13] Thus we can find easily the synodic arc

$$\Delta\lambda = \lambda_2 - \lambda_1 = 7^s + \delta_1 - \delta_2$$

and can compare it with the values assumed in the ephemerides. As an example may serve the case of last visibility of the morning star (Σ)[14]

δ_1:	15;21 in ♓	14;24 in ♈	12;50 in ♉	10;8 in ♊	
δ_2:	6;56 in ♎	7; 2 in ♏	7;45 in ♐	8;34 in ♑	etc.
$\delta_1 - \delta_2$:	8;25	7;22	5; 5	1;34	
Babyl. $\Delta\lambda$:	15;20	15;20	−0;20	−0;20	

[10] It seems to me not only pointless but seriously misleading to readers who are not in a position to control the primary sources to make such utterly fragmentary material the basis of far reaching historical conclusions and to formulate them as if they were well established results (van der Waerden [1957], p. 60 and again AA, p. 199, BA, p. 278).

[11] Cf. below II B 10, 2.

[12] For one exception cf. below II C 3, p. 554.

[13] Halma III, p. 22–25 for Σ, Ξ, Γ, Ω. For a graphical representation of these tables cf. below Fig. 125 to 127 (p. 1421 ff.). We have no way of testing Φ and Ψ.

[14] The values δ_1 and δ_2 are found by interpolation between clima III and IV; they are easy to estimate in our Fig. 125.

The result for all twelve signs is depicted in Fig. 43 which shows that the Babylonian parameters follow the actual trend fairly well. A similar result is obtained for Ξ and Γ, always with the exception of some larger deviation for one group of signs (as in ♎, ♏, ♐ for Σ) – perhaps caused by climatic conditions. Only Ω disagrees seriously since $\delta_1 - \delta_2$ shows a sinusoidal variation between $\pm 15°$ while the Babylonian $\Delta\lambda$ varies comparatively little between about 0° and $+10°$. We have, of course, no way of knowing to what extent Ptolemy's assumptions, based on visibility conditions in Alexandria, are incorrect for Babylon.

We finally consider the phases in their natural order, an investigation which again depends only on very scanty evidence, the extent fragments of ephemerides and a procedure text (ACT No. 812 which, were it fully preserved, would give us all data for the "subdivision" of the synodic motion).

A general impression of the intervals involved can be obtained by combining ACT No. 420 with contemporary sections in No. 410:

Ξ	S.E. 3,56	II 6	Δt	♊ 4;30	$\Delta\lambda$	
Ψ		X 18	$8^m\,12^d$	♒26	4,21;30°	
Ω		XI 3	15^d	[♒]		
Γ		XI 7	4^d(invis.)	♒16;40	[−9;20]	
Φ		XII 25	18^d	♒ 7;30	−9;10	(9)
Σ	S.E. 3,57	VIII 1	$8^m\;6^d$	♏ 3;20	4,25;50	
Ξ		X 6	$2^m\;5^d$(invis.)	♑18;40	1,15;20	
		total:	$20^m\;0^d$	total:	$7^s + 14;10°$	

The totals for the synodic motion from Ξ to Ξ agree with the rules concerning this phase in ♊ for System A_1.[15]

From ACT No. 812 we see that this procedure text assumed $8^m 10^r$ for the time between Ξ and Ψ in ♊, 23^r in ♒ for $\Psi \to \Omega$, 1^r for $\Omega \to \Gamma$, and $1^m 26^r$ in ♏ for $\Sigma \to \Xi$.[16] This shows that the ephemerides do not conform exactly with the rules from a procedure text.

One parallel to Ptolemy's discussion of planetary phases is provided by parameters preserved in ACT No. 812. This procedure text in describing the intervals of invisibility at inferior conjunctions ($\Omega \to \Gamma$) assigns 15^d to ♋ and ♌, $4;30^d$ in ♐, 1^d in ♑ and ♒, and 2^d in ♓.[17] This agrees excellently with what Ptolemy calls the "paradoxical" behavior of Venus, being only 2 days invisible at the beginning of ♓, in contrast to 16^d of invisibility at the beginning of ♍.[18]

The time of invisibility at superior conjunction ($\Sigma \to \Xi$) varies much less. Our procedure text[19] gives numbers between 56 and 62 days. For the linear stretch $\Xi \to \Psi$ we find from 238 to 252 days, from 17 to 23 days for retrogradation as evening star ($\Psi \to \Omega$). Values for the remaining intervals are lost but there are

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[15] Cf. ACT, p. 301.

[16] Tabulated in ACT, p. 399.

[17] The remaining data are lost. It should be remarked that intervals between evening setting and morning rising by definition cannot amount to integer days. All dates of this type are therefore subject to arbitrary interpretations, whether 1/2 day is included or excluded.

[18] Alm. XIII, 8; cf. above I C 8, 3 A.

[19] Cf. ACT, p. 399 for the time intervals, p. 397 for the corresponding arcs.

also intervals listed for larger sections, e.g. $\Gamma \to \Xi$ and $\Xi \to \Gamma$, adding up, as far as can be seen, to $9,53;30^r$ as compared with $9,53;10^r$ in (3b), p. 461. Visibility as morning star, i.e. $\Gamma \to \Sigma$, varies between 270^d and 276^d. The description of the corresponding longitudinal changes are almost completely lost, excepting for the interval $\Gamma \to \Xi$ which amounts to about 330°.

6. Mercury

A. Periods

In the case of Venus we could make the scarcity of sources responsible for our limited understanding of the Babylonian procedures. Mercury can remind us that a considerably greater amount of textual evidence need not to put us into a much more favorable position.

The visible parts of Mercury's orbits are so short that it is not surprising when the Babylonian theory restricted itself to the four phases of first and last visibility, ignoring the stationary points.[1] From our texts we know of at least three computational methods, all of System A type, while a fourth such system, also of clearly Babylonian origin, left some traces in India.[2] No evidence whatever exists for a System B for Mercury.

We already gave a description of the two systems from which ephemerides survived: System A_1 operating with three zones, A_2 with four.[3] In both systems only two phases are computed with System A methods: the first visibilities (Γ and Ξ) in A_1, the last visibilities (Σ and Ω) in A_2. The remaining two phases are treated in both cases as "satellites" of the first ones. For these two systems we have the parameters which govern the computation of the synodic arcs and thus we know all periods. Surprisingly these periods are different for all four phases:

$$\begin{array}{llrrl} A_1: & \Gamma & \Pi = 2673 = 44,33 & Z = 848 = 14,\ 8 & P = 3;9,7,38,\ldots \\ & \Xi & 1513 = 25,13 & 480 = \ 8,\ 0 & 3;9,7,30 \\ A_2: & \Sigma & 1223 = 20,23 & 388 = \ 6,28 & 3;9,7,25,\ldots \\ & \Omega & 684 = 11,24 & 217 = \ 3,37 & 3;9,7,32,\ldots \end{array} \tag{1}$$

It is clear from the values of P that it will take a very long time before these divergences reach a significant level. Nevertheless it is theoretically unsatisfactory to see the same ephemeris operating with different periods for related phases.

A third system, A_3, probably concerning Ω,[4] is based on

$$\Pi = 1119 = 18,39 \qquad Z = 355 = 5,55 \qquad P = 3;9,7,36,\ldots \tag{2}$$

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[1] For some evidence for a reference to these points cf. below p. 471 and the Indian material (below p. 473). Cf. also above II A 5, 3 B for a possible contamination of Ω and Γ with Ψ and Φ, respectively.

[2] Cf. below p. 473.

[3] Above p. 402.

[4] Cf. below p. 471.

It is strange to see that the 46-year period of Mercury, always used in the goal-year texts,[5] seems to be ignored for the construction of computing schemes of ephemerides – in marked contrast to what we found, e.g., for Venus or Jupiter.[6] Its parameters are

$$\Pi = 145 = 2{,}25 \qquad Z = 46 \qquad P = 3;9,7,49,\ldots \tag{3}$$

No division of the ecliptic in zones of different synodic arcs is known for these parameters.

The mean synodic arcs from this variety of periods all belong to the interval

$$1{,}54;12{,}24^\circ < \overline{\Delta\lambda} = 6{,}0/P < 1{,}54;12{,}40^\circ. \tag{4}$$

None of the known number periods Π produces a finite sexagesimal expression for $\overline{\Delta\lambda}$.

The 46-year period is combined with a 13-year period to a 125-year period through the relation $125 = 3 \cdot 46 - 13$, based on the assumption that 46 years exceed complete returns to the same phase by 1^d, while 13 years are 3 days longer[7].

Finally in a procedure text (ACT No. 816) a 20-year interval is discussed in relation to System A_3. The fact that this period is attested in cuneiform material is of interest because 20 years is the period assigned to Mercury in hellenistic astrology.[8]

Before turning to the rules governing the computation of ephemerides it is useful to establish in general terms some relations which connect smaller periods of a System A pattern with larger periods in the same system. In this way one can attach some exact meaning to the concept of "approximate periods" in relation to larger periods adopted for the computation of ephemerides.

Let us assume that some integer multiple nZ of Z comes near to an integer multiple of the number period Π, i.e. assume that

$$nZ + d = m\Pi \quad \text{or} \quad nZ \equiv -d \mod \Pi \tag{5}$$

where d is a comparatively small number. Since we know[9] that all points of the ecliptic at which a phase takes place are separated from one another by "steps" of length $\delta_i = w_i/Z$ (where w_i, measured in degrees, is the synodic arc associated with the arc α_i of the ecliptic) we see that (5) implies that nZ events fall $d\delta_i$ degrees short of an accurate return to the same longitude. Hence

$$d\delta_i = (m\Pi - nZ)\delta_i = (mP - n)w_i \tag{6}$$

measures the deviation from exact returns of an "approximate" period of m years.

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[5] Sachs [1948], p. 283, also ACT procedure text No. 800 (p. 363). It is probably only an accident that the periods of A_1 can be connected arithmetically with (3):

$$2673 = 1513 + 8 \cdot 145 \quad \text{and} \quad 848 = 480 + 8 \cdot 46.$$

At any rate such a relation has no astronomical meaning.

[6] Cf. above p. 460 and p. 441.

[7] Cf. Neugebauer-Sachs [1967], p. 206f.

[8] Cf. below p. 899.

[9] Cf. above II A 6.

Since for Mercury $P=\Pi/Z\approx 3$ the shortest approximate period is $m=1$ year, corresponding to a deviation from exact returns of

$$d\delta_i=(P-3)w_i. \tag{7}$$

For the computation of ephemerides this relation allows one to proceed in steps of three lines with constant differences[10] as long as one remains in the same zone α_i.

The above-mentioned 20-year period corresponds to another solution of (5). For all the values of Π listed in (1), (2), and (3) and $m=20$ the corresponding n has the value 63. Hence

$$d\delta_i=(20\Pi-1{,}3Z)\delta_i=(20P-1{,}3)w_i. \tag{8}$$

Both parameters $d\delta_i$, for three synodic periods, or one year, and for 20 years were determined by the Babylonian astronomers as we shall see presently.[11]

Instead of speaking about "approximate periods" we can also adopt a viewpoint which is analogous to the distinction between "true functions" and "tabulated functions" introduced in the discussion of linear zigzag functions, i.e. for "System B" patterns.[12] The essence is the remark that the introduction of new intervals of tabulation introduces differences $d\delta_i$ which in turn can be treated as new "synodic arcs" w'_i (actually being multiples of such), valid in the same respective zones α_i as before. Thus we can define with

$$w'_i=d\delta_i=(m\Pi-nZ)\delta_i=(mP-n)w_i \tag{9}$$

a new period

$$P'=\sum\frac{\alpha_i}{w'_i}=\frac{1}{d}\sum\frac{\alpha_i}{\delta_i}=\frac{\Pi}{d}=\frac{\Pi'}{Z'} \tag{10}$$

where Π' is either identical with Π or a divisor of it depending on d being relatively prime to Π or not.

B. System A_1 to A_3

Our sources are very unevenly distributed. Of System A_2 we have only one ephemeris (ACT No. 300a and a fragment from a duplicate, No. 300b). Covering (at least) the years from S.E. 4 to 22 (i.e. $-307/6$ to $-289/8$) this is the earliest ephemeris in the ACT material which otherwise belongs mainly to the period from S.E. 100 to about S.E. 270.[1] Of System A_1 six ephemerides and two duplicates are extant (ACT Nos. 300 to 305). The order of the phases in the texts from Babylon (Nos. 301 to 305) is Ξ, Ω, Γ, Σ, while No. 300, from Uruk, starts with Γ and Σ and is not connectible with the Babylon texts.[2]

In No. 301 the computational rules would produce, in two cases, the day number 30 while the text has day 1 of the next month. This seems to indicate

[10] The values for System A_1 are given in ACT, p. 290f., for A_2 on p. 296f. The values of d are 129 for Γ, 73 for Ξ, 59 for Σ, 33 for Ω.

[11] Cf. below p. 470.

[12] Cf. above p. 375.

[1] Cf. the diagram ACT II, p. XII; similarly for the moon ACT I, p, XVI.

[2] Cf. ACT, p. 317f.

the existence of lunar ephemerides from which one could conclude that the month in question would be hollow. This text, as well as No. 302, is vitiated by computing errors, several of which consist in a wrong determination of a sexagesimal place value. Occasionally we can see from such errors how the scribe operated with transition coefficients between adjacent zones.[3] At the end No. 301 gives a long list of rules and parameters (ACT No. 820a) which are characteristic for System A_1.

We need not repeat the rules for the computation of longitudes for Γ and Ξ in System A_1 and for Σ and Ω in System A_2 since the essential parameters are listed above in II A 5, 1 C. For the convenience of actual computations these rules can be formulated in several different ways for which cf. ACT, p. 289–297 and Aaboe [1965], p. 221.

System A_3. The set of rules which we denote as System A_3 is known from the procedure text ACT No. 816 and a peculiar "ephemeris" which gives the longitudes of consecutive evening settings (Ω) of Mercury but no dates except for year numbers which can be shown to be the regnal years of Artaxerxes I, Darius II, and Artaxerxes II year 1 and 2 (i.e. from -423 to -401).[4]

The following synodic arcs and zones constitute the basic parameters:

$$\begin{array}{lllr} w_1=1,50;56,15^\circ & \text{from } ♌30 \text{ to } ♈30 & \text{thus } \alpha_1=4,0^\circ & \\ w_2=2,11;28,53,20 & ♈30 \text{ to } ♋20 & \alpha_2=1,20 & (11) \\ w_3=1,45;11,\ 6,40 & ♋20 \text{ to } ♌30 & \alpha_3=40. & \end{array}$$

The resulting periods are listed above p. 466 (2). The fact that two thirds of the circle are associated with a synodic arc that differs only slightly from the mean synodic arc ($\approx 1,54;12^\circ$) is reminiscent of similar arrangements for Saturn and Jupiter in Indian astronomy.[5]

For the "steps", $\delta_i = w_i/Z$, one obtains from (2) and (11):

$$\begin{aligned} \delta_1 &= 0;18,45^\circ = 5/16 \\ \delta_2 &= 0;22,13,20 = 10/27 \qquad (12) \\ \delta_3 &= 0;17,46,40 = 8/27. \end{aligned}$$

The above-mentioned simplification of computing longitudes by progressing in groups of three synodic arcs is obviously the reason that one procedure text ACT No. 816, Section 1) mentions the following yearly decrements

$$\begin{array}{lll} (P-3)\,w_1 = 16;52,30^\circ & \text{in } \alpha_1 & \\ (P-3)\,w_2 = 20 & \alpha_2 & \qquad (13) \\ (P-3)\,w_3 = 16 & \alpha_3 & \end{array}$$

which one indeed obtains from (7), p. 468 with $P = 18,39/5,55$ and the above given w_i in (11).

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[3] Cf. ACT, p. 319, B(Ξ).

[4] This text will be published by Aaboe-Henderson-Neugebauer-Sachs, restored from five fragments (BM 36651 + ⋯). The Mercury ephemeris is written on the reverse, at right angles to the writing on the obverse (which concerns lunar eclipses). Similar writing at right angles is found with the Venus ephemeris No. 430 and the Mars ephemeris No. 501a, for a Mars ephemeris and solstices (Aaboe-Sachs [1966], p. 11 f.), and for lunar theory and a list of unexplained numbers (Neugebauer-Sachs [1969], p. 96).

[5] Cf. above p. 438.

The same procedure text also refers to the 20-year period for which it gives (in Section 4) the following decrements

$$4;49,17^\circ \qquad 5;32,51,30^\circ \qquad 4;34,15^\circ. \tag{14}$$

Here one runs into difficulties. Using again the periods in (2) one would find with (5)

$$20\,\Pi = 6,13,0 \qquad 1,3\,Z = 6,12,45 \qquad \text{thus } d = 15 \tag{15}$$

and therefore with (6) and the δ_i in (12)

$$4;41,15^\circ \qquad 5;33,20^\circ \qquad 4;26,40^\circ \tag{16}$$

instead of (14). Probably a scribal error is involved since a change of 32 to 42 in the second number restores nearly the expected proportionality with the δ_i:

$$4;49,17,12,11,15^\circ \qquad 5;42,51,30^\circ \qquad 4;34,17,12^\circ. \tag{17}$$

An unexpected situation arose by the discovery of the above-mentioned "ephemeris" of Mercury positions for the fifth century. The procedure text No. 816 gives twice (both in Sect. 1 and 4) ♋20;37,30 as boundary between the zones α_2 and α_3, not ♋20 as we wrote in (11) in order to obtain the proper value for P that leads to the values (13). Surprisingly it turned out that both these boundaries are combined in the ephemeris with the same set (11) for the w_i.[6] The ephemeris is arranged in three columns, each of which gives longitudes of Ω such that each line corresponds to one year. For example the first two lines are

♌24;4,10	♐14;40,57,25,18,45	♉ 5;37,12,25,18,45
♌ 8;4,10	♏27;48,27,25,18,45	♓18;44,42,25,18,45.

The decrements are 16° and 16;52,30°, respectively, as they should be for α_3 and α_1 according to (13). One would expect that the whole ephemeris is obtainable by repeated use of (13) in each column. This, however, is not the case: as soon as the receding longitudes have to cross from α_3 back into α_1 it is not ♋20 that is used as boundary but ♋20;37,30. The first column, e.g., with ♋20 as boundary would be:

♌24; 4,10
♌ 8; 4,10
♋24; 4,10
....
♋ 2;35,12,30
♊12;35,12,30
etc.

while the text begins in the new zone with ♋2;25,50.

In other words both the procedure text and the ephemeris are based on the same pair of contradictory parameters: the zones α_i and the synodic arcs from (11) determine the yearly decrements (13) but α_1 and α_3 are slightly modified for the actual computation from one year to the next. Neither one of the two norms explains the decrements (14) or (17) for 20-year steps nor do the numbers actually

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[6] This was discovered by A. Aaboe.

obtained in the ephemeris[7] give these results, as is not surprising since the combination of two different procedures will result in a variety of values for the 20-year intervals.

The difficulties with the procedure text No. 816 do not end here. Accordingly to its Sect. 1 and 4 the zones given in (11) concern both Γ and Ω while Sect. 3 seems to associate the synodic arcs with the stationary points though they are not included in the ephemerides.[8] So much at least can be established from the given numerical data that they can only refer to Ω and not also to Γ. We know from the Systems A_1 and A_2 that the synodic arcs for Γ and Ω are greatly different (cf. Fig. 23, p. 1322) while the rules for System A_3 are very similar to the rules for Ω from System A_2 (cf. Fig. 44).[9]

In the Sect. 2 and 5 decrements after one year and after 20 years are mentioned for the longitudes of Ξ. Since the two sets of numbers are in the proper ratio for these two corrections it suffices to use only one of them. For three synodic periods the following decrements are given

from ♒ 0° to ♊ 30	16°	in ♎	23°	
in ♋ and ♌	20	in ♏	18	(18)
in ♍	28	in ♐ and ♑	14 .	

Nothing is said about computing the longitudes of Ξ themselves. In order to obtain at least some estimate of the underlying synodic arcs one can try to make use of the relation (7), p. 468 and divide the numbers in (18) by $P-3 \approx 0;9,7,40$. This would give the following synodic arcs, in the same order of the arcs as in (18):

$$1,45° \quad 2,11° \quad 3,4° \quad 2,31° \quad 1,58° \quad 1,32°. \tag{19}$$

The general trend of these numbers is at least generally similar to the synodic arcs for Ξ in System A_1.[10]

Dates. For System A_3 we have no information about dates. The dates for the primary phases, Γ and Ξ in System A_1, Σ and Ω in System A_2, are determined by adding a constant amount, reckoned in tithis, to the synodic arcs, the result being the synodic time interval. From the Sect. 1 and 2 of the procedure text No. 801 we know that

$$\Delta\tau = \Delta\lambda^{\tau} + 3;30,39,4,20^{\tau} \tag{20}$$

was assumed to be the accurate relation between $\Delta\tau$ and $\Delta\lambda$. From it follows for the epact[11]

$$e = 3;30,39,4,20 \cdot P.$$

Using for P the extremal values known from (1), (2), and (3), p. 466f. one finds

$$11;3,59,6 < e < 11;4,0,35.$$

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[7] Instead of the first or third value in (14) or (17) one finds in the ephemeris decrements of 4;49,9,36,33,45 and 4;34,10 respectively.
[8] Cf. above p. 466.
[9] This is confirmed by a comparison of the ephemeris with modern data, carried out by A. Sachs for the two historically possible periods covered by the ephemeris.
[10] Cf. Fig. 23 p. 1322.
[11] Cf. above II A 4.

Obviously $e = 11;4^{\mathrm{r}}$ is the value from which (20) was derived. For the actual computation of ephemerides[12] the relation (20) was abbreviated to

$$\Delta\tau = \Delta\lambda^{\mathrm{r}} + 3;30,39^{\mathrm{r}}. \tag{21}$$

This procedure was abandoned, however, for the satellite phases. Just as their longitudes are obtained from the longitudes of the primary phases by a set of "pushes" so are the dates found by a series of increments which lead from the primary dates to the dates of the satellites, at least for the ephemerides of System A_1. For System A_2, however, the one extant ephemeris[13] shows serious deviations from (21) and we do not know how the dates of Σ and Ω were actually computed.[14]

Pushes. For the outer planets the procedure texts supply us with a great deal of information for the transition from one phase to the next one. One might think that this "subdivision of the synodic arc" would also be available for Mercury since the pushes in System A_1 give the increments (of longitude and time) from Γ to Σ and from Ξ to Ω while the pushes in System A_2 bridge the intervals from Σ to Ξ and from Ω back to Γ.[15] In fact, however, the rules of the two systems are not compatible because the results obtained by the pushes of one system are not identical with the primary sequences of the other. It is difficult to understand why one introduced pushes at all when one had independent computing schemes for all four phases.

The list of pushes from Γ to Σ for longitudes and times is fairly secure since we have them tabulated from degree to degree in a text from Uruk (ACT No. 800a).[16] For $\Xi \rightarrow \Omega$ we have only fragments of a similar table[17] but the ephemerides[18] make it possible to complete the scheme.[19]

If one computes for all pairs of pushes in longitude and in time the quotients $\Delta\lambda/\Delta\tau$ one obtains numbers between $0;51^{\circ/\tau}$ and $1^{\circ/\tau}$ as is to be expected for phases which are always near the sun. The variations of $\Delta\lambda/\Delta\tau$, however, are rather irregular and show no relation to the solar anomaly.

The vectors for all pushes belonging to System A_1 are shown in Fig. 45. The conspicuous deviations from parallelism for $\Gamma \rightarrow \Sigma$ in ♒ and for $\Xi \rightarrow \Omega$ in ♍ reflect the difficulties with the given parameters mentioned above in notes 16 and 19. The great variability in length illustrates the variations in the visibility conditions of Mercury depending on the seasons of the year.[20] In ♈ and ♉ the

[12] Cf., e.g., ACT No. 300, col. I and II. For simplifying procedures cf. ACT, p. 293.

[13] ACT No. 300a and 300b.

[14] Cf. ACT, p. 298.

[15] Cf. Fig. 21, p. 1321.

[16] A copyist error made column II run from 43° up to 44;56° instead of down from 44° to 42;4°. The value $\Delta\lambda = 30°$ in ♒15° is suspect (34° would be better) but it is secured by the pattern of cols. IV–VI–VIII and by the colophon (ACT No. 820a, p. 438) of the ephemeris No. 301. For ♋15°, No. 820a (obv. VI) has $\Delta\tau = 24^{\mathrm{r}}$ but No. 301 uses 25^{r} (cf. ACT, p. 321).

[17] ACT Nos. 800c and d.

[18] Mainly Nos. 301 and 302.

[19] Cf. ACT, p. 293–295. In the case of ♊15° there is some vacillation between $\Delta\lambda = 46°$ and 45° (cf. ACT, p. 319). The value $\Delta\tau = 20^{\mathrm{r}}$ at ♍15° (from ACT No. 800d, col. II/IV) is hardly correct since it is the only case for which $\Delta\lambda > \Delta\tau$. The ephemeris No. 301 uses 30^{r} (ACT, p. 320) which is equally implausible; one expects 22^{r} or 23^{r}.

[20] Cf. also the graph ACT, p. 299, Fig. 57c, derived from the ephemeris No. 301 for the three years S.E. 145 to 147 (i.e. −166 to −163).

morning star phases show an approach to each other of less than 15° and the same holds for the evening star phases in ♎ and ♏ (dotted line in Fig. 45). These are exactly the regions where these phases are "passed by", i.e. where the planet remains invisible.[21]

Whatever theory lies behind the given sequence of pushes, it is clear that they were constructed for the midpoints of the signs. As is quite justifiable for data which have no influence on subsequent numbers the ephemerides use several shortcuts of interpolations instead of using the exact numbers from the tables ACT No. 800a to e which are based on linear interpolations. The most drastic procedure uses for the whole sign the value given for 15° of the sign. This can result in deviations up to 7° with respect to linear interpolation.[22] This seems to be a rather large amount, particularly in view of the limited visibility of Mercury.

India. The theory of Mercury is another example for the influence which Babylonian astronomy exercised, directly or indirectly, or Indian astronomy. In the last chapter of the Pañca-Siddhāntikā a quotient $P=684/217$ is mentioned[23] which is identical with the ratio Π/Z for Ω in System A_2. Also to System A_2 (Σ) belongs the number $Z=388$, implicit in the next verses,[24] though broken up into eight components and incorrectly denoted as "days". These components are related to an eight-division of the ecliptic of rather unequal size, not attested in any System A scheme.

The most conspicuous feature of the Babylonian theory of Mercury, the "pushes" for longitudes and times between consecutive phases, has also its close parallel in India, the only difference being that these pushes are not distributed between two alternative systems by giving for each zodiacal sign all four pushes from one phase to the next. For example we have for longitudes

	$\Xi\to\Omega$	$\Omega\to\Gamma$	$\Gamma\to\Sigma$	$\Sigma\to\Xi$
in ♈	35°	−22°	21°	54°
in ♉	44	−17	23	69

etc., and similarly for the time intervals. The numerical values are not identical with the Babylonian pushes but the general trend is the same,[25] at least for $\Xi\to\Omega$ and $\Gamma\to\Sigma$ in System A_1. For System A_2 the Indian and Babylonian data show serious differences,[26] perhaps resulting from an attempt to bring A_2 into agreement with A_1 – assuming that A_2 really existed as a generally recognized Babylonian system.

Apparently aimed at the mean synodic period is a subdivision of Mercury's motion in its relation to the sun (in Pc.-Sk. XVII 70f.[27]), similar in style to the corresponding Babylonian rules for the outer planets. The Indian pattern starts with the inferior conjunction of the planet with the sun and includes also the stations[28] beyond the four phases of first and last visibilities. The total synodic motion is found to be 114° in agreement with the Babylonian $\Delta\bar{\lambda}=1,54;12,\ldots°$

[21] Cf. above p. 403.
[22] Cf. ACT, p. 294.
[23] Pc.-Sk II, p. 112, Table 24; cf. also above p. 466.
[24] Pc.-Sk. II, p. 114.
[25] Cf. Pc.-Sk. II, p. 122 and Fig. 67 on p. 123.
[26] Compare Pc.-Sk. II, p. 124, Fig. 68 and ACT, p. 298f., Figs. 57a and 57b.
[27] Pc.-Sk. II, p. 128, Table 36.
[28] Cf. above p. 471.

B. Lunar Theory

§ 1. Introduction

The theory of the moon is by far the most advanced part of Babylonian astronomy. This is evident alone from the outward appearance; while a planetary ephemeris, say for the heliacal risings (Γ), requires only 4 or 5 columns, an ephemeris for the first visibility of the moon can run to 18 columns (e.g. ACT No. 122, our Pl. IV). The planetary theory ignores latitudes whereas a large section of the lunar theory is concerned with latitudes and eclipses. Yet the basic mathematical tools are the same: fundamental period relations, obtainable by direct counting of events, which determine the periods of step functions or zigzag functions. If need be these can be modified by interpolatory devices based on arithmetical progressions of first or second order.

Our source material for the lunar theory is comparatively rich. We have more texts which concern the moon than for all planets combined. Consequently we do have a fair understanding of the techniques which provide the answer to some fundamental problems of the lunar theory, in particular the determination of the moments of the true syzygies. But we have only a very vague idea about the historical sequence of the different methods nor do we really know on what basis the majority of the functions was constructed, empirically as well as theoretically. We grasp, e.g., nowhere in the lunar theory the basic concepts as clearly as in the planetary theory of System A with its rearrangement of Π events in groups of Z in sections of different density (above II A 6).

As far as the motivation for the lunar theory is concerned we see a little more clearly. The mere fact that the majority of the "ephemerides" progress in monthly steps and that they lead up to data for the first and last visibility shows their usefulness for the true lunar calendar which is so characteristic for ancient Mesopotamia. Beside the calendaric purpose stands equally clear the interest in eclipses which is certainly related, at least historically, to the interest in omina provided by these extraordinary celestial events. Neither the lunar nor the planetary ephemerides and procedure texts contain any hint of astrological, in particular horoscopic, applications as we know them from the hellenistic and Roman period in Egypt and thereafter. One may, however, point to the fact that one would never suspect from the Almagest the existence of the Tetrabiblos. The only difference is that we know far less about Babylonian astrology of the hellenistic period than of its Greek offspring.

We distinguish in the lunar theory also two "*Systems*," A and B, respectively. But, unlike the situation in the planetary theory where System A and B meant the use of step functions versus zigzag functions for the planetary motion, it is not the model for the lunar motion itself that provides the criterium for the classification

but the solar motion. We shall see later the reason for this decisive influence of the solar model on the lunar theory.[1] At the moment we simply define that a lunar ephemeris belongs to

"*System A*" when the solar velocity is described by a step function and when the vernal equinox is located at ♈10°

but to

"*System B*" when the solar velocity follows a linear zigzag function and when the vernal equinox is located at ♈8°.

We have described these features of the solar theory in the Sections 4 and 5 of our introductory chapter[2] and need no further details at this time. Much in the solar theory remains obscure and depends on our understanding of the lunar theory. For this reason we postpone the discussion of these elements (e.g. the problem of the length of the solar year) to a later section.[3]

It is customary to assume that System A is older than System B, mainly on the basis of improvements of parameters in B compared with A. This trend is, however, not consistent nor is the chronological situation really clear since both systems coexist in the material available to us. The planetary theory provides no better chronological criteria. And it is by no means evident that parameters which are "better" according to our present knowledge are the result of improved observations and not, e.g., the accidental consequence of some requirements of the adopted computational model.

Beside the above definitions there are other features which are to a certain extent typical for the two systems, though by no means sharply. For example the majority of texts of System A are from Babylon whereas Uruk seems to favor System B. This may be, of course, purely accidental, due to the peculiar circumstances of preservation and discovery. But what is most likely not accidental is the fact that practically all texts of System A are consistently computed such that any later ephemeris is the direct continuation of the older ones. System B, in contrast, shows many variants and only small groups of texts are connectible by continued computation. Another peculiarity of System B is the existence of "auxiliary tables" for specific functions, e.g. solar velocity and resulting longitudes, lunar velocity, eclipse magnitudes, etc. The ephemerides were then compiled from such individual tables, frequently only with abbreviated or rounded values. Auxiliary tables are not totally unknown for System A but apparently much less common. On the other hand the majority of "procedure texts" concerns System A, a fact which must make one suspicious about all such statistical evidence from our exceedingly fragmentary sources, since it is hard to believe that any method of computation of ephemerides could dispense with procedure texts which formulated the necessary rules and gave the needed numerical parameters.

In the following the columns of ephemerides and related texts are denoted by capital letters such that the same letter is always associated with the same astronomical concept, e.g. F (and variants like F′, F*) with the lunar velocity. Similarly a subscript 1 means new moons, a subscript 2 full moons.

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[1] Cf. below p. 516f.

[2] Above p. 366ff. and p. 371ff.

[3] Below II B 8.

The Systems A and B do not operate with identical functions, yet many columns are equivalent in a larger sense. Hence we can list, independent of the specific systems, the following notations[4]:

Solar motion:	A: velocity, B: longitude ($\lambda_\odot$ or $180+\lambda_\odot$) C, D, etc.: length of daylight, night, etc.
lunar latitude:	E; eclipse magnitude: Ψ
lunar velocity:	F (related: Φ)
syzygies:	G and subsequent columns up to K: length of the synodic month, found in several steps; L and M: moments of syzygy
visibility:	P_1 and P_3: first and last visibility, straddling conjunction P_2, etc.: related to full moon.

We also have texts[5] which list only the character of the consecutive months during several years (i.e. the sequences full/hollow) and the values for P_1, the latter perhaps in order to allow for the interchange of a full and a hollow month in case P_1 shows a particularly high or particularly low value for the ripeness of the crescent. Such lists might also be useful for composing planetary texts for the daily motion during a given year or for a given sequence of months[6] or for the "Diaries" where each month carries an entry "30" or "1" which indicates whether the preceding month was hollow or full, respectively.[7]

§ 2. Lunar Velocity

We know of two different types of tabulations for the lunar velocity $v_☾$, measured in degrees per day: (a) a linear zigzag function F* which gives $v_☾$ in intervals from day to day, based on a mean value of $13;10,35^{\circ/d}$ (or near to it) with a period near $27\,1/2^d$; (b) a linear zigzag function F with the same mean value but with a period corresponding to a tabulation of $v_☾$ at consecutive mean conjunctions (or oppositions).

In order to estimate the latter period we must first express the period p of the true function F* in the units of the tabulation function F,[1] i.e. in mean synodic months. For the latter we can take about $29\,1/2^d$ hence for p

$$p \approx \frac{27;30}{29;30} = \frac{55}{59} \approx 0;56^m. \tag{1}$$

Consequently, the period P of the tabulation function will be near

$$P = \frac{p}{1-p} \approx \frac{0;56}{0;4} = 14^m. \tag{2}$$

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[4] For details cf. ACT I, p. 43.
[5] ACT No. 180ff.
[6] Cf., e.g., the texts discussed in II A 5, 3 A and 3 B. This was suggested by Huber [1957], p. 276.
[7] This notation has its origin in the concept that the first day of the current month is the 30th day of the preceding one if the latter had been hollow; otherwise day 1 of month N follows day 30 of month $N-1$.
[1] Cf. for these concepts above p. 375.

Since in our texts F is found to have a period of nearly $13;57^{m}$ it is clear that indeed F represents $v_{☾}$. This conclusion is confirmed by the relation of F to G, the variable length of the synodic months[2] in their dependence on the lunar anomaly.

Although we know in this way that F is in principle the tabulation function of the true function F* we find in fact a variety of functions F with slightly different parameters such that no exact agreement between the data of F* and F exists. The cause of this is in part the tendency to construct zigzag functions with convenient differences but behind some variants also lie other reasons not fully intelligible to us. As a practical consequence remains the fact that column F is an exception to the rule that in System A all ephemerides are connectible[3]; for F this is not the case, in marked contrast, e.g., to the function Φ which is also related to the lunar velocity.[4]

One reason for tabulating the lunar velocity at the syzygies is obviously connected with eclipses. The duration of an eclipse depends on the travel time of the moon across the shadow cone or across the solar disk. For this the variable lunar velocity must be known (the solar velocity can be considered constant). The existence of a function F′ which is identical with F except of the use of degrees per large hour[5] supports this explanation since it is natural to express the duration of eclipses in large hours and their fractions instead of in days. But then it is not clear why F exists at all beside F′.

Another problem for which the lunar velocity is of importance concerns the duration of the first and last visibility of the moon, a problem to which the final columns of lunar ephemerides are devoted.[6] Again one would think that F′ would be more convenient than F.

1. System B

From the ephemerides of System B we know two functions F, the second obviously only an abbreviated version of the first.

Examples:

ACT No. 122 obv./rev. VI	ACT No. 123 obv. VIII
11,30	11,34
11,16,10	11,12
11,52,10	11,48
⋮	⋮
14,16,10	14,12
14,52,10	14,48
15, 4	15, 8
14,28	14,32
etc.	etc.

[2] Cf. below II B 3, 1 and 2.

[3] Cf. p. 475.

[4] Below p. 484. Later (p. 500) we shall see to what extent an exact relation F* ↔ F can be reconstructed for System A.

[5] ACT No. 120 rev. II gives F′, exactly parallel to F in the next column.

[6] Cf. below II B 10.

From these sequences one deduces the following parameters

$$\begin{array}{rl@{\qquad}rl}
\text{accurate:}\ d= & 36,\ 0 & \text{approximate:}\ d= & 36 \\
m= & 11,\ 5,\ 5 & m= & 11,\ 5 \\
M= & 15,16,\ 5 & M= & 15,16 \\
\mu= & 13,10,35 & \mu= & 13,10;30 \\
\Delta= & 4,11,\ 0 & \Delta= & 4,11
\end{array} \tag{1}$$

hence in both cases

$$P=\frac{4,11}{18}=13;56,40^{\mathrm{m}} \tag{2a}$$

as expected.[1] Since

$$p=\frac{P}{P+1}=\frac{4,11}{4,29}=0;55,59,6,28,\ldots^{\mathrm{m}} \tag{2b}$$

we can say that these velocity functions are based on the relation

$$4,29 \text{ syn.m.}=4,11 \text{ anom.m.} \tag{2c}$$

The "exact" value of the mean velocity is

$$\mu=13;10,35^{\circ/\mathrm{d}} \tag{3}$$

and all other parameters in (1) can be normed accordingly.

2. System A

The numerical values which characterize F in System B leave no doubt that we are dealing with the lunar velocity, measured in degrees per day. In System A it is only the periods which have the expected values while the mean value $13,30,\ldots^{\circ/\mathrm{d}}$ is much higher than $13;10,35^{\circ/\mathrm{d}}$ and also the amplitude $\Delta\approx 4;53$ much exceeds $\Delta=4;11$ in System B.

The parameters in question come from procedure texts or from the ephemeris ACT No. 18 col. V. From these sources one derives

$$\begin{array}{rl@{\qquad}rl}
d= & 0;42 & \mu= & 13;30,29,31,52,30 \\
m= & 11;\ 4,\ 4,41,15 & \Delta= & 4;52,49,41,15 \\
M= & 15;56,54,22,30 & &
\end{array} \tag{4}$$

and thus

$$P=\frac{1,44,7}{7,28}\approx 13;56,39,6,\ldots^{\mathrm{m}} \qquad p=\frac{1,44,7}{1,51,35}\approx 0;55,59,6,13,\ldots^{\mathrm{m}} \tag{5a}$$

which means that

$$1,44,7 \text{ syn.m.}=1,51,35 \text{ anom.m.} \tag{5b}$$

A variant of these parameters is found in the ephemeris ACT No. 92 with

$$\begin{array}{rl@{\qquad}rl}
m= & 10;59,53,26,15 & \mu= & 13;26,18,16,52,30^{\circ/\mathrm{d}} \\
M= & 15;52,43,\ 7,30 & &
\end{array} \tag{6}$$

[1] Cf. above p. 476 (2).

but with d and Δ as in (4), hence with the periods (5a). We do not know why such changes were introduced.

The majority of ephemerides contains only an abbreviated column F, obviously derived from (4) by roundings to minutes

$$\begin{array}{ll} d = \ 0;42 & \mu = 13;30,30^{\circ/\mathrm{d}} \\ m = 11;\ 4 & \Delta = \ 4;53. \\ M = 15;57 & \end{array} \tag{7}$$

This modifies the periods to

$$P = \frac{4,53}{21} \approx 13;57,8,\ldots^{\mathrm{m}} \qquad p = \frac{4,53}{5,14} \approx 0;55,19,\ldots^{\mathrm{m}} \tag{8}$$

which is too inaccurate to allow continuation of this abbreviated function F over longer intervals. Hence even contemporary texts may use slightly different values for F as is shown in the following example

ACT		No. 4 = No. 4a	No. 5
S.E. 2,26	I	11;52	11;55
	II	12;34	12;37
	III	13;16	13;19
		etc.	

For a visibility computation or for an eclipse a difference of a few minutes in the lunar motion per day is, of course, without significance.

A peculiar variant of a velocity function has been found by A. Aaboe[2]. The parameters

$$\begin{array}{ll} m = 10;50,23,26 & \mu = 13;17,27,39,22,30 \\ M = 15;44,31,52,30 & d = \ 0;42,11,15 \end{array} \tag{9a}$$

are chosen in such a fashion that the period

$$P = 4,11/18 \tag{9b}$$

agrees exactly with the norm for F in System B. Nevertheless the present function is also related to System A by being "truncated" at $M' = 15$ and $m' = 11;15$ which is the equivalent of the truncation of the function Φ[3].

All zigzag functions for the lunar velocity have a comparatively high mean value which the truncation changes only insignificantly. On the other hand the "Saros Text" which concerns the relationship between Φ and G, and thus surely belongs to System A, gives in Section 11 for the length of one half of the anomalistic month[4]

$$1/2\ \mathrm{m}_{\mathrm{an}} = 13;46,38,15^{\mathrm{d}} \qquad (\text{thus } 1\ \mathrm{m}_{\mathrm{an}} = 27;33,16,30^{\mathrm{d}}) \tag{10}$$

[2] Aaboe [1968], p. 30–34.

[3] Cf. below p. 501 f.

[4] Cf. also below p. 501; Neugebauer [1957, 1], p. 18 f.

with a longitudinal progress[5] of 3,1;22°. Hence the mean motion in longitude is

$$\bar{v} = 3,1;22/13;46,38,15 \approx 13;10,34,34, \ldots^{\circ/d} \qquad (11)$$

very nearly the same as 13;10,35 in System B.

3. Daily Motion

We have a group of texts (ACT No. 190ff.) which give either the lunar velocity F* for consecutive days, or F* and the longitudes resulting from its summation, or the longitudes, day by day, alone. If one computes the differences in the last case one finds that they agree with F* in the other texts. If such texts contain dates then they are meant as actual calendaric dates (not tithis) as is evident from the irregular alternation of full and hollow months.

The moment for which the lunar velocity is tabulated (in degrees per day) is in all probability midnight, concluding from an entry "(month) II, night (of day) 1" in one of these texts.[1] The use of midnight epoch in lunar ephemerides is the norm in System B (column L).[2] Also the fact that F* is based on a mean value $13;10,35^{\circ/d}$ relates these texts to System B.[3] A difference in amplitude, however, (4,8 instead of 4,11) excludes F* as true function underlying F.

The parameters of F* can be easily derived from ACT No. 190, a text which begins and ends with the same number and hence represents one whole number period in 248 lines. Indeed, the text is based on

$$\begin{array}{ll} d = 0;18 & \mu = 13;10,35 \\ m = 11;\ 6,35 & \Delta = 4;\ 8 \\ M = 15;14,35 & \end{array} \qquad (1)$$

and thus

$$P = \frac{4,8}{9} = 27;33,20^d, \qquad \Pi = 4,8^d \qquad (2a)$$

which means

$$9 \text{ anom.m.} = 248^d \qquad (2b)$$

and, of course

$$1 \text{ anom.m.} = 27;33,20^d. \qquad (2c)$$

From (1) and (2c) one finds for the lunar motion during one anomalistic month

$$27;33,20 \cdot 13;10,35 = 6,3;4,57,46,40^\circ$$

or about $+3;5^\circ$ as motion of the lunar apogee during one anomalistic month. For the daily motion of the apogee this gives about 0;6,42,45° in comparison with about 0;6,41,4° for the difference between mean motion in longitude and in anomaly according to the Almagest (IV, 4).

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[5] Sect. 4.

[1] ACT No. 194a, obv. II, 25.

[2] Cf. below p. 492.

[3] Above p. 478.

The parameters (1) and (2) and their use in a linear zigzag function are well known in Greek astronomy as, e.g., Geminus' "Introduction" shows.[4] They also appear in Indian astronomy, e.g. in the Pañca-Siddhāntikā (II, 2–6) in a chapter which is based on the Vasiṣṭha-Siddhānta.[5] In order to determine the longitude of the moon at a given date one starts from an epoch (date and longitude) at which the lunar velocity had its minimum value. First one determines the number of anomalistic periods contained in the time interval elapsed since epoch. This is done in two steps, first counting the number of intervals of length

$$3031^d = 110 \text{ anom.m.} \tag{3}$$

and secondly, for the residue, the number of intervals of $27;33,20^d$ length. The relation (3) we shall find again not only in the Tamil tradition[6] but also in Greek papyri of the Roman period.[7] The moon's progress during each interval (3) is assumed to be $5,37;32°$ (which is the equivalent of a mean velocity of $13;10,34,52,46,\ldots^{\circ/d}$). The single anomalistic months (2) each contribute $3;4,50°$ in longitude (corresponding to $13;10,34,43,3,\ldots^{\circ/d}$ as mean motion). This provides us with the increment in longitude since epoch for the completed anomalistic months.

In general there will remain a fraction of one anomalistic month, beginning at the moon's apogee. For this residual time interval a second order procedure is constructed which gives the corresponding lunar motion.[8] This procedure is based on a linear zigzag function that represents the lunar velocity. Its period is again (2) but it results from different parameters:

$$m = 11;42^{\circ/d} \qquad M = 14;39 + 1/7,0 \qquad d = 1;30/7 \approx 0;12,51,\ldots \qquad \Delta \approx 2;57. \tag{4a}$$

The corresponding mean value is

$$\mu = 13;10 + 4/7,0 \approx 13;10,34,17,\ldots^{\circ/d}. \tag{4b}$$

Hence this last step differs in detail considerably from the Babylonian function F* as defined by (1). I see no plausible motive for replacing the parameters (1) by (4).

4. Summary

The parameters of the lunar velocity functions which we have discussed so far are listed in Table 11. The basic relations for F*

$$1 \text{ anom.m.} = 27;33,20^d \quad \text{or} \quad 9 \text{ anom.m.} = 248^d \tag{1}$$

will be paralleled in System A[1] by a more accurate value

$$1 \text{ anom.m.} = 27;33,16,30^d. \tag{2}$$

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[4] Manitius, p. 204 to 211. Cf. also below p. 602f.

[5] Cf. Pc.-Sk. I, p. 14. The Vasiṣṭha-Siddhānta is probably also the source of Pc.-Sk. XVII, 1–60 where one finds much undoubtedly Babylonian material for the theory of the planets (cf. above II A 7, 1).

[6] Cf. below V A 2, 1 D 1.

[7] Cf. below V A 2, 1 A.

[8] For the details of Pc.-Sk. II, p. 18f.

[1] Cf. below p. 501 (9).

Table 11

		System B			System A		
	F^*	F		F′	Proc. texts No. 18: F	No. 92: F	ephemerides: F
μ	$13;10,35^{\circ/d}$	$13;10,35^{\circ/d}$	$13;10,30^{\circ/d}$	$2;11,45,50^{\circ/H}$	$13;30,29,31,52,30^{\circ/d}$	$13;26,18,16,52,30^{\circ/d}$	$13;30,20^{\circ/d}$
m	11; 6,35	11; 5, 5	11; 5	1;50,50,50	11; 4, 4,41,15	10;59,53,26,15	11; 4
M	15;14,35	15;16, 5	15;16	2;32,40,50	15;56,54,22,30	15;52,43, 7,30	15;57
Δ	4; 8	4;11		0;41,50	4;52,49,41,15		4;53
d	0;18	0;36		0; 6	0;42		
p		$\frac{4,11}{4,29}\approx 0;55,59,6,28,\ldots^{m}$			$\frac{1,44,7}{1,51,35}\approx 0;55,59,6,13,\ldots^{m}$		$\frac{4,53}{5,14}\approx 0;55,19,\ldots^{m}$
P	$\frac{4,8}{9}=27;33,20^{d}$	$\frac{4,11}{18}=13;56,40^{m}$			$\frac{1,44,7}{7,28}\approx 13;56,39,6,\ldots^{m}$		$\frac{4,53}{21}\approx 13;57,8,\ldots^{m}$

That (2) is specifically related to System A lends support to the above made suggestion[2] to associate (1) with System B.

The basic period relation for the function F is in System B

$$4,11 \text{ syn.m.} = 4,29 \text{ anom.m.} \tag{3}$$

and in System A

$$1,44,7 \text{ syn.m.} = 1,51,35 \text{ anom.m.} \tag{4}$$

Again the parameters of System A seem to be more refined than in System B. There is certainly nothing "primitive" in the mathematical structure through which the anomaly of the moon is related to the various components of the lunar ephemerides of System A and the theory of eclipse cycles.[3] But because of the intricacy of these relations we are still far from a real understanding of the theoretical arguments which motivated these procedures. Any pronouncement about the relative age of System A and B still remains guesswork.

§ 3. The Length of the Synodic Months

In the ephemerides of both System A and B is found a column which we denote by G (usually following F) which represents the excess of the variable length of the synodic month over 29^d, this excess being measured in large hours. This fact can easily be deduced from the subsequent columns which introduce a correction depending on the solar velocity and thus lead from G to K which is the difference column (always beyond 29^d) for the moments of consecutive true conjunctions, respectively oppositions.[1] Hence the basic meaning of column G and its units can be considered known.

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[2] Above p. 480.

[3] Below p. 497ff.

[1] The details will be described below in II B 3, 3 to 5.

We shall concentrate in the following on the computational aspects of column G. This raises very few difficulties for ephemerides of System B because G is a simple zigzag function with the period of F. In System A, however, G is flattened near its extrema and its computation is made dependent on an auxiliary zigzag function, called Φ, which is usually the first numerical column right after the date column at the beginning of an ephemeris. This column Φ is perhaps the most intriguing element in the whole lunar theory since it is connected with the periodicity of eclipses, the so-called "Saros." At the moment, however, this aspect of the theory will be completely ignored and Φ will be treated only as an auxiliary function by means of which one finds the value of G which, in turn, is needed to determine the time intervals between the syzygies.

1. System B, Column G

We find tabulated in ephemerides (e.g. in ACT No. 122, col. VII) a zigzag function which we denote by G with the following parameters

$$\begin{array}{ll} d=0;22,30^{\mathrm{H/m}} & \mu=3;11,\ 0,50^{\mathrm{H}} \\ m=1;52,34,35 & \Delta=2;36,52,30. \\ M=4;29,27,\ 5 & \end{array} \tag{1}$$

Its period is the same as the period of F[1]

$$P_{\mathrm{G}}=\frac{4,11}{18}=13;56,40=P_{\mathrm{F}}. \tag{2}$$

This shows that G accounts only for the influence of the lunar anomaly upon the length of the synodic month and it is therefore not surprising that additional columns are needed to also take the solar anomaly into consideration.[2]

The mean value μ in (1) enables us to express the length of a mean synodic month in days

$$1 \text{ mean syn.m.}=29^{\mathrm{d}}+3;11,0,50^{\mathrm{H}}=29;31,50,8,20^{\mathrm{d}}. \tag{3}$$

This parameter is well known in Greek astronomy; we found it in the Almagest[3] and we find it again and again in ancient and mediaeval sources.[4]

The fact that F and G have identical periods is easy to explain. A synodic month will be longest (shortest) when the progress of the moon during this month is as small (great) as possible. The progress of the moon during one anomalistic month is always the same[5]; what counts for the variability of the motion is the velocity during the two days beyond the anomalistic month. This contribution will be a minimum if F* has its maximum exactly at the midpoint of the synodic interval (cf. Fig. 46), i.e. when the tabulation function F has its minimum in the midpoint of the synodic month. Similarly G will have a minimum for a

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[1] Cf. above p. 478 (2).

[2] Cf. below p. 492f.

[3] Above p. 69 (1).

[4] E.g. with Bīrūnī, with Maimonides, in the Hexapterygon, etc.

[5] As shown on p. 480, about 363;5°.

month in which F has a maximum. Consequently the period of G must be the same as the period of F but the phases must be opposite.

In an ephemeris the value of G which concerns the interval between the syzygies $N-1$ and N will be listed in the line for the month N. Hence we should expect that the maxima of the zigzag function G are tabulated behind the minima of F at a distance of exactly 1/2 interval. In fact, however, this is not the case in the extant ephemerides. In the group ACT Nos. 120 to 122[6] the maxima of G follow the minima of F by 0;37,20,33,20 instead of the expected 0;30 of an interval. In ACT No. 123[7] the opposite extrema of F and G coincide which is at least a plausible type of error. But for a phase difference of 0;37, ... I see no explanation.

2. System A, Columns Φ and G

A. The Function Φ

For our present purpose, the formulation of the rules which lead from Φ to G, it is convenient to take all parameters of Φ as integers in the last tabulated digit. In this norm we have for the linear zigzag function Φ:

$$\begin{array}{ll} d = 2,45,55,33,20 & \mu = 2,\ 7,26,26,20,\ 0 \\ m = 1,57,47,57,46,40 & \Delta = 19,16,51,\ 6,40. \\ M = 2,17,\ 4,48,53,20 & \end{array} \tag{1}$$

Later[1] we shall see that Φ is measured in large hours (or time degrees) such that, e.g., $\mu = 2;7,\ldots^{\mathrm{H}}$.

It follows from (1) that

$$P = \frac{1,44,7}{7,28} = 13;56,39,6,\ldots^{\mathrm{m}} \tag{2a}$$

and hence[2]

$$P_\Phi = P_{\mathrm{F}}. \tag{2b}$$

Since the numbers of G are derived in a unique fashion from the numbers in Φ it is clear that G has the same period as Φ and hence that

$$P_{\mathrm{G}} = P_{\mathrm{F}} \tag{2c}$$

as in System B.

Experience has shown that all lunar ephemerides of System A operate with the same function Φ such that the number of lines required for Φ is one of the safest chronological parameters for System A, each line representing one synodic month. The chronological usefulness of Φ is further enhanced by the fact that Φ for an ephemeris of new moons (henceforth called Φ_1) is never connectible with Φ for full moons (Φ_2). Consequently any short section of Φ on a fragment of an ephemeris not only immediately gives us the date of the text[3] but, by its

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[6] Belonging to the interval from S.E. 179 to 210 (−132 to −101).

[7] S.E. 235 (−76/75).

[1] Below p. 498.

[2] Above p. 478 (5).

[3] Since the number period of Φ is $1,44,7^{\mathrm{m}}$ (cf. (2a)), i.e. more than 500 years, any date is uniquely determined within the historical limits of our material.

connectibility either with Φ_1 or with Φ_2, also its character as an ephemeris for conjunctions or oppositions, respectively.

The strict separation between Φ_1 and Φ_2 is also of theoretical interest. We take, e.g., from ACT No. 18 obv. 3 the value

$$\Phi_1 = 2{,}0{,}27{,}13{,}20{,}0$$

which is the last value before a minimum and refers to the conjunction near the end of S.E. 4,23 II (cf. Fig. 47a). If x is the distance of this point from the minimum of Φ_1 we have

$$xd = \Phi_1 - m = 2{,}0{,}27{,}13{,}20{,}0 - 1{,}57{,}47{,}57{,}46{,}40 = 2{,}39{,}15{,}33{,}20.$$

Similarly we obtain for Φ_2 (cf. Fig. 47b) for the opposition in S.E. 4,23 III (rev. 3)

$$x'd = M - \Phi_2 = 2{,}17{,}4{,}48{,}53{,}20 - 2{,}15{,}48{,}31{,}6{,}40 = 1{,}16{,}17{,}46{,}40.$$

Hence we have

$$(x - x')d = 2{,}39{,}15{,}33{,}20 - 1{,}16{,}17{,}46{,}40 = 1{,}22{,}57{,}46{,}40 = 1/2\, d$$

or exactly

$$x - x' = 1/2. \tag{3}$$

We now arrange the conjunctions and oppositions in their chronological order (cf. Fig. 48) such that the conjunctions lie exactly halfway between the oppositions and vice versa. Then (3) implies that the minima of Φ_1 lie exactly at the same points as the maxima of Φ_2 and vice-versa. This makes it evident that Φ_1 and Φ_2 are never connectible.

The symmetry between Φ_1 and Φ_2 shown in Fig. 48 is the direct consequence of tabulating the same true function Φ^* at two sets of intervals exactly 1/2 interval apart (cf. Fig. 49). The period of this true function Φ^* is found from (2a):

$$p = \frac{1{,}44{,}7}{1{,}51{,}35} = 0;55{,}59{,}6{,}13, \ldots^{m} \tag{4}$$

and is of course identical with the period of F* because of (2b).[4]

B. Column G near the Extrema

From several procedure texts we know the exact numerical relations which coordinate certain values of Φ with values of G. For intermediate values linear interpolation has to be used. These rules are amply confirmed by the ephemerides themselves. The result is a periodic function G, of course with $P_G = P_\Phi = P_F$, which in its major increasing and decreasing sections is linear but with a constant segment as minimum and with a much shorter constant segment as maximum. The linear segments are connected with the constant extrema by means of sequences of second order. Hence G can be described as a zigzag function with rounded extrema (cf. below Fig. 51, p. 1337).

Table 12 defines the relation between Φ and G near the minimum of G. This scheme is symmetric with respect to G such that the same value of G can be

[4] Cf. above p. 478 (5).

Table 12

	$\Phi(\downarrow)$	$\Phi(\uparrow)$	G	ΔG	k	
1.	2;13,20	2;15,48, 8,53,20	2;40	0	0	1.
	2;13, 2,13,20	2;15,30,22,13,20	2;40,17,46,40	0;0,17,46,40	1	
	2;12,44,26,40	2;15,12,35,33,20	2;40,53,20	0;0,35,33,20	2	
	2;12,26,40	2;14,54,48,53,20	2;41,46,40	0;0,53,20	3	
5.	2;12, 8,53,20	2;14,37, 2,13,20	2;42,57,46,40	0;1,11, 6,40	4	5.
	2;11,51, 6,40	2;14,19,15,33,20	2;44,26,40	0;1,28,53,20	5	
	2;11,33,20	2;14, 1,28,53,20	2;46,13,20	0;1,46,40	6	
	2;11,15,33,20	2;13,43,42,13,20	2;48,17,46,40	0;2, 4,26,40	7	
	2;10,57,46,40	2;13,25,55,33,20	2;50,40	0;2,22,13,20	8	
10.	2;10,40	2;13, 8, 8,53,20	2;53,20	0;2,40	9	10.

associated either with an increasing or with a decreasing section of Φ.[1] The coefficients k of interpolation result from the fact that the differences in G are related to the differences in Φ by

$$\Delta\mathrm{G}=k\cdot\Delta\Phi, \qquad k=1,2,3,\ldots \tag{1}$$

where $\Delta\Phi$ has a constant value

$$\Delta\Phi=0;0,17,46,40. \tag{1b}$$

Since

$$0;0,17,46,40=\frac{3}{28}\cdot 0;2,45,55,33,20=\frac{3}{28}\,d_{\Phi}$$

(cf. p. 484 (1)) it is convenient to introduce two new constants

$$\varphi=0;0,17,46,40 \quad\text{and}\quad \varepsilon=\frac{3}{28}=\frac{1}{9;20} \tag{2a}$$

such that

$$\varphi=\varepsilon\cdot d_{\Phi}. \tag{2b}$$

Since d_{Φ} is the increment of Φ for one synodic month (2b) tells us that φ is the increment during 3/28 months, i.e. about 0;6,25 of a mean synodic month. The astronomical significance of these parameters will be discussed in a later section (p. 497ff.).

In Table 12 the values of Φ increase from the last line up with the constant difference φ. We shall see[2] that in a different context Φ is "truncated" at the value 2;13,20. In Table 12 this boundary is transgressed because in the ephemerides values of Φ occur up to its maximum 2;17,... Nevertheless the importance attached to the number 2;13,20 is shown by the fact that "2,13,20" is used as the name for the function Φ. One could, e.g., say[3] 2,13,20 *šá* u$_4$-1-kam "The 2,13,20 of the first day" meaning simply Φ_1 (in contrast to "the 2,13,20 of the 14th day" for Φ_2).[4]

[1] For a tabulation of the whole scheme cf. ACT, p. 60. In a subsequent section (Table 14, p. 509) we will find the same values of $\Phi(\downarrow)$ in column S, lines 18 to 27.

[2] Cf. below p. 506.

[3] ACT No. 204 rev. 9 (p. 249).

[4] When I first collected the passages in which 2,13,20 is used like a noun (ACT, p. 212 and Neugebauer [1957], p. 18f.) I did not fully realize that "2,13,20" simply meant "Φ". It was A. Sachs who finally clarified the situation which has a parallel in the use of "the 18" for the 18-year eclipse cycle (now called "Saros").

Table 13

	$\Phi(\uparrow)$			G	
1.			1;59,48, 8,53,20	4;54,48,53,20	1.
			1;59,30,22,13,20	4;56	
			1;59,12,35,33,20	4;56,35,33,20	
			1;58,54,48,53,20	4;56,35,33,20	
5.	1;58,33,42,13,20	to	1;58,37, 2,13,20	4;56	5.
	1;58,15,55,33,20			4;54,48,53,20	

Table 13 gives values of $\Phi(\uparrow)$ which are associated with the maximum of G.[5] An apparently strange feature of this list is the fact (line 5) that not only one value of Φ leads to G=4;56 but all values from Φ=1;58,33,42,13,20 to 1;58,37,2,13,20. From the procedure texts which contain the rules on how to compute G from Φ it was not clear why the function Φ was moved parallel to itself by an amount of 0;11,15 ε. Only through the understanding of much more general computational methods could such an apparent irregularity find its explanation. A similar anomaly can then be detected near the maximum of Φ in consequence of which the minimum of G extends over an interval of length 16;57,30 ε (cf. Fig. 50a). We shall come back to these apparent anomalies in a later section (II B 4, 3 C 1).

C. The Function Ĝ

The fact that the function G is linear in its major part suggests ignoring the rounded sections near the extrema and expanding G into an ordinary linear zigzag function Ĝ (cf. Fig. 51) which has, of course, the same period as Φ and G. The part for which G is identical with Ĝ is defined by[1]

$\Phi(\downarrow)$	$\Phi(\uparrow)$	G=Ĝ
2;10,40	2;13, 8, 8,53,20	2;53,20
1;58,31,6,40,0	2; 0,59,15,33,20	4;46,42,57,46,40

This interval has the length of 41 ε. The parameters of Ĝ are

$$\begin{array}{ll} d=0;25,48,38,31,\ 6,40 & \mu=3;34,58,23,42,13,20 \\ m=2;\ 4,59,45,11,\ 6,40 & \Delta=2;59,57,17,\ 2,13,20. \\ M=5;\ 4,57,\ 2,13,20 & \end{array} \tag{5}$$

It is important to note that this difference d of Ĝ has a simple numerical relation to the difference of Φ. Indeed

$$d_{\hat{G}}=0;25,48,38,31,6,40=9;20\cdot 0;2,45,55,33,20=\frac{1}{\varepsilon}\,d_{\Phi}.$$

Consequently

$$d_{\Phi}=\varepsilon\, d_{\hat{G}} \tag{6}$$

[5] Since Φ and G are not exactly in phase, all values of Φ near the maximum of G belong to the increasing branch of Φ (cf. p. 1336, Fig. 50b).

[1] Cf. the table in ACT I, p. 60.

a relation which will be shown to be fundamental for the understanding of the astronomical meaning of the function Φ.[2]

From (5) one correctly obtains for the period

$$P_{\hat{G}} = \frac{1,44,7}{7,28} = P_\Phi = P_F \tag{7}$$

while it follows from (6) and (7) that

$$P_G = 2\Delta_{\hat{G}}/d_{\hat{G}} = 2\varepsilon\,\Delta_{\hat{G}}/d_\Phi = P_\Phi = 2\,\Delta_\Phi/d_\Phi$$

hence,

$$\Delta_\Phi = \varepsilon\,\Delta_{\hat{G}} \tag{8}$$

a relation which one can directly confirm numerically.

Since Φ and F are exactly in phase, as can be seem from the ephemerides which have a column Φ as well as an unabbreviated column F,[3] one would expect that the extrema of Ĝ follow the opposite extrema of Φ at a distance of exactly 1/2 mean synodic month,[4] i.e. at 4;40 ε. In fact, however, the ephemerides show only a difference of

$$4;10\,\varepsilon = 1/2(1-\varepsilon) \text{ mean syn. m.} \tag{9}$$

We shall come back to this displacement of G with respect to Φ.[5]

We can now make an explicit check of the identity of the period of G or Ĝ and Φ. We have seen before (p. 487) that G near its extrema was constant for 16;57,30 ε and 1;11,15 ε, respectively, with transitions of 9 ε and 6 ε, respectively at, each side. Hence we have

$$\begin{array}{rl}
\text{near minimum of G:} & 9+16;57,30+\ 9=\ \ 34;57,30\,\varepsilon \\
\text{for } G=\hat{G}: & \qquad\qquad 41+41=1,22 \\
\text{near maximum of G:} & \underline{6+\ \ 1;11,15+\ 6=\ \ 13;11,15} \\
\text{thus:} & \qquad\qquad P_G=2,10;\ 8,45\,\varepsilon
\end{array}$$

or (cf. (7)):

$$P_G = \frac{2,10;8,45}{9;20} = \frac{1,12\cdot 1,44,7}{1,12\cdot 7,28} = P_\Phi.$$

3. System A, Column J

In the ephemerides column G is followed by a column which we call J and which represents a step function with discontinuities exactly at the same points (♍13 and ♓27) known as discontinuities for the solar motion.[1] As we shall see[2] the values of J will be combined additively with G. Since the variations of G have

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[2] Below p. 498f.
[3] E.g. ACT No. 18, col. [−I] and V.
[4] Cf. p. 483.
[5] Below p. 504.
[1] Above p. 372.
[2] Below p. 491.

only the periodicity of the lunar anomaly while J depends only on the solar velocity it is clear that J is a correction of G for solar anomaly. One finds furthermore that the values g of J are

$$\begin{aligned} &g=0 && \text{on the fast arc} \\ &g=-0;57,3,45^{\mathrm{H}} && \text{on the slow arc.} \end{aligned} \tag{1}$$

Obviously G is computed under the assumption of high solar velocity (i.e. $V=30^{\circ/\mathrm{m}}$) and therefore no correction is needed on the fast arc. On the slow arc (solar velocity $v=28;7,30^{\circ/\mathrm{m}}$) the moon will overtake the sun sooner than on the fast arc, thus the months are shorter and g is negative.

The following numerical details can easily be deduced from the ephemerides. If J_1 and J_2 represent J for new moons and full moons, respectively, we must describe the location of the discontinuities with reference to the lunar (not solar!) longitudes as follows

$$\text{for } J_1: \begin{array}{l} \uparrow = ♍ 13 \\ \downarrow = ♓ 27 \end{array} \qquad \text{for } J_2: \begin{array}{l} \uparrow = ♓ 13 \\ \downarrow = ♍ 27 \end{array} \tag{2}$$

$\uparrow$ meaning a transition from slow to fast, $\downarrow$ the opposite. For intervals which belong completely either to the fast or to the slow arc the values g in (1) apply. If $g(\uparrow)$ denotes a value of J for an interval which contains a shift from low to high velocity according to (2) then

$$g(\uparrow)=-0;2,1,44\cdot s \tag{3a}$$

where s is the arc from the beginning of the interval until $\uparrow$. Similarly

$$g(\downarrow)=-0;2,1,44\cdot s' \tag{3b}$$

where s' is the arc from $\downarrow$ until the end of the interval.

Example. ACT No. 18 obv. 5: $B_1=15;11,15$ ♌
$$\frac{\uparrow = 13\,♍}{s=27;48,45}$$

thus $g=-0;2,1,44\cdot 27;48,45=-0;56,25,42,30^{\mathrm{H}}$ as given in the text.

Remark. It is characteristic for the ephemerides of System A that all transition values of J_1 are four digit numbers (not counting the initial zero), the last digit being always 30. The transition values of J_2 are always three digit numbers only.

The coefficient 0;2,1,44 in (3a) and (3b) is easy to explain. In both cases s and s' is the part of the interval that belongs to the slow arc. On this arc where the sun moves only $v=28;7,30°$ per month the correction per degree is exactly

$$g/v=-0;57,3,45/28;7,30=-0;2,1,44^{\mathrm{H}}. \tag{4}$$

Hence (3a) and (3b) is the correction required for arcs of s or s' degrees, respectively.

It still remains to explain the value $g=-0;57,3,45^{\mathrm{H}}$ on the slow arc. We shall see[3] that System A uses for the mean synodic month the value

$$29;31,50,19,11,\ldots^{\mathrm{d}}=29^{\mathrm{d}}+3;11,1,55,\ldots^{\mathrm{H}}. \tag{5}$$

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[3] Below p. 501 (10).

Let μ_G be the mean value of G, i.e. the mean value of the excess over 29^d under the assumption of high solar velocity. This assumption does not hold on the slow arc on which the sun travels during

$$2,46°/28;7,30^{°/m} = 5;54,8^m. \tag{6}$$

If we call

$$\mu_G = 3;11,1,55 + x^H \tag{7}$$

then we are sure that x must be positive. The excess accumulated by using (7) during one whole year over the correct total of 12;22,8 months of length (5) must be eliminated during the $5;54,8^m$ on the slow arc. The corresponding monthly correction should then be the value of g we wish to explain. Hence one would have

$$x\,\frac{12;22,8}{5;54,8} = 0;57,3,45$$

or

$$x \approx 0;27,13,45^H$$

and thus with (7)

$$\mu_G \approx 3;38,15,40^H. \tag{8}$$

In other words: we would have explained the value of g in (1) if we could find for μ_G the value (8). Unfortunately it is not evident how a "mean value" of the function G should be defined since G is a function with smoothly rounded extrema (cf. Fig. 51, p. 1337). The modern way of defining the mean value of such a function would be to add the areas of all the small trapezoids of base ε and sides G, known through the definition of G. Doing this and dividing the total by $P_G = 2,10;8,45\,\varepsilon$ (cf. p. 488) one finds

$$\mu_G \approx 3;38,15,1,\ldots^H. \tag{9}$$

The agreement with (8) is so good that one can consider $3;38,15^H$ at least as a close approximation to the Babylonian value for μ_G from which g in (1) was determined.[4]

4. System A, Columns C′, K, and M

The time difference between two consecutive conjunctions or oppositions is known from $G + J + 29^d$. Column M gives the moments of the syzygies but not with respect to a fixed epoch (noon or midnight) but with reference to sunset:

$$\begin{aligned} &\text{new moons: } M_1 = \text{time before sunset (šú)} \\ &\text{full moons: } M_2 = \text{time after sunset (du).} \end{aligned} \tag{1}$$

Column K is the difference column for M. Consequently K is not simply the sum of G and J but it includes a correction, which we call C′, that takes into account the variation of the length C of daylight, C being computed according to the pattern which is characteristic for System A[1].

[4] Using (9) instead of (8) one would find $g = -0;57,2,10^H$ instead of $-0;57,3,45^H$.

[1] Above p. 369.

The rules for these computations can easily be deduced from Fig. 52a (for M_1) and b (for M_2). We divide the time axis in segments representing alternatingly night (heavy line) and daylight and express the time T between two consecutive syzygies of the same kind in two ways: first as $G+J+29^d$ (the days reckoned between fixed epochs, e.g. noon), secondly with respect to the sunset epoch.

In the case of M_1 T reaches from the conjunction at the end of month n to the conjunction at the end of month $n+1$ (Fig. 52a), hence

$$\begin{aligned} T &= 29^d + G(n+1) + J(n+1) \\ &= 29^d + (M_1(n) - 1/2\,C(n)) + (1/2\,C(n+1) - M_1(n+1)) \end{aligned}$$

thus

$$M_1(n+1) = M_1(n) - (G(n+1) + J(n+1) + 1/2(C(n) - C(n+1))).$$

Similarly for full moons (Fig. 52b):

$$\begin{aligned} T &= 29^d + G(n+1) + J(n+1) \\ &= 29^d + (6^H - 1/2\,C(n) - M_2(n)) + (1/2\,C(n+1) - (6^H - M_2(n+1))). \end{aligned}$$

We now define

$$C'(n+1) = -1/2(C(n+1) - C(n)) = -1/2\,\Delta C(n+1), \tag{2}$$

i.e. a function proportional to the monthly change in the length of daylight. If we furthermore define K by

$$K = G + J + C' \tag{3}$$

then we have the following rules for the computation of M_1 and M_2

$$\begin{aligned} M_1(n+1) &= M_1(n) - K_1(n+1) \\ M_2(n+1) &= M_2(n) + K_2(n+1) \end{aligned} \tag{4}$$

of course reckoned modulo 6^H. For the calendar dates which belong to M cf. below p. 534.

Examples. From ACT No. 18 obv., adding place values (zeros) and units; for S.E. 243

	II = C	ΔC	VIII = $-1/2\Delta C = C'$	VI = G_1
1.	3; 8,27,30^H			
2.	3;24,19,30	+0;15,52^H	−0;7,56^H	4; 2,50,37, ...^{H}
3.	3;33,11,30	+0; 8,52	−0;4,26	4;28,39,15, ...

	I = B_1	VII = J_1	$J_1 + C'$	$G_1 + J_1 + C'$	IX = K	X = M_1
1.	22;41,15 ♈	−0;52,7, 1,30^H				3;32,57^H šú
2.	20;48,45 ♉	−0;57,3,45	−1;4,59,45^H	2;57,50,52^H	2;57,51	0;35, 7 šú
3.	18;56,15 ♊	−0;57,3,45	−1;1,29,45	3;27, 9,30	3;23,54	3;11,13 šú

Note two computing errors[2]: IX, 3 has 3;23,54 instead of 3;27,9 and X,2 has 35,7 instead of 35,6.

[2] This text has many more similar errors; cf. ACT I, p. 100.

5. System B, Columns H to M

A. Summary

As we have seen[1] in System B the mean value of column G is associated with the mean synodic month since

$$29^{d}+\mu_{G}=29^{d}+3;11,0,50^{H}=29;31,50,8,20^{d}. \tag{1}$$

Because the period of G is the period of the anomalistic month[2] we can conclude that G is based on the assumption of variable lunar velocity in combination with constant mean solar velocity, not high solar velocity as in System A.[3] A corrective column (again called J) for solar anomaly should therefore show positive and negative corrections with a mean value zero. We shall see[4] that this is indeed the case for column J which is based on a difference column H such that the values in J vary in a sinusoidal fashion around zero.

The next column (K) gives the (algebraic) sum of G and J

$$K=G+J \tag{2}$$

which serves as difference column for a column L which provides the moments of the syzygies in midnight epoch, both for conjunctions and for oppositions. That the epoch in L is fixed follows from the absence of a correction (like C′ in System A) depending on the variation of the length of daylight. We also note that always

$$K=\Delta L \tag{3}$$

without change of sign between new and full moons in contrast to the rules for M_1 and M_2 in System A (above p. 491 (4)). A reference to "before" and "after" appears, however, in the next column (M) but then not only with respect to sunset but also for sunrise.[5]

The only interesting part in this sequence of columns from G to M is, of course, the construction of a sinusoidal function J from a linear zigzag function (H) which provides the differences of J. In modern terms J could be described as a function of period P, composed of parabolic arcs whose derivatives form a linear zigzag function of period $P/2$. But such a description does not meet the essence of the problem since we are not dealing with continuous functions but with discrete values of a number theoretical function. And it is by no means trivial to construct alternating difference sequences such that the summatory function somehow shows the proper periodicity.

B. Columns H and J

The correction J for solar anomaly of the time between consecutive conjunctions or oppositions must have the same period as the solar velocity, hence

$$P_{J}=P_{A} \tag{1}$$

[1] Above p. 483 (3).
[2] Above p. 483 (2).
[3] Above p. 489.
[4] Below p. 493.
[5] Cf. below p. 497.

where P_A refers to column A which is a linear zigzag function for the monthly differences of the solar (or lunar) longitudes given in column B.[1] What precisely is to be understood as "P_J" will be discussed presently.

Some theoretical (or perhaps observational) considerations must provide the amplitude Δ_J of the effect of the variation in solar velocity. As we have seen (p. 492) the mean value of this correction J must be zero. The ephemerides show that the simplest pattern was adopted by assuming symmetry of the extrema, i.e.

$$M_J = -m_J, \qquad \mu_J = 0. \tag{2}$$

For the problem of constructing the difference column H we may consider these extrema to be known.

It is, of course, a purely mathematical device to adopt for J a variation based on a difference sequence H which forms a linear zigzag function. The problem which we must discuss here is the determination of the parameters of such a function H on the basis of the data (1) and (2).

If H is a linear zigzag function of period P_H (cf. Fig. 53[2]) the summatory function J will alternatingly be near m and M whenever H is near zero. Today we would call the value of H negative for each descending branch of J but the Babylonian norm makes H always ≥ 0 but asks for a shift from addition to subtraction when J crosses an extremum. Qualitatively one could say that $P_J = 2P_H$ but P_J cannot be defined in a simple fashion. If $\mathrm{J}(n)$ is a value of J just before M_J would be crossed the next time, $\mathrm{J}(n+1)$ the value after this, then we have according to the ordinary reflection rule[3]

$$\mathrm{J}(n) + \mathrm{J}(n+1) = 2M_J - \mathrm{H}(n+1)$$

because $\mathrm{H}(n+1)$ is the difference given by H for the transition from n to $n+1$. Since at the next maximum another difference will be prescribed by H the points for which linear interpolation would reach M_J are not equidistant. Hence there is no constant P_J definable in this direct fashion.

What leads to a workable definition is based on the same idea which we have encountered repeatedly in the planetary theory: conditions for periodic functions must be satisfied exactly only for the number period whereas true intervals may well deviate from the mean ones as long as they lead to the proper total over the number period.

Since H is a linear zigzag function with exactly defined period

$$P_H = 2M_H/d_H \tag{3}$$

also a number period Π_H and a wave number Z_H is well defined by

$$P_H = \Pi_H/Z_H \qquad (\Pi_H, Z_H) = 1. \tag{4}$$

Hence we can define a number period for J as well by requiring

$$\Pi_J = 2\Pi_H \tag{5}$$

which replaces the loose condition $P_J \approx 2P_H$.

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[1] Cf. above p. 476.

[2] The amplitudes of H and J in Fig. 53 are drawn in the correct ratio.

[3] Cf. p. 374 (3). Cf. also Fig. 55, p. 1339, inset.

We now determine the total effect of H on J over the whole number period Π_J. If $\zeta(i)$ represents the ordinates of J continued through alternating strips of width $\Delta_J = 2M_J$ (cf. Fig. 54) and if $y(i)$ is a value from H then we have because of (5)

$$\zeta(\Pi_J) - \zeta(0) = 2 \sum_{1}^{\Pi_H} y(i). \tag{6}$$

If we assume that Π_H is an even number then we know[4] that

$$\sum_{1}^{\Pi_H} y(i) = \mu_H \Pi_H = 1/2 \Delta_H \Pi_H, \tag{7}$$

hence

$$\zeta(\Pi_J) - \zeta(0) = \Delta_H \Pi_H. \tag{8}$$

The number of waves of J over a number period is given by

$$Z_J = (\zeta(\Pi_J) - \zeta(0))/2\Delta_J. \tag{9}$$

This wave number Z_J must be half the number of waves of H within Π_J, i.e. half of $2 Z_H$ because of (5), or

$$Z_J = Z_H. \tag{10}$$

Consequently, from (8), (9), and (10)

$$\zeta(\Pi_J) - \zeta(0) = \Delta_H \Pi_H = 2\Delta_J Z_J = 2\Delta_J Z_H$$

or

$$2\Delta_J = \Delta_H \Pi_H/Z_H = \Delta_H P_H. \tag{11}$$

Finally we must take into consideration the astronomical meaning of J as correction due to the solar velocity. Each wave of J must therefore correspond to a complete period of the solar velocity, i.e. to one period P_A of the linear zigzag function in column A which represents the solar velocity. Hence a number Z_J of waves within the number period Π_J of J must also exactly restore the velocity of the sun; therefore

$$\Pi_J = Z_J P_A \tag{12}$$

and with (5) and (10)

$$\Pi_J = 2\Pi_H = 2P_H Z_H = Z_J P_A$$

hence

$$P_H = 1/2 P_A \tag{13}$$

and with (11)

$$\Delta_H = 2\Delta_J/P_H = 4\Delta_J/P_A. \tag{14}$$

This shows that the amplitude of H can be found from the amplitude of J and from the period of the solar velocity, i.e. from the length of the (anomalistic) year. Finally it follows from (13) and (14) that

$$d_H = 2\Delta_H/P_H = 4\Delta_H/P_A = 16\Delta_J/P_A^2 \tag{15}$$

is obtainable through the same quantities. Hence H is fully known and computable as soon as an initial value is given.[5] What is now left is only to substitute numerical data in these general formulae.

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[4] Cf. above p. 383.

[5] It may be remarked that H=0 actually occurs in our texts; cf. ACT No. 160 obv. I, 21 = No. 161 obv. II, −3 (S.E. 124 IX).

The parameters of the columns H and J are not only deducible from ephemerides but H alone as well as H and J are tabulated for long sequences of years in auxiliary tables (ACT Nos. 160 to 174). Column H is a linear zigzag function based on the following parameters

$$m_H=0 \qquad M_H=\Delta_H=0;21 \qquad d_H=0;6,47,30 \tag{16a}$$

thus

$$P_H=\frac{16,48}{2,34}=6;11,2,34,\ldots. \tag{16b}$$

The integer units of d_H and M_H are large hours as is evident from the combination of H with J and G and finally, K, L and M. And $\Pi_H=16,48$ is an even number as we had hoped for in our computations (p. 494 (7)).

The extrema of J can be found, e.g., from the following passages in an auxiliary table[6]

4,57,30	+31, 9
1,50	+31,57,12

Hence

$$31,9,0+31,57,12=1,3,6,12=2M_J-d=2M_J-1,50,0$$

or

$$M_J=1/2\ 1,4,56,12=32,28,6.$$

Similarly near the next minimum of H

6,12,30	−31, 5,18
0,35	−31,40,18
7,22,30	−25,53,24

Hence

$$-31,40,18-25,53,24=-57,33,42=2m_J+d=2m_J-7,22,30$$

or

$$m_J=-1/2\ 1,4,56,12=-32,28,6=-M_J.$$

Consequently

$$\mu_J=0 \qquad M_J=-m_J=0;32,28,6 \qquad \Delta_J=1,4,56,12. \tag{17}$$

Beside these accurate values for M_J and m_J one also finds abbreviated extrema

$$M_J=-m_J=0;32,28, \tag{18}$$

thus ignoring the final 6 in (17). For theoretical investigations only (17) is significant.

Having found for the amplitudes of H and J the values

$$\Delta_H=0;21 \qquad \Delta_J=1;4,56,12$$

it is advisable first to determine the value P_A of the year which had been used in (14). In this way one finds

$$P_A=4\Delta_J/\Delta_H=4\cdot 1,4;56,12/21=12;22,8 \tag{19}$$

[6] ACT No. 165 obv. III and IV, 12, 13 and 18 to 20.

exactly. This shows that System B also makes use of the parameter

$$1 \text{ year} = 12;22,8 \text{ mean syn.m.} \tag{20}$$

which plays such an important role both in System A of the lunar theory[7] and in the planetary theory.[8]

Here, however, the exact arithmetical agreement with the theory ends. If we compute d_H with (15) one finds

$$d_H = 4\,\Delta_H/P_A = 4 \cdot 21/12,22;8 = 0;6,47,28,26,\dots \tag{21}$$

instead of 0;6,47,30 (cf. (16a)). Furthermore we should have with (13)

$$P_H = 1/2\, P_A = 6;11,4 \tag{22}$$

instead of 6;11,2,34, ... in (16b), the cause being, of course, the rounding in the value of d in (16b) instead of using the relation (13).[9] Or to formulate it differently: if one accepts (16b) one obtains on the basis of (13)

$$P_A = 2P_H \approx 12;22,5,8,\dots. \tag{23}$$

The fact that the value for Δ_J is based on (20) whereas $2P_H \approx 12;22,5$ is theoretically unpleasant since a phase difference of about $0;35^h$ each year will slowly bring H and J out of phase. To this internal, however small, discrepancy must be added that the solar velocity in column A introduced a third period, namely,[10]

$$P_A = 12;22,8,53,20 \tag{24}$$

the cause of deviation being again the use of a rounded difference (0;18). Hence the solar velocity A will also eventually step out of phase with H and J which were constructed in order to account for the influence of A. In practice these differences are without interest for the results of the ephemerides, but theoretically System B is much less precise here than System A in spite of the fact that one might think that zigzag functions and sinusoidal sequences represent the actual phenomena better than step functions.

How sensitive the function J is with respect to a displacement of the zeros of H in relation to the extrema of J is shown by an error which occurred in a text (ACT No. 126), where the ancient scribe mistook (in line 37) a negative value of J for a positive one and then continued from it for the rest of the text (cf. Fig. 55[11]), completely destroying the sinusoidal shape of J.

C. Column M

From Column L one knows the moments of the syzygies in terms of large hours after midnight. In column M the same moments are given with respect to

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[7] Above p. 378 (15b).

[8] Above p. 396 (5b).

[9] If one uses the rounded extrema (18) one finds instead of (19)

$$P_A = 4 \cdot 1,4;56/21 \approx 12;22,5,43.$$

[10] Cf. below p. 533 (2). We ignore here the period $P_A = 12;22,13,20$ of the abbreviated column A.

[11] H and J are drawn in the same scale, except for the inset which shows the situation near M_J in greater detail.

the nearest sunset or sunrise, reckoned forwards or backwards with noon and midnight as boundaries.

In order to transform L into M one needs the half duration of night, a quantity which is tabulated as column D, obtainable from

$$D = 1/2(6^H - C) \tag{1}$$

C being the length of daylight. With C and D known the transformation can be effected by the following formulae[1]

$$\left.\begin{array}{ll} \text{before sunrise (kur } \textit{šamáš}\text{):} & M = D - L \\ \text{after sunrise (me nim):} & M = L - D \\ \text{before sunset (šú } \textit{šamáš}\text{):} & M = C + D - L \\ \text{after sunset (ge}_6\text{ du):} & M = L - (C + D) \end{array}\right. \tag{2}$$

depending on the quarter of the day in which the syzygy is located. The results are usually given to two sexagesimal digits only, i.e. to integer time degree. The numbers are often slightly rounded which has no adverse effect since a rounding of L or M in one line does not influence the next line.

§ 4. The "Saros" and Column Φ

At the time of the publication of the ACT the astronomical significance of column Φ was still a complete mystery. Two years later, however, A. Sachs discovered fragments of a tablet on which he read the sentence[1] "17,46,40 (is) the addition and subtraction for 18 years." Since "18 years" is the standard term for the famous eclipse cycle of 223 lunations, known as the "*Saros*",[2] it was now clear that the basic parameter[3]

$$\varphi = 0;0,17,46,40 = \frac{3}{28} d_\Phi = \varepsilon d_\Phi \tag{1a}$$

of column Φ originated in the theory of eclipses, a fact no one who had studied the Babylonian lunar theory had ever suspected. But now it was easy to verify the statement in the new text[4] (henceforth called "Saros Text" or "Text S"). If one asks how many (n) months separate a value $\Phi(n)$ from an initial value $\Phi(0)$ if they differ, as the text says, by the amount $-\varphi$ (the negative sign is chosen

[1] The terminology found in the ephemerides is usually abridged to kur, nim, šú, du, respectively, cf. above p. 490 (1) and ACT, p. 80.

[1] Section 12 in the now published text; cf. note 4.

[2] This terminology is not of ancient origin; cf. Neugebauer [1937, 3], p. 241 to 245 and [1938, 1], p. 407 to 410 where I have shown that the interpretation of the word "Saros" as a name for an eclipse cycle originated first with Halley in 1691. The term became common among astronomers probably through Newcomb [1897], p. 7; to Schram this usage seemed to be new, or at least not well founded (Schram [1881], p. 182 note).

[3] Cf. above p. 486 (2).

[4] Two larger pieces, BM 36705 and 36725, were joined by A. Sachs and form the basis of the publication Neugebauer [1957]. Several years later A. Aaboe joined another small fragment, BM 37484, to the main parts; cf. Aaboe [1968], p. 35–38.

only in order to obtain a positive value for n), i.e. if

$$\Phi(n)-\Phi(0)=-\varphi, \tag{1b}$$

then one finds (cf. Fig. 56) from

$$n d_\Phi+\varphi=\alpha\cdot 2\Delta \tag{2a}$$

for the integers n and α

$$\begin{aligned} n &\equiv 3{,}43 \quad \text{mod. } 1{,}44{,}7 \\ \alpha &\equiv 16 \quad \text{mod. } 7{,}28 \end{aligned} \tag{2b}$$

where α counts the number of complete periods P_Φ needed to reach $\Phi(n)+\varphi$ from $\Phi(0)$. The number

$$n=S=3{,}43=233 \text{ mean syn.m.} \tag{3}$$

is indeed the length of the "Saros" cycle.

It follows from (2a) and (1) that

$$n-\varepsilon=\alpha P_\Phi$$

and hence with (2b)

$$S-16P_\Phi=\varepsilon.$$

Since $\hat{G}$ and Φ have identical periods we also have

$$S-16P_{\hat{G}}=\varepsilon$$

or

$$S d_{\hat{G}}-32\Delta_{\hat{G}}=\hat{G}(S)-\hat{G}(0)=\varepsilon d_{\hat{G}}.$$

But we have seen before[5] that

$$\varepsilon d_{\hat{G}}=d_\Phi$$

thus it results that

$$d_\Phi=\hat{G}(S)-\hat{G}(0). \tag{4}$$

This shows that Φ represents the periodic variation in the length of the lunations, one Saros apart, a variation which is caused by the difference in lunar velocity at the beginning and at the end of one Saros.

A slight restriction should be made to this statement insofar as the above equations are only correct for $\hat{G}$ whereas the actual duration of the lunations depends on G which deviates near the extrema from $\hat{G}$. The basic idea, however, which undoubtedly underlies the construction of Φ is nevertheless correctly expressed by (4).

From (4) it is also clear that the units of Φ are large hours, in particular that

$$d_\Phi=0;2,45,55,33,20^{H}\approx 0;11^{h}.$$

This justifies the norm adopted before (p. 487).

The relation (4) can also be written in a slightly different form.[6] Let $n=3{,}43$ again be the number of months in one Saros; then (4) remains valid when we add

[5] Above p. 487 (6).

[6] Due to van der Waerden, AA, p. 149.

and subtract 3,42 months:

$$d_\Phi = \hat{G}(1) + \hat{G}(2) + \cdots + \hat{G}(n) + n \cdot 29$$
$$- \hat{G}(0) - \hat{G}(1) - \cdots - \hat{G}(n-1) - n \cdot 29.$$

The first line is the length of the Saros $S(1)$ that begins at the end of month 1 whereas the second line corresponds to the Saros $S(0)$ that begins one month earlier. Thus

$$d_\Phi = S(1) - S(0) \tag{5}$$

or, in other words: the difference of Φ is the difference between two consecutive Saroi.

1. The Functions Φ^* and F*

It is evident that the column F which gives the lunar velocity in degrees per day and which has a period of about 13;56 mean synodic months[1] can only be the tabulation function of a true function F* which has the anomalistic month as period, about 27 1/2 days in length. Since F and Φ have exactly identical periods one can expect that Φ also is the tabulation function of a true function Φ^* whose period must then be the anomalistic month.

This expectation finds its confirmation within our material. The ephemerides ACT No. 80 and 81 (for S.E. 178) proceed in steps of single days for several months. The first numerical column is clearly a function Φ^* since it is a linear zigzag function with exactly the same extrema and amplitude

$$m = 1,57,47,57,46,40 \qquad M = 2,17,4,48,53,20 \qquad \Delta = 19,16,51,6,40 \tag{1}$$

which we know from Φ.

In order to prove that Φ is exactly the tabulation function of this function Φ^* we only have to show that the new difference

$$d^* = 1,22,39,15,33;20\,^{2}$$

produces the difference of Φ

$$d = 2,45,55,33,20$$

or vice versa. Now it follows from p. 375 (6b) that

$$\frac{2\Delta}{d} = P = \frac{p}{1-p} = \frac{2\Delta}{d^*} \cdot \frac{d^*}{d^* - 2\Delta}$$

hence

$$d^* = 2\Delta + d. \tag{4}$$

In this formula d^* and d are measured in the same units, i.e. with respect to mean synodic months. But Φ^* progresses not in months but in "days," hence d^* must also be referred to these smaller units. Assuming that these "days" are actually

[1] Cf. above p. 478.

[2] Φ^* is given to six sexagesimal places in contrast to the five digits of Φ. In order to compare the two functions we norm the last digit of d^* as minutes.

tithis we have to replace (4) by

$$d^* = \frac{2\Delta + d}{30}. \tag{5}$$

And indeed

$$d^* = 1,22,39,15,33;20 = 0;2(2 \cdot 19,16,51,6,40 + 2,45,55,33,20)$$

which shows not only that Φ^* is the true function underlying Φ but also that the steps of Φ^* are tithis. Hence we find as period

$$P_{\Phi^*} = \frac{2\Delta}{d^*} = \frac{19,16,51,6,40}{41,19,37,46;40} = \frac{10,24,42}{22,19} = 27 + \frac{22,9}{22,19} \approx 27;59,33,\ldots^{\tau} \tag{6}$$

which represents the length of the anomalistic month in tithis. Since

$$\text{anom.m.}: \text{syn.m.} = P_{\Phi^*}: 30 = \frac{10,24,42}{30 \cdot 22,19} = \frac{1,44,7}{1,51,35} \tag{7}$$

we find here again the characteristic ratio for the lunar anomaly in System A as before, p. 482 (4).

There remains the problem of initial values. The values of Φ^* in the texts ACT Nos. 80 and 81 show that only for the 28th day (S.E. 178 month VI) can Φ^* be connected with Φ_1 of the ordinary ephemerides, while the value of Φ^* for the 13th day is connectible with Φ_2. These two dates, 15 tithis apart, are therefore the values which appear in the tabulation function for new moons (conjunctions) and full moons, respectively, exactly corresponding to the phase difference found for Φ_1 and Φ_2 in the ephemerides.[3] Hence all values of Φ^* and of Φ_1 and Φ_2 will be completely known as soon as one single value of Φ^* is given.[4]

Having shown that Φ^* in ACT Nos. 80 and 81 progresses in steps of single tithis we know that the same must hold for all subsequent columns of these tablets, a fact which can be confirmed by direct computation.[5] Unfortunately where we should find F* only a few initial digits are preserved, much too few for a determination of the parameters.[6] But we can reconstruct the parameters of F* by using (5) with the known accurate values for Δ and d of F[7] and obtain in this way

$$d_{F^*} = 0;2(2 \cdot 4;52,49,41,15 + 0;42) = 0;20,55,18,45 \tag{8}$$

for the difference per tithi of the true function F* to which F belongs as the tabulation function. Of course the resulting period P_{F^*} is identical with P_{Φ^*} in (6). It follows furthermore from the coincidence of the phases of F and Φ that the phases of F* and Φ^* must also coincide. It is easy to show that this condition can indeed be satisfied on the basis of the numerical values of Φ^* in ACT Nos. 80 and 81.

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[3] Cf. above p. 485.

[4] The information for a "single value" must include, of course, the direction of the branch (increasing or decreasing) to which this value belongs.

[5] Cf. ACT I, p. 119.

[6] As far as can be seen there also seem to be errors obscuring this column. The correct value in No. 81 obv. VI, 6 would be 11,10,38,26,15 but not 11,12[.

[7] Above p. 478 (4).

The units in which the lunar velocity is measured must be the same in F and F*. The common mean value $\mu = 13;30,\ldots$ is rather high for a motion in degrees per day but it would be worse with degrees per tithi. Hence, in spite of the tabulation interval being 1^{τ} the units of F* must be degrees per day.

The Saros Text also contains information about the length of the anomalistic month expressed in days. In Sect. 2 one reads "from high velocity to low lunar velocity ... 1,22,39,49,30 (is) its duration." Obviously this number must be interpreted as $1,22;\ldots^{H} \approx 13;\ldots^{d}$ hence the whole length of the anomalistic month

$$1 \text{ anom.m.} = 1/3\ 1,22;39,49,30 = 27;33,16,30^{d}. \tag{9}$$

The modern value

$$27.5546^{d} \approx 27;33,16,23^{d}$$

shows that (9) is much better than the round value $27;33,20^{d}$ found in System B.[8]

We can consider (9) to be the parameter accepted as accurate for the theory in System A. This can be confirmed by another statement in text S. As we know[9] System A assumes that 1,44,7 syn.m. = 1,51,35 anom.m. Thus from (9)

$$1 \text{ syn.m.} = \frac{1,51,35}{1,44,7} \cdot 27;33,16,30 = 29;31,50,19,11,6,45,\ldots^{d}$$

while our text S says[10] $\ldots 11,4,56^{d}$. Conversely: this value given by the Saros text would require instead of (9) only $27;33,16,29,59,59,22,\ldots^{d}$. It is therefore clear that we may consider

$$1 \text{ syn.m.} = 29;31,50,19,11^{d} \tag{10}$$

as the value which, together with (9), satisfies the fundamental ratio of synodic to anomalistic month.

The identity of the periods of F and Φ (and F* and Φ^*) implies that the differences have a fixed ratio. This ratio is explicitly mentioned in the procedure text ACT No. 200 Sect. 5 in the form

$$d_{F} = 15;11,15\, d_{\Phi}. \tag{11}$$

This relation can also be used to transform any value of F into the corresponding value of Φ and vice-versa. For example the value

$$M'_{\Phi} = 2;13,20 \tag{12a}$$

is 0;3,44,48,53,20 lower than the maximum M_{Φ}. If we then subtract

$$0;3,44,48,53,20 \cdot 15;11,15 = 0;56,54,22,30$$

from the maximum $M_{F} = 15;56,54,22,30$, above p. 478 (4), we obtain

$$M'_{F} = 15 \tag{12b}$$

as the value corresponding to (12a). Similarly one finds that

$$m'_{\Phi} = 1;58,31,6,40 \tag{13a}$$

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[8] Above p. 480; cf. also p. 479.

[9] Above p. 478 (5).

[10] Sect. 2, line 15.

and

$$m'_{\mathrm{F}} = 11;15 \tag{13b}$$

are equivalent values. These two pairs, (12) and (13), are of interest because F and Φ are truncated at these values, that is to say values of either one of the zigzag functions F and Φ above M' and below m' are replaced by these constant values. We shall come back to these "truncations" presently[11].

2. The Saros

An eclipse cycle requires not only that it comprise a common period for lunations and nodal months but lunar as well as solar anomaly should be restored as nearly as possible in order to obtain eclipses of equal circumstances. All these conditions are remarkably well satisfied in the "Saros" cycle which contains

$$223 \text{ syn.m.} \approx 242 \text{ drac.m.} \approx 239 \text{ anom.m.} \approx 241 \text{ sid.m.} \approx 18 \text{ years.} \tag{1}$$

We shall here and in subsequent sections demonstrate that all these relations were known to the Babylonian astronomers. It should be noted, however, from the outset that these relations are not all of equal accuracy. We therefore define an interval S which we call "the Saros interval" by exactly

$$S = 3{,}43 \text{ mean syn.m.} \tag{2}$$

Let us first specify what it means to say that the Saros is about 18 years. If one year contains approximately $365;15^{\mathrm{d}}$ and 12;22,8 months then

$$3{,}43/12;22{,}8 \approx 18;1{,}45^{\mathrm{y}} \approx 18^{\mathrm{y}} + 10;40^{\mathrm{d}}.$$

Hence

$$S = 18^{\mathrm{y}} + 11^{\mathrm{d}} \tag{3}$$

must be the approximate length of one Saros.

By assuming a definite model for the distribution of the syzygies the estimate (3) can be supplemented by exact data for the changes of longitudes of the syzygies after one Saros.[1] Operating with System A we have the synodic arcs[2]

$$w_1 = 30° \qquad w_2 = 28;7{,}30°$$

respectively, and the period

$$P = 12;22{,}8 = \frac{46{,}23}{3{,}45} = \frac{\Pi}{Z}$$

which gives for the basic intervals[3]

$$\delta_1 = \frac{w_1}{Z} = \frac{30}{3{,}45} = 0;8° \qquad \delta_2 = \frac{w_2}{Z} = \frac{28;7{,}30}{3{,}45} = 0;7{,}30° \tag{4}$$

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[11] Below p. 506.

[1] The following elegant derivation is due to Aaboe-Henderson [1975].

[2] Cf. above p. 377f. (10), (13) and (14).

[3] Cf. for this concept above p. 433f.

respectively. During one Saros of 3,43 months the conjunctions move 1,21 steps ahead because

$$3{,}43 \cdot Z = 3{,}43 \cdot 3{,}45 = 13{,}56{,}15 \equiv 1{,}21 \bmod. \Pi. \tag{5}$$

Therefore the conjunction points at the end of a Saros will be

$$\begin{array}{ll} 1{,}21 \cdot \delta_1 = 1{,}21 \cdot 0;8 = 10;48^\circ & \text{on the fast arc} \\ 1{,}21 \cdot \delta_2 = 1{,}21 \cdot 0;7{,}30 = 10;7{,}30^\circ & \text{on the slow arc}^4 \end{array} \tag{6}$$

ahead of the position at the beginning of the Saros (assuming that both ends belong to the same velocity zone). This, then, is the equivalent of (3).

For the anomalistic months we have the relation [5]

$$P_F = P_\Phi = \frac{1{,}44{,}7}{7{,}28}$$

or

$$1{,}44{,}7 \text{ syn.m.} = 1{,}51{,}35 \text{ anom.m.} \tag{7}$$

From this, theoretically exact equation, one derives the approximate relation contained in (1):

$$S = 3{,}43 \text{ syn.m.} \approx 3{,}59 \text{ anom.m.} \tag{8}$$

by dividing both sides in (7) by 28:

$$(3{,}43 + 3/28) \text{ syn.m.} = (3{,}59 + 3/28) \text{ anom.m.}$$

Thus (8) results from (7) by committing an error of

$$\delta = 3/28 \,(1 \text{ syn.m.} - 1 \text{ anom.m.})$$

thus approximately

$$\delta \approx 3/28 \cdot 2^d \approx 0;13^d \approx 5^h.$$

Again, Babylonian parameters allow us to sharpen this estimate. We know from (10), p. 501 that

$$3{,}45 \cdot 29;31{,}50{,}19{,}11 = 1{,}49{,}45;20{,}1{,}17{,}53^d \tag{9}$$

and from (9) on the same page that

$$3{,}59 \cdot 27;33{,}16{,}20^d = 1{,}49{,}45;32{,}43{,}20^d \tag{10}$$

hence

$$3{,}43 \text{ syn.m.} - 3{,}59 \text{ anom.m.} \approx -0;12{,}42^d \approx -5;5^H. \tag{11}$$

This shows the excellent agreement between synodic and anomalistic periods within one Saros. The values (9) and (10) also explain why in modern literature "Saros" is often described as an interval of

$$S \approx 1{,}49{,}45^d = 6585^d. \tag{12}$$

Knowing now the significance of the parameter $\varepsilon = 3/28$ we can also motivate another previously established relation. As we have seen [6] the function G has its

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[4] This parameter is actually mentioned in the procedure text ACT No. 207, Sect. 7 (p. 250).

[5] Above p. 484 (2a).

[6] Above p. 488.

extrema not, as one would expect, 1/2 interval behind the opposite extrema of Φ but the phase difference amounts to exactly $1/2\,(1-\varepsilon)$. In Fig. 57 the dotted line represents a position of G with a phase difference of 1/2 whereas the actual G (solid line) is $\varepsilon/2$ nearer to Φ. In the first case a minimum of G would have a distance $a+1/2$ from a value $\Phi(0)$ while this distance would increase to $a+\varepsilon+1/2$ from the value $\Phi(S)$, one Saros later. From the preceding discussion it has become clear that this increment ε is inherent in the definition of the "Saros." But by moving G nearer to Φ the distance at $\Phi(0)$ is increased to $a+1/2(1-\varepsilon)$ but at the end lowered to $a+1/2(1+\varepsilon)$ near $\Phi(S)$. Hence the deviation from the ideal phase difference 1/2 is, as it were, split between the beginning and end of one Saros.

We still have left in (1) the relations for the draconitic and for the sidereal months. Since the return of the moon to the nodes is the crucial element for an eclipse cycle, which is the main concern of the "Saros", we shall discuss this aspect in greater detail in the next section.[7]

With respect to the sidereal months we have only a sentence in a procedure text which is associated with System B. There we find the statement[8] "5,27;51,20 day is the number (of days in) 12 months of the moon, returning to its longitude." Consequently

$$1 \text{ sidereal month} = 5{,}27;51{,}20/12 = 27;19{,}16{,}40^{d} \tag{13}$$

and

$$4{,}1 \cdot 27;19{,}16{,}40 = 1{,}49{,}44;25{,}56{,}40^{d} \tag{14}$$

which is only about one day shorter than the number of days in one Saros.[9] This completes our inquiry into the precise meaning of the relations (1).

Corollaries. 1. Let x be the number of full months in one Saros, y the number of hollow months, i.e.

$$x+y=3{,}43.$$

From (10) we know that the corresponding number of days must be

$$30x+29y=1{,}49{,}45.$$

From these two equations it follows that "18 years" (i.e. one Saros) contain

$$x=1{,}58 \text{ full months}, \qquad y=1{,}45 \text{ hollow months} \tag{15}$$

as stated in procedure texts.[10]

This implies that about $118/223 \approx 53\,\%$ of the months in a true lunar calendar must be full months. This ratio is, e.g., very nearly satisfied by the Islamic calendar.[11]

2. It is a well-known rule of thumb in antiquity that eclipses occur either at 6-months distance or occasionally at 5-month intervals.[12] If s represents the number of 6-month intervals in one Saros, f the number of 5-month intervals,

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[7] Cf. below II B 5, 3.
[8] Cf. ACT No. 210, Sect. 3, line 8; the numbers are in part restored.
[9] Cf. above p. 503 (9).
[10] Cf. ACT No. 210, Sect. 3, line 18.
[11] Cf., e.g., Ginzel, Hdb. I, p. 254.
[12] Cf. below II B 7; also above p. 129.

then one has to solve the diophantine equation

$$6s + 5f = 3{,}43$$

which yields the result that

$$s = 33 \text{ six-month intervals}, \quad f = 5 \text{ five-month intervals} \tag{16}$$

are contained in one Saros.[13]

3. Φ, Friends and Relations

Since the publication of the ACT in 1955 a whole family of functions has come to light, all more or less directly related to Φ and hence to the "Saros." The identification of these functions and the elucidation of their interrelations is mainly due to the work of A. Aaboe, published in [1968], [1969], and [1971], and largely used in what follows. As usual this increase in our source material gave rise to many new questions, foremost the problem of motivating the construction of rather intricate sets of numerical tables outside the familiar framework of ephemerides and eclipse tables. It is precisely in the absence of an evident connection of the new material with the previously known procedures that one did not suspect the far reaching influence of the function Φ on the formation of auxiliary tables. This fact is a renewed reminder that we are far away from a real "history" of Late Babylonian astronomy.

A. Summary

The fundamental relation which connects the function Φ with the eclipse cycle of 223 months rests on the definition[1]

$$1 \text{ Saros} = 223 \text{ syn.m.} = 6585^d + \Phi^H \tag{1}$$

counted from the end of the month for which Φ is tabulated. From this relation is derived by a clever mathematical device to be described presently, a set of three functions, G, Λ, and W, such that

$$\begin{aligned} 1 \text{ synodic month} &= 29^d + G^H \\ 12 \text{ synodic months} &= 354^d + \Lambda^H \\ 6 \text{ synodic months} &= 177^d + W^H. \end{aligned} \tag{2}$$

These relations are derived under the assumption that the solar velocity has the round value $30^{\circ/m}$, such that (2) is valid only on the "fast arc" of the ecliptic. On the "slow arc" corrective terms must be applied of which only J was known from the ephemerides.[2] Thus we now have to use three terms on the slow arc, all measured in "large hours":

$$G + J, \quad \Lambda + Y, \quad W + Z. \tag{3}$$

[13] This statement is found in a text to be published by Aaboe-Henderson-Neugebauer-Sachs [1975].

[1] This relation was first suggested by van der Waerden (AA, p. 150, BA, p. 228); cf. also Aaboe [1968], p. 10.

[2] Cf. above II B 3, 3.

Both G and Λ were further modified (to G′ and Λ') by introducing constants of not yet fully understood purpose.[3] Finally a function X is associated with Λ such that

$$354^d + \Lambda + Y + X = 1 \text{ year.} \tag{4}$$

Hence X plays the role of an "epact", i.e. of the difference between 12 lunar months and the solar year.[4]

Of all these functions only Φ, G, and J are regularly represented in the ephemerides, Φ as a basis for the computation of G, and G+J in a vital role for the determination of the intervals between consecutive syzygies.[5] All three functions Λ, W, and G appear in ephemeris-like texts[6] which proceed by intervals that cover in the shortest possible fashion 223 months:

$$223 = 18 \cdot 12 + 6 + 1,$$

i.e. schematically

$$1 \text{ Saros} = 18 \cdot \Lambda + W + G(+6585^d).$$

The values of these three functions appear in this order (thus covering 20 lines for each Saros) in one column. I do not understand the purpose of such texts. To be useful for eclipses would require intervals of 6 or 5 months but not a sequence of 18 12-month intervals, followed by 6 months and 1 month. Hence these texts are not subsidiary to eclipse tables in spite of their obvious connection with the Saros.

B. Mathematical Methodology

All functions of the Φ-family are held together by an interesting methodological principle which has become known only from this group of tables whose major outward characteristic is the use of "truncated" zigzag function, i.e. zigzag functions which are not used up to their maximum M, or down to their minimum m, but go instead only to some constant level $M' < M$ and $m' > m$. We shall call M' and m' the "effective" extrema.[7]

Of greater importance, however, is a general principle of producing new functions by combining two versions of the same function, a given interval apart. Take two samples, R and S, of the same zigzag function, such that S runs c intervals[8] to the right of R (cf. Fig. 58, p. 1341) and let U be a function whose differences ΔU are always R − S. If d is the difference of the function R (and thus also of S) we see that $\Delta U = +dc$ where both R and S increase, $-dc$ where both decrease, while for the intervals of length c between two consecutive minima or maxima of R and S the function ΔU increases or decreases with the constant

[3] Cf. Aaboe [1971], p. 10f., p. 11f., p. 17, Fig. 3; [1969], p. 9, Fig. 1.

[4] Aaboe [1968], p. 28f.; refined [1969], p. 11.

[5] An isolated case is an ephemeris for the years Philip Arrhidaeus 4 to 7 (−318 to −315) in which Λ and W follow the columns K and M. Cf. Aaboe [1969], p. 19.

[6] ACT No. 55, restored by Aaboe [1971], p. 22. Similar texts are ACT Nos. 75 and 76. ACT No. 55 concerns S.E. 180 to 202 (= −131 to −109).

[7] It was van der Waerden who in AA, p. 152f. suggested that the computation of G should be based an a truncated function Φ, but without giving any details. The actual existence of truncated functions F and Φ was established, and their working explained, by Aaboe [1968] (p. 8, Fig. 2 and p. 18, Fig. 4).

[8] We need not specify the units of these intervals in the present context.

difference $2d$. Consequently U will be part of a linear zigzag function $\hat{\mathrm{U}}$ with the same period as R and S and with the difference dc, except near the extrema where ΔU changes linearly, hence U follows a "parabolic" curve (i.e. $\Delta\Delta$U $=$const.).

We can now lengthen the zone in which U is rounded near its extrema by introducing an "effective" maximum $M'<M$ (all considerations hold correspondingly for $m'>m$ where $m'-m$ do not need to be the same as $M-M'$). Let us assume that the translation c of S with respect to R is $>\frac{2}{d}(M-M')$. Then we have a situation as shown in Fig. 59, p. 1341 in which the transition of ΔU from $+dc$ to $-dc$ takes place with the slope d in two intervals of length $\frac{2}{d}(M-M')$ and with slope $2d$ in a middle interval of length $c-\frac{2}{d}(M-M')$.

If $c=\frac{2}{d}(M-M')$ the middle interval vanishes and the whole transition difference is d for an interval of length $2c=\frac{4}{d}(M-M')$. Of special interest is the case where $c<\frac{2}{d}(M-M')$. Then (cf. Fig. 60) the transition of ΔU again covers an interval of length $c+\frac{2}{d}(M-M')$ but $\Delta\mathrm{U}=0$ in a middle section of length $\frac{2}{d}(M-M')-c$. In this case the function U has a constant extremum, connected by two parabolic arcs with the linear sections of U. It is this situation which prevails in the relation between G and Φ.[9]

It is clear that the truncation of Φ (or of the equivalent F) is not based on some observations concerning the effective extrema of these functions. Every function U whose differences ΔU are the differences between R and S must show a rounded transition near the extrema of, at least, width c (cf. Fig. 58). What the truncation of R and S produces is only a lengthening and lowering of the arched sections, i.e. a smoother transition from the increasing to the decreasing branch of $\hat{\mathrm{U}}$.

The numerical choice of M' or m' is certainly dictated much more by computational expediency than by the desire to obtain a specific maximum or minimum of U.[10]

One might think that the choice of effective extrema was also influenced by the possibility of introducing an "effective" mean value μ', correcting the ordinary mean value μ. The numerical data show, however, that this is not the case for Φ and F since μ' is only little lower than μ (which seems too high[11]).

C. Numerical Details

As we have seen[12] the astronomical meaning of Φ singles out two parameters, φ and ε, which we norm as follows

$$\varphi=0;0,17,46,40^{\mathrm{H}}, \quad \varepsilon=3/28 \quad \text{or} \quad \bar{\varepsilon}=9;20. \tag{1}$$

[9] Cf. Fig. 50, p. 1336 and Fig. 61, p. 1342.

[10] This is evident in the numerical values used for M' and m' of Φ; cf. below p. 508 (2).

[11] Cf. above p. 478f.

[12] Above II B 4.

Measured in units of φ the truncation parameters of Φ[13] become simple numbers:

$$M' = 7{,}30\,\varphi, \qquad m' = 6{,}40\,\varphi, \qquad \Delta' = 50\,\varphi. \tag{2}$$

Since the period of Φ is fixed and

$$d_\Phi = \bar{\varepsilon} \cdot \varphi \tag{3}$$

the amplitude of the non-truncated function has to absorb the unavoidable fractions:

$$\Delta_\Phi = 1{,}5;4{,}22{,}30\,\varphi \tag{4}$$

and

$$M - M' = 12;38{,}45\,\varphi, \qquad m' - m = 2;25{,}37{,}30\,\varphi. \tag{5}$$

We now apply the method described in the preceding section to produce the difference column for the functions G and Λ which represent the variation in length of a one-month and a 12-month interval,[14] respectively.

1. The Table for G. Since G concerns monthly intervals we should make $c = 1$. Hence the two functions R and S would show, in their parallel branches, the difference $\pm d_\Phi$ and this would be the difference for $\hat{\mathrm{U}}$ as well. We know, however, that the astronomical relationship between Φ and $\hat{\mathrm{G}}$ requires[15] $d_{\hat{\mathrm{G}}} = \bar{\varepsilon}\, d_\Phi$. In order to make $\hat{\mathrm{G}}$ increase $\bar{\varepsilon}$-times faster than Φ we tabulate Φ at correspondingly shorter intervals, $\varepsilon = 3/28$ syn.m. in length. Because of (3) the increment of Φ will then be

$$\varepsilon\, d_\Phi = \varphi = 0;0{,}17{,}46{,}40. \tag{6}$$

Furthermore we know that G should increase as Φ decreases. Hence we put the minimum of G where R and S have their maxima. As the minimum of G was chosen

$$\mathrm{G}_{\min} = 2;40^{\mathrm{H}} = 9{,}0\,\varphi. \tag{7}$$

A truncation of R and S will produce (cf. Fig. 60) a whole interval in which $\Delta\mathrm{G} = 0$, thus $\mathrm{G} = \mathrm{G}_{\min}$.

Our goal is the construction of a table, based on the these assumptions, such that we can find for a given value of Φ the corresponding value of G, if necessary, by linear interpolation. As we know from the procedure texts we may expect that only one copy of Φ is actually listed as independent variable,[16] although in fact two copies, R and S, of Φ are needed to produce

$$\Delta\mathrm{G} = \mathrm{R} - \mathrm{S}. \tag{8}$$

Here we have assumed that R and S are two identical copies of the same linear zigzag function Φ. In fact, however, we shall demonstrate that the table for G is based on a very elegant modification of the basic idea which we have described in the preceding section. The essence of this modification has its origin in the fact that the period of Φ is not an integer multiple of our tabulation interval ε but

$$P_\Phi = \frac{1{,}44{,}7}{7{,}28}\ \text{syn.m.} = 130;8{,}45\ \varepsilon \qquad (\varepsilon = 1/9;20\ \text{syn.m.}). \tag{9}$$

[13] Cf. above p. 501 (12a) and (13a).

[14] Cf. above p. 505 (2).

[15] Cf. above p. 487 (6).

[16] E.g. ACT, p. 60.

Obviously one would like to have a table which is strictly periodic for the values of the argument. To achieve this exactly would mean the use of the number period Π of Φ, hence a table with thousands of entries. But fortunately (9) tells us that P_Φ is almost an integer multiple of ε, the excess being only 0;8,45 ε. It is exactly this amount which is eliminated in the actual construction of the $\Phi \to$ G table. But as a consequence of this modification neither R nor S are exact replicas of Φ, nor are R and S identical. What remains valid is only the relation (8).

Table 14

	R	S	ΔG	G	
1.	2;13, 2,13,20^{H}	2;13,20	−0;0,17,46,40^{H}	2;40^{H}	1.
2.	2;13,20	2;13,20	0	2;40	2.
	2;13,20	2;13,20	0	2;40	
	2;13,20	2;13, 2,13,20	0;0,17,46,40	2;40,17,46,40	
20.	2;13,20	2;12,44,26,40	0;0,35,33,20	2;40,53,20	20.
	2;13,20	2;12,26,40	0;0,53,20	2;41,46,40	
	2;13,20	2;12, 8,53,20	0;1,11, 6,40	2;42,57,46,40	
	2;13,20	2;11,51, 6,40	0;1,28,53,20	2;44,26,40	
	2;13,20	2;11,33,20	0;1,46,40	2;46,13,20	
25.	2;13,20	2;11,15,33,20	0;2, 4,26,40	2;48,17,46,40	25.
	2;13,20	2;10,57,46,40	0;2,22,13,20	2;50,40	
	2;13,20	2;10,40	0;2,40	2;53,20	
	2;13, 8, 8,53,20	2;10,22,13,20	0;2,45,55,33,20	2;56, 5,55,33,20	
	2;12,50,22,13,20	2;10, 4,26,40	0;2,45,55,33,20	2;58,51,51, 6,40	
	2; 1,34,48,53,20	1;58,48,53,20	0;2,45,55,33,20	4;43,57, 2,13,20	
	2; 1,17, 2,13,20	1;58,31, 6,40	0;2,45,55,33,20	4;46,42,57,46,40	
	2; 0,59,15,33,20	1;58,31, 6,40	0;2,28, 8,53,20	4;49,11, 6,40	
70.	2; 0,41,28,53,20	1;58,31, 6,40	0;2,10,22,13,20	4;51,21,28,53,20	70.
	2; 0,23,42,13,20	1;58,31, 6,40	0;1,52,35,33,20	4;53,14, 4,26,40	
	2; 0, 5,55,33,20	1;58,31, 6,40	0;1,34,48,53,20	4;54,48,53,20	
	1;59,48, 8,53,20	1;58,37, 2,13,20	0;1,11, 6,40	4;56	
	1;59,30,22,13,20	1;58,54,48,53,20	0;0,35,33,20	4;56,35,33,20	
75.	1;59,12,35,33,20	1;59,12,35,33,20	0	4;56,35,33,20	75.
	1;58,54,48,53,20	1;59,30,22,13,20	−0;0,35,33,20	4;56	
	1;58,37, 2,13,20	1;59,48, 8,53,20	−0;1,11, 6,40	4;54,48,53,20	
	1;58,31, 6,40	2; 0, 5,55,33,20	−0;1,34,48,53,20	4;53,14, 4,26,40	
	1;58,31, 6,40	2; 0,23,42,13,20	−0;1,52,35,33,20	4;51,21,28,53,20	
80.	1;58,31, 6,40	2; 0,41,28,53,20	−0;2,10,22,13,20	4;49,11, 6,40	80.
	1;58,31, 6,40	2; 0,59,15,33,20	−0;2,28, 8,53,20	4;46,42,57,46,40	
	1;58,31, 6,40	2; 1,17, 2,13,20	−0;2,45,55,33,20	4;43,57, 2,13,20	
	1;58,48,53,20	2; 1,34,48,53,20	−0;2,45,55,33,20	4;41,11, 6,40	
	2;10, 4,26,40	2;12,50,22,13,20	−0;2,45,55,33,20	2;56, 5,55,33,20	
	2;10,22,13,20	2;13, 8, 8,53,20	−0;2,45,55,33,20	2;53,20	
	2;10,40	2;13,20	−0;2,40	2;50,40	
	2;10,57,46,40	2;13,20	−0;2,22,13,20	2;48,17,46,40	
125.	2;11,15,33,20	2;13,20	−0;2, 4,26,40	2;46,13,20	125.
	2;11,33,20	2;13,20	−0;1,46,40	2;44,26,40	
	2;11,51, 6,40	2;13,20	−0;1,28,53,20	2;42,57,46,40	
	2;12, 8,53,20	2;13,20	−0;1,11, 6,40	2;41,46,40	
	2;12,26,40	2;13,20	−0;0,53,20	2;40,53,20	
130.	2;12,44,26,40	2;13,20	−0;0,35,33,20	2;40,17,46,40	130.
1.	2;13, 2,13,20	2;13,20	−0;0,17,46,40	2;40	1.

In order to describe this situation it is best to present the actual computation of the table for G step by step (cf. Table 14 and Figs. 61 and 62, p. 1342f.). We take as initial value of R in line 1 the value

$$M'_{\Phi} - \varphi = 2;13,20 - 0;0,17,46,40 = 2;13,2,13,20$$

where $M'_{\Phi} = 2;13,20^{\mathrm{H}}$ is the effective maximum for both R and S. Since φ is the constant difference for both functions the next value, in line 2, will be exactly M'_{Φ} (point A in Fig. 61). For the computation of ΔG the value M'_{Φ} is maintained for the next 25 lines. If we nevertheless compute for this stretch the non-truncated zigzag function R we obtain in line 27 the value B = 2;13,25,11,6,40, i.e. a value which is still hidden by the process of truncation. The next value, now below M'_{Φ}, would be C = 2;13,7,24,26,40. Instead, we find in line 28 the number D = 2;13,8,8,53,20. All subsequent values, down to line 77, are continued from this value. This implies that the function R has been moved to the right (cf. Fig. 61), parallel to itself, by an amount

$$x_1 = 0;2,30\,\varepsilon. \tag{10a}$$

The value E = 1;58,37,2,13,20 in line 77 is the last value still above the lower truncation at $m'_{\Phi} = 1;58,31,6,40$. Hence $\mathrm{E} - \varphi = \mathrm{F} = 1;58,19,15,33,20$ is no longer tabulated. In order to reach the value $\mathrm{H} = m'_{\Phi}$ which is given in line 82 we must replace F by the smaller value G = 1;58,15,55,33,20, an operation which amounts to a motion of R to the left by the amount

$$x_2 = -0;11,15\,\varepsilon. \tag{10b}$$

In this way we reach the point which is the basis for all subsequent values and leads exactly to the value M'_{Φ} in line 130 + 1. Hence this modified function R has the period 130 ε. This is confirmed by the fact that

$$x_1 + x_2 = -0;8,45\,\varepsilon \tag{10c}$$

is an amount which eliminates in (9) the excess of P_{Φ} over 130 ε.

The function S begins with 18 lines of the constant value M'_{Φ}. Therefore $\Delta\mathrm{G} = 0$ from line 2 to 18 and G remains at its minimum 2;40 from line 1 to 18. From now on we make S the mirror image of R. Hence the difference of S from line 18 to 19 is φ, corresponding to the difference of R from line 2 to 1. We continue S without modification down to line 72. The next value, being the first value below the truncation at m'_{Φ} corresponds to the point E of R in Fig. 61. In line 122 we are at a situation similar to D and C such that we come back to A in line 18.

What remains, after having completely determined R and S, is trivial arithmetic. ΔG is simply R − S (cf. Fig. 62) and G is obtained by adding ΔG line by line, starting from the minimum value $2;40^{\mathrm{H}}$ in line 1. The resulting function G is strictly periodic in 130 lines, and symmetric about its maximum $4;56,35,33,20^{\mathrm{H}}$ in lines 74 and 75 (cf. Table 14). This is in perfect agreement with the procedure texts.

We can now draw some consequences from this analysis of the structure of the table for $\Phi \rightarrow$ G. First: it is now clear why the "Φ"-values in this table cannot be connected with either Φ_1 or Φ_2 in the ephemerides. The reason being that neither *R* nor *S* is exactly a function Φ.

Secondly: it is not strictly correct to say that R and S are tabulated for an interval $\varepsilon = 1/9;20$ syn.m. For example the interval in R from line 27 to 28 is shorter than ε, although it is associated with a complete interval ε for S. The inverse situation occurs for S in lines 122/123. I think it is very much to the credit of the Babylonian computers that they were willing to introduce a minute mathematical inconsistency in order to gain greatly in arithmetical simplicity.

Thirdly we see how the table $\Phi \rightarrow$ G, attested in the procedure texts, was compiled. As representative of Φ was chosen our function S such that the endpoint of $M'_\Phi = 2;13,20$ corresponds to G$=2;40$ (cf. Table 14, line 18). Decreasing from here on, the function $\Phi \approx$ S was computed down to m'_Φ (line 67). Now truncation takes place, invisible in the procedure texts which continue the zigzag function into the space between effective and real extrema. The first value above m'_Φ would be 1;58,33,42,13,20 but it is raised (in line 17) to 1;58,37,2,13,20, corresponding to the shift x_2 in R (cf. Fig. 61). The new branch of S is then continued up to 2;13,8,8,53,20 (line 122) where truncation hides the new shift x_1 which brings S back to its initial value.

Finally we realize how the procedure texts introduced an ambiguity which caused misunderstanding among modern and possibly ancient users as well. The traditional lists give not only the new value 1;58,37,2,13,20 at the discontinuity in line 73 but the original value 1;58,33,42,13,20 as well. Hence the impression is created of an interval separating two entries whereas in reality all values of Φ between these two limits are equivalent. By accident this situation is actually preserved in an ephemeris[17] which associates G$=4;56$ with $\Phi(\uparrow) = 1;58,36,28,53,20$. But it is by no means necessary to use exactly this procedure instead of interpolating linearly either in the preceding or in the following interval. Similarly one could have located the discontinuity at one step earlier or later.

Another modification of G is attested in an "atypical" text which describes the function G near its extrema.[18] Here a constant maximum 4;54,48,53,20 is assumed for six consecutive entries (lines 72 to 77 in Table 14), instead of the increase to 4;56,35,33,20. Such more or less arbitrary modifications of the strict pattern may eventually help to explain small discrepancies in computations which involve G, e.g. in eclipse tables.

2. The Table for Λ. For the construction of a table which leads from Φ to Λ no new principle must be introduced. In order to find the variable excess of 12 months over 354 days we need only to multiply the one-month difference $9;20\,\varphi = d_\Phi$ by 12 and we obtain

$$12 d_\Phi = 1,52\,\varphi.$$

Now $1,52\,\varphi$ is near the amount of $2\Delta_\Phi = 2,10;8,45\,\varphi$[19]. Indeed

$$12 d_\Phi - 2\Lambda_\Phi = 18;8,45\,\varphi = 0;5,22,35,33,20^{\mathrm{H}} \tag{11}$$

is exactly the amount of $\Delta\Lambda = \mathrm{R} - \mathrm{S}$ in linear sections (cf. Table 15 lines 13 to 44 and 89 to 120 in the column for $\Delta\Lambda$). In Fig. 64 the distance between parallel branches of R and S is measured by (11).

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[17] ACT No. 5 obv. VI, 15.

[18] Neugebauer-Sachs [1969], p. 110. Cf. for the same text below p. 548, p. 552f.

[19] Cf. above p. 508 (4).

Table 15

	R	S	$\Delta\Lambda$	Λ	
1.	2; 0,20,22,13,20^H	2; 0,38, 8,53,20^H	−0;0,17,46,40	−0;20,13,20	1.
2.	2; 0,38, 8,53,20	2; 0,20,22,13,20	0;0,17,46,40	−0;19,55,33,20	2.
8.	2; 2,24,48,53,20	1;58,33,42,13,20	0;3,51, 6,40	−0; 5,42,13,20	8.
9.	2; 2,42,35,33,20	1;58,31, 6,40	0;4,11,28,53,20	−0; 1,30,44,26,40	9.
10.	2; 3, 0,22,13,20	1;58,31, 6,40	0;4,29,15,33,20	0; 2,58,31, 6,40	10.
11.	2; 3,18, 8,53,20	1;58,31, 6,40	0;4,47, 2,13,20	0; 7,45,33,20	11.
12.	2; 3,35,55,33,20	1;58,31, 6,40	0;5, 4,48,53,20	0;12,50,22,13,20	12.
13.	2; 3,53,42,13,20	1;58,31, 6,40	0;5,22,35,33,20	0;18,12,57,46,40	13.
14.	2; 4,11,28,53,20	1;58,48,53,20	0;5,22,35,33,20	0;23,35,33,20	14.
44.	2;13, 4,48,53,20	2; 7,42,13,20	0;5,22,35,33,20	3; 4,53,20	44.
45.	2;13,20	2; 8	0;5,20	3;10,13,20	45.
61.	2;13,20	2;12,44,26,40	0;0,35,33,20	3;55,15,33,20	61.
62.	2;13,20	2;13, 2,13,20	0;0,17,46,40	3;55,33,20	62.
63.	2;13,20	2;13,20	0	3;55,33,20	63.
70.	2;13,20	2;13,20	0	3;55,33,20	70.
71.	2;13, 2,13,20	2;13,20	−0;0,17,46,40	3;55,15,33,20	71.
88.	2; 8	2;13,20	−0;5,20	3; 4,53,20	88.
89.	2; 7,42,13,20	2;13, 4,48,53,20	−0;5,22,35,33,20	2;59,30,44,26,40	89.
119.	1;58,48,53,20	2; 4,11,28,53,20	−0;5,22,35,33,20	0;18,12,57,46,40	119.
120.	1;58,31, 6,40	2; 3,53,42,13,20	−0;5,22,35,33,20	0;12,50,22,13,20	120.
121.	1;58,31, 6,40	2; 3,35,55,33,20	−0;5, 4,48,53,20	0; 7,45,33,20	121.
122.	1;58,31, 6,40	2; 3,18, 8,53,20	−0;4,47, 2,13,20	0; 2,58,31, 6,40	122.
123.	1;58,31, 6,40	2; 3, 0,22,13,20	−0;4,29,15,33,20	−0; 1,30,44,26,40	123.
124.	1;58,31, 6,40	2; 2,42,35,33,20	−0;4,11,28,53,20	−0; 5,42,13,20	124.
125.	1;58,33,42,13,20	2; 2,24,48,53,20	−0;3,51, 6,40	−0; 9,33,20	125.
130.	2; 0, 2,35,33,20	2; 0,55,55,33,20	−0;0,53,20	−0;19,55,33,20	130.
1.	2; 0,20,22,13,20	2; 0,38, 8,53,20	−0;0,17,46,40	−0;20,13,20	1.

The actual computation of the table for Λ is again based on the idea of eliminating the small fraction 0;8,45 φ by replacing Φ by functions R and S which have small discontinuities in symmetric positions but hidden by the truncation at M'_Φ. In contrast to the construction of ΔG we now have only one discontinuity in the amount of $x = -0;8,45\,\varphi$ (cf. Fig. 63) by raising the value B = 2;13,22,35,33,20 (line 45 of the untruncated function R) to C = 2;13,25,11,6,40. Continuing R as an ordinary zigzag function, starting at C, brings us exactly to $M'_\Phi = 2;13,20$ in line 70 and to line 131 = line 1. For the function S the shift to a parallel branch is located between the lines 88 and 89. This completely determines $\Delta\Lambda$. The function Λ itself starts from its minimum value $-0;20,13,20^H$ in line 1. The result is a strictly symmetric sinusoidal function with a flat maximum $3;55,33,20^H$ from line 62 to 70 (cf. Fig. 64).

3. Additional Functions. The function Λ is not the only expansion of G. A function W has come to light which concerns six-month intervals in the same way as Λ deals with 12 months. The generating functions R and S should then

have the distance.

$$6 d_{\Phi} = 56 \varphi \tag{12}$$

between parallel branches. Unfortunately, however, 56φ is near the value $\Delta_{\Phi} \approx 1{,}5 \varphi$ which means that R and S are almost one half-period apart (cf. $\Phi \approx$ R and W $\approx$ S in the schematic Fig. 65). For two functions in almost opposite phases the linear function $\hat{\mathrm{U}}$ vanishes, leaving only second-order branches for U. Aaboe tentatively reconstructed[20] the table for W. It is not necessary to discuss it in detail here because evidence for its use is still very fragmentary.

For the function Λ, however, it has slowly become clear how it enters the framework of the ephemerides, perhaps for checking purposes of the ordinary year-by-year ephemerides, or for eclipse intervals. To make such a direct application to ephemerides possible Λ must be modified by corrections which take care of the shift from high to low solar velocity. Hence the progress in 12-month steps is effected by a function

$$\mathrm{K} = \Lambda + \mathrm{Y} + \mathrm{C}' \tag{13}$$

where

$$\mathrm{C}'_{\mathrm{n}} = 1/2\big(\mathrm{C}(n-12) - \mathrm{C}(n)\big) \tag{14a}$$

and

$$\mathrm{Y}_{\mathrm{n}} - \mathrm{Y}_{\mathrm{n}-1} = \mathrm{J}(n) - \mathrm{J}(n-12), \tag{14b}$$

C and J being the proper values taken from the ordinary ephemeris.[21]

By distributing the influence of J, which is zero on the fast arc, over the whole ecliptic Aaboe determined a "mean" correction $\bar{\mathrm{J}}$, and similarly a "mean" $\bar{\mathrm{Y}}$. Then he could show that

$$\max \sum_{1}^{12} \mathrm{G} + 12 \bar{\mathrm{J}} = \max \Lambda + \bar{\mathrm{Y}} \approx 4;5{,}36^{\mathrm{H}}. \tag{15}$$

This shows in what sense Λ is the equivalent of twelve consecutive true synodic months.

Applying the mean correction $\bar{\mathrm{Y}}$ to the "epact" function X[22] as well, Aaboe found for the length of the "year"

$$354^{\mathrm{d}} + \Lambda + \bar{\mathrm{Y}} + \mathrm{X} \approx 365;11{,}13^{\mathrm{d}}, \tag{16}$$

a parameter which is, however, not explicitly attested in our sources.[23] A similar consideration suggests[24] that a mean correction S for the solar anomaly should be applied to the Saros such that accurately

$$1 \text{ Saros} = 6585^{\mathrm{d}} + \Phi^{\mathrm{H}} + \bar{\mathrm{S}} \tag{17a}$$

with the value

$$\bar{\mathrm{S}} \approx -0;9{,}48^{\mathrm{H}} \approx -0;1{,}38^{\mathrm{d}}. \tag{17b}$$

Textual evidence for such a correction is still missing.

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[20] Aaboe [1971], p. 15f. and Table 5.
[21] Cf. Aaboe [1969].
[22] Cf. above p. 506 (4).
[23] Cf. below p. 528f.
[24] Aaboe [1969], p. 12f.

§ 5. Lunar Latitude

1. Retrogradation of the Lunar Nodes

The lunar nodes recede in the ecliptic; this is an obvious deduction from the experience that lunar eclipses occur in sequences of 6-month intervals. Since 12 lunations cover about 11° less than full rotations the nodes must have moved back by a comparable amount to make an eclipse possible.

In order to describe a retrograde nodal motion we assume the simplest possible model for the lunar latitude, a linear zigzag function. For the moment we do not need to make any assumption about its extrema (i.e. about its period) but consider only an increasing section near a node. Let w be the (constant) solar progress between consecutive (mean) syzygies, and d the corresponding increase in the moon's latitude. If we assume

$$w = c\,d, \qquad c = \text{constant} \tag{1}$$

then we have a fixed nodal position (cf. Fig. 66a). If, however, the lunar orbit recedes, say k degrees per month,[1] while keeping a fixed inclination to the ecliptic, then we have (cf. Fig. 66b)

$$w + k = c\,d \tag{2}$$

(of course with an increment d now larger than before).

In this relation we can consider w to be known from the model for the solar motion. The most important parameter to be determined is the monthly recession k of the nodes because on it depends the frequency of eclipses. The parameters on the right hand side of (2) must then be chosen such that the empirical ratio between synodic and draconitic months is satisfied. The details obviously depend on the assumptions made about the solar and lunar motion.

The ascending and descending nodes move exactly in the same fashion; this is implied in the use of a linear zigzag function. The consequences of this basic assumption, e.g., for the theory of eclipse magnitudes, are fully confirmed by the empirical data. This gives us the right to speak of a "nodal line" connecting opposite nodes, though no such geometric concept is attested in the Babylonian theory.

2. System A, Column E

The ephemerides System A measure the latitude of the moon in units called "barley corn" (še), related by

$$72\,\text{še} = 1° \tag{1}$$

to angular measurements based on degrees.[1] For the sake of convenience we

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[1] Here and in the following I reckon k as a positive quantity, i.e. I do not introduce signs for the directions of motions.

[1] For the arguments on which this interpretation of the units of E is based cf. Neugebauer [1945]. Like all angular measurements these units also are originally measures of length, in this case such that 72 barley corns make 1 uš = 2/5 cubits (thus 1 cubit = 180 še).

will also give the degree-values for the latitudinal parameters although they do not appear explicitly in the ephemerides.[2]

The basic model for the latitude function E is a linear zigzag function with double slope in a "nodal zone" between $+\kappa$ and $-\kappa$ of latitude

$$\kappa = 2{,}24 = 2^\circ \tag{2}$$

and with

$$M = -m = 7{,}12 = 6^\circ \tag{3}$$

as extrema (cf. Fig. 67). In the ephemerides the values of E are tabulated for the syzygies which result from conjunctions and oppositions of the mean moon with the "true sun" as determined by the pattern characteristic for "System A", i.e.[3]

$$\begin{aligned} w_1 &= 30^{\circ/m} && \text{from } ♍13 \text{ to } ♓27 \\ w_2 &= 28;7{,}20^{\circ/m} && \text{from } ♓27 \text{ to } ♍13. \end{aligned} \tag{4}$$

Similarly E operates with two differences, D on the fast arc, d on the slow arc

$$\begin{aligned} D &= 2,\ 6;15{,}42 = 1;45{,}13{,}5^\circ \\ d &= 1{,}58;45{,}42 = 1;38{,}58{,}5^\circ \end{aligned} \tag{5}$$

which change exactly at the same points as the solar velocity in (4). The values in (5) refer to the sections outside the nodal zone. Inside of it these differences have to be doubled.

As we have shown[4] a retrograde motion of k degrees per month of a lunar orbit of fixed slope $(1/c)$ results in a relation

$$w + k = c\,d. \tag{6}$$

Applying this to the model of System A as defined by (4) and (5) we have

$$\begin{aligned} 30^\circ + k &= c \cdot 1;45{,}13{,}5^\circ \\ 28;7{,}30^\circ + k &= c \cdot 1;38{,}58{,}5^\circ \end{aligned} \tag{7}$$

hence

$$c = 18 \tag{8a}$$

and

$$k = 1;33{,}55{,}30^\circ \tag{8b}$$

for the monthly retrogradation of the nodal line.[5] In (7) we made the important assumption that the nodal motion k is independent of the change in the solar velocity. We shall come back to this point presently. The value of c in (8a) is valid only outside the nodal zone. Inside of it the differences (5) are doubled, thus c will be changed to

$$c_0 = 9 \tag{8c}$$

while k remains unchanged.

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[2] Hence parameters not expressly denoted as degrees are reckoned in še.

[3] Cf. above p. 372.

[4] Cf. above p. 514.

[5] Assuming $29;31{,}50{,}8{,}20^d$ for the length of the mean synodic month one obtains $-0;3{,}10{,}50^{\circ/d}$ as an estimate for the daily motion of nodes. The values (8a), (8b), (8c) were first determined by van der Waerden by different arguments (AA, p. 145; BA, p. 216 to 220).

The relations (7) lead to a convenient procedure for computing the latitude column E in an ephemeris. In (7) both sides are expressed in degrees. If we multiply the differences D and d on the right hand side by 72, and therefore divide c by 72, we obtain the differences ΔE expressed in še:

$$\Delta \mathrm{E} = 4(w+k)^{\circ} \tag{9a}$$

outside the nodal zone, and

$$\Delta \mathrm{E} = 8(w+k)^{\circ} \tag{9b}$$

inside of it. Here w represents either w_1 or w_2 from (4) or an intermediate value for an interval which contains a discontinuity of the solar motion. Hence, computing an ephemeris, one can find from column B, which gives the lunar longitudes at the syzygies, the motion w and thus from (9) the corresponding differences for column E, expressed in še.

Examples.[6] No recursion to column B is necessary for intervals which belong completely to the fast or to the slow arc because then the differences (5) are directly applicable. The advantages of the rules (9) concern intervals in which the solar motion shifts from w_1 to w_2 or vice versa.

Example 1. col. B: XI ♓ 22;48 col. E: $-4{,}31;24{,}27(\downarrow)$
XII ♈ 21;11,15
thus: $w = 28;23{,}15$
$k = 1;33{,}55{,}30$
$w+k = 29;57{,}10{,}30$ $4(w+k) = 1{,}59;48{,}42$.

Hence the next value in E is $-(4{,}31;24{,}27 + 1{,}59;48{,}42) = -6{,}31;13{,}9$ še, as given in the text.

Example 2. col. B: V ♌ 24; 3,45 col. E: $+1{,}21;15{,}6(\uparrow)$
VI ♍ 22;48
thus: $w = 28;44{,}15$
$k = 1;33{,}55{,}30$
$w+k = 30;18{,}10{,}30$.

Since the given value of E lies inside the nodal zone we must use (9b) and form

$$8(w+k) = 4{,}2;25{,}24.$$

Now $1{,}21;15{,}6 + 4{,}2;25{,}24 = 5{,}23;40{,}30$ reaches beyond the nodal zone by $5{,}23;40{,}30 - 2{,}24 = 2{,}59;40{,}30$. The slope in the outer part should only be half of the slope inside the nodal zone; thus 1,29;50,15 is the amount beyond $\kappa = 2{,}24$. Hence $+3{,}53;50{,}15$ is the next value of E, in agreement with the text.

We still have to comment on the fact that (7) associates the same motion k with the solar velocities $w_1 = 30^{\circ/\mathrm{m}}$ and $w_2 = 28;7{,}30^{\circ/\mathrm{m}}$. This implies, in order to obtain mean synodic months of constant length, that the lunar motion must also depend on the arcs of the ecliptic. In order to compensate for this artificial lunar anomaly two different draconitic periods p_1 and p_2 had to be introduced[7] such

[6] From ACT No. 6aa. Example 1 from Obv. 18/19 (correcting a scribal error in V, 18), Example 2 from Obv. 11/12.

[7] In ACT No. 81 we have an ephemeris for the lunar latitude E* per tithi which has exactly the parameters required by our p_1 on the fast arc; cf. also ACT, p. 54f.

that

$$(6{,}0+w_1)p_1=(6{,}0+w_2)p_2. \tag{10}$$

Then the same time always elapses between two consecutive passages of the same node. Now we know[8] that the period of a "true function" (e.g. the daily motion in latitude) is related by

$$p=\frac{2\Delta}{2\Delta+d}$$

to the parameters of the "tabulated function" (tabulated with mean synodic months as intervals), where Δ is the amplitude of a simple zigzag function of constant difference d and with the period $P=p/(1-p)$. Hence (10) requires for the same Δ two differences D and d on the fast and on the slow arc, respectively, such that

$$\frac{6{,}0+w_1}{6{,}0+w_2}=\frac{2\Delta+D}{2\Delta+d}. \tag{11}$$

Hence we see that the existence of two differences for the latitude function E, accurately associated with the arcs of the solar velocity, is the equivalent of the requirement (7) that the nodes recede with constant angular velocity, independent of the different spacing of the syzygies.

To check (11) numerically we can free ourselves from the inconvenience of a "nodal zone" by using for Δ the values $2M-\kappa$ (dotted function in Fig. 68, with the same period as E). Thus, reckoning with degrees, we have $\Delta=10°$ and with (5) from (11)

$$\frac{6{,}30}{6{,}28;7{,}30}=\frac{21;45{,}13{,}5}{21;38{,}58{,}5} \tag{12}$$

which is very nearly correct since the left ratio is about 1;0,17,23, the right one 1;0,17,21.[9]

3. The Saros

The constant

$$k=-1;33{,}55{,}30° \tag{1}$$

which measures the nodal motion during each synodic month is closely related to the "Saros" which contains 3,43 months.[1] Indeed

$$-1;33{,}55{,}30°\cdot 3{,}43=-5{,}49;5{,}16{,}30=-6{,}0°+10;54{,}43{,}30°. \tag{2}$$

In other words, in one Saros the nodes recede almost a complete circle, reaching at the end of this period a point with about 11° greater longitude than at the start.

On the other hand the mean progress of the moon per month must be

$$6{,}0{:}12;22{,}8=29;6{,}19{,}0{,}57,\ldots° \tag{3}$$

[8] Cf. above p. 375 (6b).

[9] For the numerical details of obtaining d and D from (11) cf. Neugebauer [1937, 3], p. 257f.

[1] Cf. above p. 502 (2).

assuming for the sidereal year exactly 12;22,8 mean synodic months. Consequently the moon's progress in 3,43 months is

$$29;6,19° \cdot 3,43 = 1,48,10;28,37° \equiv 10;28,37° \text{ mod. } 6,0. \tag{4}$$

Hence the mean moon changes its distance from a node by only about 0;26° in 223 months. This almost perfect coincidence is the basis for "the 18"[2] as an eclipse cycle.

Using the estimate of 6585 days[3] for the Saros one obtains for the draconitic month

$$1,49,45:4,2 = 27;12,38,40, \ldots^{d}. \tag{5}$$

Unfortunately we do not have an explicit statement anywhere about the length of the draconitic month in days, only period relations. For example we know from System B that[4]

$$1,38,43 \text{ drac.m.} = 1,30,58 \text{ syn.m.} \tag{6}$$

Combining this with the parameter 29;31,50,8,20^{d} for the length of the synodic month one obtains

$$1 \text{ drac.m.} = 27;12,43,56, \ldots^{d}. \tag{7}$$

Here, as in so many cases, it is impossible to say which set of parameters, if any, had been considered as "canonical", i.e. numerically exact, value. It is only clear that the close agreement between the estimates (5) and (7) justifies the use of 242 draconitic months in the list of period relations which conventionally define the "Saros".[5]

In this context it is of interest to establish a connection between the fundamental parameters of the latitudinal and the synodic motion. Let k again denote the monthly retrograde motion of the nodes, m the mean synodic motion of the sun, and p the draconitic month in units of the synodic month, i.e. the ratio of draconitic to mean synodic month. The nodal motion n during the time p (cf. Fig. 69) must satisfy the relation

$$n : p = k : 1. \tag{8a}$$

On the other hand the moon travels $n+m$ degrees in the time between the end of the draconitic month and the end of the synodic month. Since the monthly progress of the moon is $6,0+m$ we must therefore have

$$(n+m) : (1-p) = (6,0+m) : 1 \tag{8b}$$

hence with (8a)

$$\frac{kp+m}{1-p} = 6,0+m$$

or

$$k+m = 6,0\,\frac{1-p}{p} = \frac{6,0}{P} \tag{9}$$

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[2] Cf. for this term above p. 486, n. 4.

[3] Cf. above p. 503 (12).

[4] Cf. below p. 523 (2c).

[5] Cf. above p. 502 (1).

where P is the period of the tabulated function which originates from the true function of period p. The relation (9) can also be expressed in the form: the argument of latitude $k+m$ completes in P mean synodic months one revolution.

The equation (9) tells us that the value of $k+m$ is fixed as soon as one has adopted a ratio between draconitic and synodic month. Hence, if one has somehow established the speed of the nodal motion (e.g. from the longitudinal differences between eclipses) the solar mean motion is fixed. Or vice versa: the solar mean motion determines the nodal motion as soon as one has an eclipse cycle. In short: P, k, and m are three fundamental parameters of which only two can be independently determined, the third follows from (9). Or again: whatever compromise one makes for the sake of computational convenience in the value for the solar velocity or the nodal motion, their sum must maintain the value of $6{,}0/P$.

If we now look at attested parameters it is clear that we may assume that the period relations were considered to be basic parameters. Thus we compute

$$\begin{array}{llll}
\text{Saros:} & P=\dfrac{3{,}43}{19}=11;44{,}12{,}37{,}53,\ldots & 6{,}0/P=30;40{,}21,\ldots & \\[2ex]
\text{System A:}^{6} & P=\dfrac{3{,}5{,}32}{15{,}48;24{,}27{,}11}=11;44{,}15{,}18,\ldots & 6{,}0/P=30;40{,}14{,}33,\ldots & (10)\\[2ex]
\text{System B:}^{7} & P=\dfrac{1{,}30{,}58}{7{,}45}=11;44{,}15{,}29,\ldots & 6{,}0/P=30;40{,}14{,}4,\ldots &
\end{array}$$

According to (9) the numbers 6,0/P should represent $k+m$. The only secure value known to us is $k=1;33{,}55{,}30$ derived from the parameters of System A. For m we have tentatively determined in (3) the value 29;6,19. Thus one should have

$$k+m=1;33{,}55{,}30+29;6{,}19=30;40{,}14{,}30^{\circ} \qquad (11)$$

in very good agreement with (10).

Ptolemy's eclipse tables in Alm. VI, 3[8] are based on

$$k+m=30;40{,}14{,}9 \qquad (12)$$

which agrees well with System B in (10) which is not surprising in view of the general role of System B in Greek astronomy. In the Babylonian ephemerides of System B we have neither a secure value for k nor for m. The best parameter is perhaps the mean value 29;6,19,20 of column A[9] from which we would obtain

$$30;40{,}14{,}9-29;6{,}19{,}20=1;33{,}54{,}44^{\circ} \qquad (13)$$

for the monthly motion of the nodes. Unfortunately we have no independent evidence for this parameter and other values for the solar mean motion give slightly

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[6] Cf. Neugebauer [1937, 3], p. 237f., or ACT, p. 47 (1).
[7] Cf. below p. 523 (2a).
[8] Heiberg I, p. 471, 32. For these tables in general cf. above I B 6, 1.
[9] Cf. below p. 533 (1).

different results. As usual, parameters in System B lack strict consistency and are therefore of little help in our search for basic relations.

The values for k and m one can relate to the Saros on the basis of (10) are very doubtful. One may guess that $m \approx 29;6,20°$ is a plausible value and hence we would have to assume a nodal motion

$$k \approx 1;34^{\circ/\mathrm{m}} \tag{14}$$

but again explicit confirmation is lacking.

4. Other Latitude Functions

Two small fragments of ephemerides (ACT Nos. 90 and 91) and one larger piece (No. 92) contain a latitude function which is a simple zigzag function without any special consideration of a nodal zone. Its parameters are

$$\begin{aligned} M &= -m = 7,12 = 6° = 1,12^{\mathrm{f}} \\ d &= 2,27;15 = 24;32,30^{\mathrm{f}} \end{aligned} \tag{1a}$$

from which one obtains the periods

$$P = \frac{1,55,12}{9,49} = 11;44,6, \ldots \qquad p = \frac{1,55,12}{2,5,1} = 0;55,17,19, \ldots. \tag{1b}$$

The extrema agree with E in System A and also the periods (1b) are near to the mean periods of E[1]

$$P_0 = 11;44,15, \ldots \qquad p_0 = 0;55,17,22, \ldots.$$

This is not the only connection with System A; No. 92 also shows a column Φ. Contrary to all our experience, however, these values of Φ are not connectible with the other ephemerides of System A and thus deprive us of a method of dating the text. Also many other details separate these three ephemerides from the familiar System A.

The main purpose of computing lunar latitudes is, of course, the determination of those syzygies which come near enough to the nodes to make eclipses possible. The same information can be obtained, however, from any function which has the same period as the lunar latitude. We shall see this idea fully developed in connection with eclipse magnitudes[2] but it can also be extended to functions which seem to have no independent astronomical significance. An example of this type is found in a badly preserved text (ACT No. 93) which uses a linear saw function of a period $1/2\,P$ (P being the period of the latitude) such that the discontinuities are located exactly at the nodes[3] (cf. Fig. 70, p. 1346). The beginning

[1] Cf. above p. 519 (10).
[2] Cf. below II B 6.
[3] Aaboe [1973] has studied this method in the wider frame of ancient astronomy.

runs as follows

1.	35
	38; 4
	23; 8
	26;12
5.	29;16
	32;20
	35;24
	38;28
	23;32
10.	26;36

This suffices for the determination of the parameters

$$m=21 \qquad M=39 \quad \text{thus} \quad \Delta=18 \qquad d=3;4. \tag{2}$$

Hence we have for the distance of opposite nodes

$$1/2\,P=\frac{2,15}{23}=5;52,10,26,\ldots \tag{3a}$$

thus

$$P=11;44,20,52,\ldots \qquad p=\frac{4,30}{4,53}=0;55,17,24,\ldots \tag{3b}$$

which gives us the underlying period relation for the latitude.[4] Apparently eclipses are considered possible only in the first interval following the discontinuity, i.e. the interval corresponding to the increment $d=3;4$. It seems to be tempting to consider $m+d=24;4^{f}$ as the width κ of a nodal zone (and $\Delta=18$ as an eclipse magnitude exactly at a node?). I cannot explain, however, the choice of the parameters (2) since the periods (3) could have been produced in many other ways.

At the intervals which contain a node the text is adding numbers which are obviously eclipse magnitudes. For example we find in the above given section [5]

in line 3: 10;13,15,26,40
in line 9: 7;48,17,46,40.

How these quantities were determined we do not know. All one can say is that they are not simply proportional to the nodal distance which is provided by the tabulated values of the saw function. This is not the only case of eclipse magnitudes which are not explicable from the latitudes alone.[6]

§ 6. Eclipse Magnitudes

It is not surprising to see quantities which measure the size of an eclipse associated with the "nodal zone" of the lunar latitude. It is, however, a significant element for our evaluation of the abstract mathematical level of Babylonian

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[4] The latitude function as drawn in Fig. 70 is, of course, arbitrary in the choice of the slope.

[5] For the terminology cf. ACT, p. 123.

[6] Cf. below p. 526f.

astronomy to find "eclipse magnitudes" computed for all syzygies, the majority of which are obviously far from any eclipse possibility. Nevertheless the construction of such a generalized function (which we shall denote by Ψ) is arithmetically advantageous and will automatically single out those syzygies where eclipses, however small, are possible. In System B one even went so far as to forego the computation of lunar latitudes since Ψ has the same period and can easily provide the latitude whenever needed.

Though the arithmetical structure of Ψ is very simple much remains obscure that concerns the underlying astronomical concepts. For example the same procedures apply to lunar and to solar eclipses though eclipse limits, durations, and magnitudes should show great differences. As a consequence we have no secure data, e.g., about the apparent diameters of sun, moon, or the shadow.

Eclipse magnitudes are expressed in the ephemerides in three types of units: fingers, degrees, and fractions of the greatest possible magnitude. For the fingers and degrees we have the customary relations

$$1^{\mathrm{f}} = 72\ \text{še} = 0;5^{\circ}. \tag{1}$$

We purposely avoid the term "digit" instead of "finger" because we do not know how many fingers correspond to the apparent diameters of the luminaries.[1]

It is not necessary to give explicit arithmetical rules here for the computation of eclipse magnitudes. Such rules are the simple consequences of the geometric pattern, shown in Fig. 72 or 73, originating from the parallelism between Ψ and E inside the nodal zone.

1. System A

The basic notion of eclipse magnitude is still recognizable in such ephemerides of System A in which a value of Ψ is only listed for those syzygies where E belongs to the nodal zone.[1] Fig. 71 illustrates the relations between E and Ψ for an ascending and for a descending node. The width of the nodal zone is determined by $\pm\kappa$ where[2]

$$\kappa = 2{,}24\ \text{še} = 24^{\mathrm{f}} = 2^{\circ} \tag{1}$$

whereas the distance of Ψ from E is given by

$$c = 17;24^{\mathrm{f}} = 1;27^{\circ} \tag{2}$$

which is the value for $E=0$; hence c is the magnitude of the greatest eclipse possible. On the other hand

$$E = \pm 24^{\mathrm{f}} = \pm 2^{\circ} \tag{3}$$

is the extremal latitude for any eclipse.

The magnitudes for all possible eclipses, corresponding to a latitude within the nodal zone, increase from $-6;36^{\mathrm{f}}$ to $+41;24^{\mathrm{f}}$ as E either increases from $-\kappa$

[1] Cf., however, below p. 551.

[1] It is possible that two consecutive values of E belong to this zone; then the value nearest to zero is chosen.

[2] Cf. above p. 515.

to $+\kappa$ or decreases from $+\kappa$ to $-\kappa$. Contrary to modern usage Ψ does not have its maximum for $E=0$ (which would be $c=17;24^f$) since values from c to $c+\kappa$ represent in fact decreasing magnitudes. In the ephemerides[3] the small negative values of Ψ are denoted by the ideogram be whereas ḫab (or no ideogram) indicates positive Ψ.

In other ephemerides of System A[4] the function Ψ is expanded to a function Ψ', defined for all syzygies and simply parallel to E in the nodal zone, maintaining the slope of E inside the nodal zone outside of it as well. Consequently Ψ' is a zigzag function with

$$M=-m=2{,}0^f=10^\circ \tag{4}$$

while E varies between $\pm 6^\circ$ (cf. Fig. 72). Of course the slope of E and Ψ' depends on the position of the syzygies on the fast or on the slow arc, respectively, following the rules of the latitude theory of System A.

2. System B

The same procedure which we just described for the function Ψ' in System A also underlies the computation of eclipse magnitudes in ephemerides of System B.[1] Though we have in our material no ephemerides with a column E we can conclude from the values given in Ψ that the model for the latitude must have looked like Fig. 73 where the dotted line must essentially represent E, at least inside the nodal zone; modifications of the slope are to be assumed outside of it in order to avoid much too high extrema.

The parameters for the true function represented in Fig. 73 are in the texts expressed in degrees but it is convenient to have also the equivalent values in fingers:

$$\begin{aligned} M &= -m = 9;52,15^\circ = 1,58;27^f, \\ d &= 3;52,30^\circ = 46;30^f \text{ per month}, \\ c &= 1;30^\circ = 18^f. \end{aligned} \tag{1}$$

From (1) it follows that the period of the function E is given by

$$P=\frac{4(M+c)}{d}=\frac{1,30,58}{7,45}=11;44,15,29,\ldots \tag{2a}$$

thus for the true function

$$p=\frac{1,30,58}{1,38,43}=0;55,17,26,\ldots \tag{2b}$$

which implies that

$$1,38,43 \text{ drac.m.} = 1,30,58 \text{ syn.m.} \tag{2c}$$

This is one of the Babylonian ratios which were known to Hipparchus.[2]

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[3] E.g. ACT No. 10, col. II.

[4] E.g. ACT No. 6, col. I.

[1] Cf., e.g., ACT No. 100, col. V.

[2] Cf. above p. 310 (3).

The same ratios are also preserved[3] in a form in which c has the value 1. Thus one obtains the new parameters by multiplying the degree-values in (1) by the factor 0;40:

$$M=-m=6;34,50, \quad d=2;35, \quad c=1. \tag{3}$$

Since c is the eclipse magnitude at an exact coincidence of the moon with the node (E = 0) we can say that (3) measures eclipses in fractions of the greatest eclipse possible. Again it is very disturbing that the same parameters apply to new- and full-moons alike.

The lunar latitudes which correspond to the parameters (1) or (3) show no connection with the solar anomaly or, in other words, they tacitly assume mean velocities both for the sun and the moon. On the other hand we have seen that the solar model of System A introduces two latitudinal differences which vary exactly with the frequency of the solar anomaly in order to guarantee a uniformly rotating nodal line.[4] In System B, however, the solar velocity varies continually according to the simple model of a linear zigzag function, of course with the year as period. Consequently the lunar latitudes should also show continually varying differences with the same period.

Such difference sequences are indeed attested, both with accurate and with abbreviated parameters[5]:

$$\begin{array}{lll} m=44;46,55,33,20 & d=\ 0;33,20 & \\ M=48;13,\ 4,26,40 & P=12;22,8 & \\ \mu=46;30 & \Pi=46,23 & Z=3,45 \end{array} \tag{4a}$$

and abbreviated

$$\begin{array}{lll} m=44;46,55,30 & d=\ 0;33,20 & \\ M=48;13,\ 4,30 & P=12;22,8,24 & \\ \mu=46;30 & \Pi=\ 3,26,9 & Z=16,40. \end{array} \tag{4b}$$

That the parameters in (4a) are meant to be exact is shown by the use of the period 12;22,8 which is basic for both lunar theories and the planets. The units in (4a) and (4b) are fingers.

The function which has (4a) or (4b) as its differences we call Ψ'. In fact it should be the latitudes which have these differences but since the eclipse magnitudes follow E in strict parallelism one can as well directly operate with Ψ'. The shift near the nodes is based on the constant

$$c=18^{\mathrm{f}} \tag{5}$$

which is the same as in (1). Also the extrema agree exactly with (1) and

$$\mu=46;30^{\mathrm{f}} \tag{6}$$

in (4a) and (4b) has the same value as d in (1). Therefore the periods P and p of the refined function also remain the same as in the simple pattern (1).[6]

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[3] Cf. ACT No. 123, col. V.

[4] Cf. above p. 516.

[5] ACT No. 121 Rev. I and No. 123 Obv./Rev. VI, respectively.

[6] For the determination of these parameters cf. Neugebauer [1937, 3], p. 303–313.

§ 7. Eclipse Tables

Several texts from the Seleucid-Parthian period are preserved which can be understood as excerpts from ephemerides, such that the lunar latitude at the syzygies belongs to the nodal zone. Of the fragments of such eclipse tables nine[1] are computed with System A, three with System B, both for lunar and solar eclipses. By far the most important text (because it is fairly well preserved) is the table for lunar eclipses ACT No. 60, covering the 25 years from -174 to -151. Its eclipse magnitudes can be directly compared with modern tables and reveal in general very good agreement.[2] The close numerical agreement shows that the "fingers" of the Babylonian eclipse magnitudes (with $c=17;24^{f}$ for $E=0$) are not very different from modern "digits" (with about 22″ corresponding to c).

It is probably one of the oldest empirical discoveries in astronomy that lunar eclipses are spaced regularly in 6-month intervals with an occasional 5-month gap between very small eclipses. These shorter gaps are usually expressly noted in Babylonian eclipse tables by the remark "5 months". As we have seen[3] one Saros of 223 months contains 33 six-month intervals and 5 five-month intervals.

In principle the checking of eclipse tables, excerpted from ephemerides, should not be difficult, since it requires only the completion of the ephemeris and the adding up of six or five lines. Applying this to column G of ACT No. 60 reveals, however, discrepancies which seem particularly associated with the extrema. We know today that 6-month intervals for G could have been computed directly in the form of a function W, obtained from two truncated versions of the function Φ.[4] I have not carried out the laborious computations required to reconstruct the text on the basis of this in itself plausible explanation.

Of great historical interest are eclipse tables which reach back into the Persian period, though copied or compiled at a later date, as the correct sequence of regnal years shows. One of these tables[5] concerns solar eclipses for the Saros from -474 to -456, the other lunar eclipses[6] (preserved for the period from -416 to -380). The methods for the computation of longitudes and of eclipse magnitudes are very similar in both texts. Only dates (regnal years and months), lunar longitudes and eclipse magnitudes are listed. This material is arranged in parallel columns for one Saros each.

The computation of the longitudes at conjunction near the same node is illustrated by the following pattern for 12-month or 11-month intervals

No. 2	♊ 12;40
4	♊ 2;10
6	♉ 21;40
8	♈ 11;10
10	♈ 0;40
	etc.

[1] As Aaboe has shown ACT No. 55 is not an eclipse table, though related to the Saros; cf. above p. 506.
[2] Cf. the graph in ACT p. 108 and the table in ACT III, Pl. 38.
[3] Cf. above p. 505 (16).
[4] Cf. above II B 4, 3 C 3.
[5] Published Aaboe-Sachs [1969], p. 11 to 20.
[6] To be discussed by Aaboe-Henderson-Neugebauer-Sachs [1975].

The change of longitude is $-10;30$ for all 12-month intervals and 30° less for an 11-month interval (from No. 6 to 8), i.e. $-40;30°$. Since one Saros contains

$$223 = 14 \cdot 12 + 5 \cdot 11 \tag{1}$$

months we see that the sun's progress has been assumed to be

$$19 \text{ rot.} - 14 \cdot 10;30 - 5 \cdot 40;30 = 19 \cdot 6,0 - 5,49;30 = 18 \cdot 6,0 + 10;30°. \tag{2}$$

This is a quite good result. We know, e.g., from System A that the mean distance between conjunction points amounts to

$$\bar{\delta} = 6,0:46,23 = 0;7,45,38,\ldots° \tag{3a}$$

and that these points move during one Saros 1,21° ahead.[7] Thus we would have a mean progress of

$$1,21 \cdot \delta \approx 10;28,37° \tag{3b}$$

in good agreement with (2). Nevertheless, in spite of the satisfactory result within one Saros period, the method itself is rather primitive as a means of determining longitudes of syzygies.

The next problem consists in determining the change of the eclipse magnitudes for 12-month intervals. In System B we found a simple scheme of computing eclipse magnitudes by means of a linear zigzag function, modified near the nodes such that $c = 18^{\mathrm{f}}$ represents the maximal eclipse magnitude.[8] The difference of this zigzag function was

$$d = 46;30^{\mathrm{f}} \tag{4a}$$

and $M = -m = 1,58;27^{\mathrm{f}}$ for the extrema. We now round this latter parameter to

$$M = 1,58;30^{\mathrm{f}} \tag{4b}$$

such that the corresponding linearized latitude function[9] would have the amplitude

$$\Delta = 2(M + c) = 4,33^{\mathrm{f}}. \tag{4c}$$

Consequently we find for a 12-month interval

$$12 \cdot d = 9,18 \equiv 12^{\mathrm{f}} \quad \text{mod. } 2\Delta. \tag{5}$$

Hence in each group of nodes 12 months apart the eclipse magnitudes should increase by 12^{f}. This is exactly what we find in our table of solar eclipses.

Indeed, associated with the above given longitudes we find the following eclipse magnitudes

No.	2	6
	4	18
	6	6
	8	$-\overline{0}$
	10	12
	etc.	

[7] Cf. above p. 503 (5).

[8] Cf. above p. 523 (1).

[9] Dotted curve in Fig. 73, p. 1347.

This sequence also shows that eclipse magnitudes are not counted beyond the maximum $c = 18^f$ but that they are counted downward with the same difference of 12^f, in the same fashion as in Greek and in modern astronomy. As to be expected, modern data for our texts agree well with the trend of these magnitudes.

What we have established so far is the fact that the eclipse magnitudes in each group of consecutive 12-month intervals form a linear zigzag function with difference 12 and maximum 18. What I cannot explain, however, is the rule by which the initial value in each group was determined. In other words I do not know how to bridge the 11-month intervals.

The same table of solar eclipses also contains a small column of numbers – henceforth called θ – following the column of eclipse magnitudes.[10] It is easy to verify that these numbers θ are associated with the eclipse magnitudes (increasing or decreasing, with the same θ) according to the following simple scheme (Ψ being the eclipse magnitudes):

$\Psi = 1^f$	$\theta = 19$	$\Psi = 8^f$	$\theta = 40$	$\Psi = 15^f$	$\theta = 47$	
2	22	9	41	16	50	
3	25	10	42	17	53	
4	28	11	43	18	56	(6)
5	31	12	44			
6	34	13	45			
7	37	14	46			

This scheme consists of three linear sequences for θ with the differences 3, 1, and 3, respectively. If we extrapolate the first sequence with the difference 3 down to $\theta = 0$ we find that $\Psi = -5;20^f$ would be the corresponding eclipse magnitude. For System B we have no information about a nodal zone but

$$-\kappa = -5;20 - 18 = -23;20^f = -1;56,40°$$

would seem to be an acceptable value, comparable to $\kappa = 24^f = 2°$ in System A.

The greatest possible eclipse $\Psi = 18^f$ is associated with $\theta = 56$. It is tempting to interpret θ as sexagesimal fraction 0;56, similar to $c = 1$ for the greatest eclipse in System B.[11] This would imply the statement that the greatest solar eclipse can only reach 0;56 of obscuration. Proceeding with this speculation one could explain such a statement as based on the experience of an annular eclipse or as the effect of observing the corona at a total eclipse. We know from the "Eudoxus Papyrus" that Greek astronomers also denied the possibility of a total obscuration of the sun.[12]

The fact that Ψ varies strictly linearly, i.e. proportional to the moon's latitude, makes it necessary to count Ψ in "linear digits" whatever their numerical relation to the apparent diameter may be. In contrast the quantities θ are only piecewise linear. If we compare them with Ptolemy's "area digits" for solar eclipses,[13] distributed over the whole range of possible eclipses from -6 to 18 and normed

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[10] Cf. Aaboe-Sachs [1969], p. 17, cols. III and IV.

[11] Cf. above p. 524 (3).

[12] Cf. below p. 688.

[13] Cf. above p. 141 and Fig. 122, p. 1244.

such that $c=1$, we obtain the graph of Fig. 74. It may well be that the comparatively good agreement is purely accidental. The fact, however, that we have two sets of numbers for the same eclipses seems to be undisputable.

§ 8. Solar Mean Motion and Length of Year

The question "how many days correspond to one 'year' in Babylonian astronomy?" is difficult to answer. In principle we have three ways to solve this problem: (a) more or less explicit statements in procedure texts, (b) values for the solar mean motion, (c) luni-solar cycles in combination with values for the length of the mean synodic month. Because of the resulting numerous ways to determine the length of the year one does not find a specific value which one could consider as the theoretically accepted basis, comparable to the relation 1 year = 12;22,8 months (or the equivalent epact $e=11;4^{\tau}$) which underlies not only the planetary theory of synodic times[1] but also the lunar ephemerides of both Systems.[2] In other words the basic definition of the "year" rests on a luni-solar cycle and not on an independent counting of days.

The only explicit statement about the length of the year is found in the procedure text ACT No. 210, Sect. 3:

1,49,34;25,27,18 days of 18 years of the sun,
returning to its longitude in 18 rotations. (1)

The return "to the same longitude" refers to sidereal returns,[3] as is clear from a similar phrase about the moon. Thus it follows from (1) that

$$1 \text{ sid. year} = 365;14,44,51^{d}. \tag{2}$$

Since, however, this value is unrelated to any other use of "years" in our material one cannot consider (2) as a fundamental parameter.

How wide a range for the length of the year may be acceptable is seen when one divides 6,0° by the solar mean motion $\bar{v}$. The commonly used value[4]

$$\bar{v} = 0;59,8^{\circ/d} \tag{3a}$$

leads to

$$1^{y} \approx 365;16,34^{d} \tag{3b}$$

which seems to be a rather high value for the sidereal year.[5]

On the other hand we have texts from Uruk[6] which describe the day by day motion of the sun with a progress in longitude

$$\bar{v} = 0;59,9^{\circ/d} \tag{4a}$$

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1 Cf., e.g., above p. 396 (5b), p. 439, p. 471f., p. 524 (4a).
2 Cf., e.g., above p. 378 (15b) or p. 496 (20).
3 The interval (1) is not the Saros which would contain 1,49,45^{d} (cf. above p. 503 (12)).
4 Cf., e.g., below p. 531, Table 16, column III. The same value is found in a great variety of later sources, e.g. in Vat. gr. 208 (fol. 64^{r}); Pc.-Sk. IX, 11; al-Khwārizmī, al-Bāttanī, Maimonides, Bar Hebraeus.
5 A year of 365;15^{d} would require $\bar{v} \approx 0;59,8,15,16,\ldots^{\circ/d}$.
6 ACT Nos. 185 to 187, for S.E. 124 = −187/6.

which would mean

$$1^y \approx 365;10,23^d. \tag{4b}$$

That such a low value is by no means excluded can be seen from a tablet from Babylon[7] which explicitly assumes for the length of the year

$$1^y = 364;30 + 0;40 = 365;10^d. \tag{5a}$$

During the interval of $364;30^d$ a solar motion of $5,59;20°$ is assumed from which the text derives correctly, and without rounding

$$\bar{v} = 0;59,8,58,16,17,46,40^{°/d} \tag{5b}$$

for the solar mean motion.

The lunar theory of System A contains an "epact" function X which seems to assume a "year" of about $365;11,13^d$ but we have no explicit evidence for this parameter.[8]

A variety of values can be deduced from luni-solar cycles, again without obtaining any generally accepted parameter. One might, e.g., combine the canonical value of 12;22,8 synodic months per year with the well attested value for the synodic month from the lunar System B.[9] Then one obtains

$$1^y = 12;22,8 \cdot 29;31,50,8,20 = 365;15,38,17,44,26,40^d \tag{6}$$

but we have no evidence for exactly this parameter anywhere in the lunar theory.

It is perhaps better to determine a range within which one may expect to find values for the length of the year. If we assume that luni-solar cycles vary between 12;22,6 and 12;22,9 months per year[10] and if we consider the variations in the length of the month small in comparison with the value used in (6) we find a range of about

$$365;14,39^d \leqq 1 \text{ year} \leqq 365;16,8^d. \tag{7}$$

All these "years" are certainly to be taken as sidereal years, even if their derivation, from a modern viewpoint, would make them "tropical" or "anomalistic" years. Our sources leave no doubt that such distinctions lie outside the framework of Babylonian astronomy.

In an arithmetical scheme for the moments of solstices and equinoxes for the decades around 600 B.C.[11] we find a constant time interval of

$$\delta = 0;16^{\tau} \approx 0;15,45^d \tag{8a}$$

from year to year for each of the four cardinal points. This difference is obviously to be added to the progress of the integer dates by $12 \cdot 29;30 + 11 = 365$ days. Thus we obtain

$$1^y = 365;15,45^d \tag{8b}$$

for the length of the year. Again we have no other evidence for this parameter.

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[7] Sachs-Neugebauer [1956].

[8] Cf. above p. 513 (16).

[9] Cf. above p. 483 (3).

[10] Cf., e.g., 12;22,6,20 for the 19-year cycle (above p. 355) or $P = 12;22,8,53,20^m$ in System B (below p. 533 (2)).

[11] Cf. below p. 543 (5b).

Very little weight can be attached to a remark by al-Bīrunī who calls the value

$$1^{y} = 365;15,30^{d} \tag{9}$$

"Babylonian", although it also occurs in the Pañca-Siddhāntikā III, 1.[12]

§ 9. Variable Solar Velocity

1. Type A and B

In consequence of the classification of the lunar ephemerides into two "Systems" A and B we extended this terminology to the associated models for the anomalistic solar motion.[1] It so happens that the "System A" solar motion is represented by a two-zone step function while "System B" operates with zigzag functions. Thus the terminology for the sun fortunately became parallel to a terminology valid for the planetary theory as well.

This does not mean, however, that we have patterns for the solar motion only in its relation to lunar ephemerides. The solar anomaly was also recognized independently of its relation to the syzygies and these schemes naturally make use of linear zigzag function, the simplest device for the description of periodic variations. Here again two types can be distinguished, interestingly enough reminiscent of the scheme for the oblique ascensions of the zodiacal signs. Thus we call "Type A" zigzag functions with constant differences both on the increasing and decreasing branch, as distinct from "Type B" where the middle differences are doubled.[2] For Type A we have evidence in several cuneiform texts; Type B is attested so far only in Indian sources.

All Babylonian Type A schemes for the solar velocity come from procedure texts. We do not know when and how these schemes were applied but from the context it seems clear that they were associated with eclipse computations. We are also ignorant of how the numerical data are to be interpreted in their relation to the zodiacal signs. It is possible that they are valid for the whole sign, or that they are meant only for the midpoints (which seems to me the more plausible interpretation). We also cannot exclude the possibility that the extrema in ♊ and ♐ (and hence all other entries) refer to the 20th degree of their signs as midpoints of the velocity zones of System A.[3] For the sake of simplicity I assume in the following strict linearity between midpoints of signs.

Our Table 16 shows three sets of Type A schemes, derived from ACT No. 200, Sect. 9 (col. I in the table), No. 220 (col. II), and No. 200, Sect. 8 (col. III), respectively. The scheme in col. II not only gives the solar velocity in degrees per day but also in fingers per day (as usual $1^{\circ} = 12^{f}$). This pattern is of interest because it comes from the colophon of an eclipse text[4] (which is probably the reason for using

[12] Nallino, Batt. I, p. 40; Pc.-Sk. II, p. 24.
[1] Cf. above p. 475.
[2] Compare for the oblique ascensions p. 368 and Fig. 5, p. 1316.
[3] Cf. Fig. 7, p. 1317.
[4] ACT No. 135; cf. ACT, p. 277.

fingers as units) and because it refers the individual values to one fixed one, here the maximum. This form of presentation is reminiscent of the style of Indian treatises and serves to justify our taking into account similar schemes in the Pañca-Siddhāntikā.

Table 16

	I	II		III	
♊	0;55,12	$0;55,30^{\circ/d}$ =	$12;30-1;24^{f/d}$	0;55,32	m
♋ ♉	0;56,12	0;56,40	12;30 − 1;10	0;56,44	
♌ ♈	0;57,12	0;57,50	12;30 − 0;56	0;57,56	
♍ ♓	0;58,12	0;59	12;30 − 0;42	0;59, 8	μ
♎ ♒	0;59,12	1; 0,10	12;30 − 0;28	1; 0,20	
♏ ♑	1; 0,12	1; 1,20	12;30 − 0;14	1; 1,32	
♐	1; 1,12	1; 2,30	12;30	1; 2,44	M
Δ	0; 6	0; 7	$= 1;24^{f}$	0; 7,12	Δ
d	0; 1	$0;\ 1,10^{\circ}$	$= 0;14^{f}$	0; 1,12	d

Indeed, the Type B for the solar velocity we know only from India. The Pañca-Siddhāntikā (presumably based here on the Pauliśa-Siddhānta)[5] gives in III, 17 a list of solar velocities in consecutive signs, obviously beginning with Aries. The velocities are expressed in terms of deviations from $1^{\circ/d}$ as follows

in ♈, ♉, ♊, ♋:	1 − 0;3	thus $0;57^{\circ/d} = m$
♌:	− 0;2	0;58
♍:	− 0;1	emend to − 0, thus 1;0
♎, ♏, ♐, ♑:	+ 0;1	$1;1 = M$
♒:	+ 0	1;0
♓:	− 0;1	emend to − 0;2, thus 0;58.

The proposed emendations are necessary to restore the required symmetry (cf. Fig. 75). The resulting scheme is then based, on its increasing and decreasing section, on the differences 0;1 0;2 0;1 in familiar Babylonian fashion.

2. System A and A′

We return to the solar motion in its relation to syzygies, i.e. to the lunar ephemerides. In this context "System A" is of basic importance because it is built on the period

$$1 \text{ year} = 12;22,8 \text{ syn.m.}$$

A. Aaboe unexpectedly identified[1] a variant of System A in a text which probably belongs to the time of Artaxerxes II (≈ -397). The basis is again a two-zone step function but modified such that it obtains the period

$$P = 12;23 \quad \text{thus } \Pi = 12,23 \qquad Z = 1,0. \tag{1}$$

[5] Cf. Pc.-Sk. I, p. 12 and p. 14.

[1] Aaboe [1966]; Aaboe-Sachs [1969], p. 7f.

I call this variant henceforth "System A′." As in System A the fast arc is traversed with $w_1 = 30^{\circ/m}$ but this arc is now shortened to

$$\text{from} \quad \uparrow = \text{♍}\,29 \quad \text{to} \quad \downarrow = \text{♓}\,13;30 \tag{2}$$

i.e. we now have

$$\alpha_1 = 2,44;30^\circ \qquad \alpha_2 = 3,15;30^\circ \tag{3}$$

as compared with $\alpha_1 = 3,14^\circ$ and $\alpha_2 = 2,46^\circ$ in System A. This change also moves the line of symmetry from ♊ 20/♐ 20 to ♊ 21;15/♐ 21;15. The velocity on the slow arc is somewhat higher than $28;7,30^{\circ/m}$ in System A:

$$w_2 = 28;20^{\circ/m}. \tag{4}$$

Consequently one finds for the basic intervals

$$\begin{aligned} \delta_1 &= w_1/Z = 0;30^\circ \qquad \text{thus} \quad \pi_1 = \alpha_1/\delta_1 = 5,29 \\ \delta_2 &= w_2/Z = 0;28,20 \qquad\qquad\quad \pi_2 = \alpha_2/\delta_2 = 6,54. \end{aligned} \tag{5}$$

These changes produce a nearly chiastic relation between the two systems:

$$\begin{array}{lll} \text{A:} & \uparrow = \text{♍}\,13 & \downarrow = \text{♓}\,27 \\ \text{A′:} & \downarrow = \text{♓}\,13;30 & \uparrow = \text{♍}\,29 \end{array}$$

hence

$$\begin{array}{lll} \text{A:} & 6;28^{m} \text{ on fast arc} & 5;54,8^{m} \text{ on slow arc} \\ \text{A′:} & 6;54^{m} \text{ on slow arc} & 5;29^{m} \text{ on fast arc.} \end{array} \tag{6}$$

It is difficult to say whether this is an accidental or deliberate modification of the original model.

Of special interest in System A′ is the fact that in the ratio

$$w_1/w_2 = 18/17 \tag{7}$$

the denominator is "irregular", i.e. 1/17 cannot be represented by a sexagesimal number with a finite number of digits – in contrast to System A where $w_1/w_2 = 16/15$. If one looks at the transition coefficients which control the crossing from one velocity zone to the next[2] one will get the impression that an irregular number in the denominator will upset the whole procedure of line by line computation.

If fact, however, this can easily be avoided, and it is indeed avoided, in the very text from which we know about the existence of a System A′. It is sufficient to satisfy two conditions: (a) the zones α_i must contain an integer number π_i of basic intervals of length δ_i, and (b) the discontinuities which separate the arc α_i from the neighboring zones must belong to the set of Π "conjunction points"[3] which represent all possible longitudes. Obviously the condition (b) is easily satisfied as soon as (a) is true. Then it is only necessary to begin the ephemeris at a point which is an integer number of basic intervals distant from the next discontinuity. Then all discontinuities will become conjunction points and the problem of transition vanishes.

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[2] Cf., e.g., p. 377.

[3] To use Newcomb's terminology; cf. above p. 434.

In our specific text π_1 and π_2 are integers[4] and one of the points given in the text, ♍ 6;20, has the distance 22;40° from ↑ = ♍ 29. Indeed 22;40 = 48·0;28,20 = 48 δ_2. Consequently every synodic arc which straddles a discontinuity will be of length $r\delta_i + s\delta_{i+1}$ where r and s are integers such that $r+s=Z$, since the discontinuities are boundaries between adjacent basic intervals.

This shows that "irregular" ratios of velocities are permissible in System A-type models as long as one arranges the conjunction points properly. Since the number period Π is usually a large number the basic intervals will be small and the proper adjustment of the initial value in an ephemeris will not cause more than very minor changes from whatever value one may have chosen on other grounds, e.g. observations.

3. System B

The description of a variable solar velocity by a linear zigzag function seems to be the most obvious solution as soon as one realizes that the sun's progress varies from month to month. Nevertheless a great deal of theoretical deliberation, unknown to us, must have been involved in the choice of the characteristic parameters which define column A of System B which tabulates the variable solar motion in degrees per month:

$$\begin{array}{lll} m=28;10,39,40 & d=0;18 & \mu=29;\ 6,19,20 \\ M=30;\ 1,59 & & \Delta=\ 1;51,19,20 \end{array} \quad (1)$$

thus

$$P=12;22,8,53,20 \quad \Pi=2,46,59 \quad Z=13,30. \quad (2)$$

Beside these accurate parameters one finds another set of values, obviously obtained by rounding from (1):

$$\begin{array}{lll} m=28;10,40 & d=0;18 & \mu=29;\ 6,20 \\ M=30;\ 2 & & \Delta=\ 1;51,20 \end{array} \quad (3)$$

thus

$$P=12;22,13,20, \quad \Pi=5,34, \quad Z=27.$$

In both cases the convenient difference $d=0;18$ could have easily been combined with the canonical value $P=12;22,8$ by making $\Delta=1;51,19,12$. It is difficult to understand why one introduced the parameters (1) when a slight modification of the last sexagesimal digit of m or M would have eliminated in (2) the last two digits of P.

§ 10. Visibility

With column M an ephemeris has reached its first goal, the determination of the moments of the syzygies. But M does not depend only on the preceding columns. Since it contains a calendar date it requires knowledge of the character of the month (i.e. "full" or "hollow"), hence information about the visibility of

[4] Almost all System A planetary schemes operate with integer π_i; cf. above p. 423, Table 9.

the new crescent in relation to the moment of the conjunction. Therefore the completion of a column M requires, in principle, also the solution for each month of the visibility problem.

The methods for solving this problem are only imperfectly known to us, for System A from procedure texts, and for System B, from four columns (N, O, Q, R). In both cases we end up with a final column P which gives the duration of the visibility. Obviously there existed criteria for concluding from the numbers in P whether the moon could be expected to become visible or not. Our material is too fragmentary to reconstruct these criteria in an accurate quantitative form.

1. The Dates of the Syzygies

For the determination of the calendar dates of the syzygies one must remember that the intervals between consecutive syzygies can never be as low as 29 days or as high as 30 days. We now use a diagram in which points on the same vertical in consecutive lines represent moments exactly 30 days apart. Then the preceding statement means that points representing syzygies always move to the left from line to line (i.e. from month to month) but less than 6^H. Hence we assume in Fig. 76 a conjunction that belongs to the night of the 28th in month n. Visibility criteria may tell us (as is plausible in this case) that the next crescent will be seen two evenings later, thus defining day 1 of month $n+1$. Hence "day 30" of month n will be day 1 of $n+1$ and the date, 30 days after month n day 28, will be day 29 of $n+1$. According to computation (column K) the next conjunction may fall into daytime called day 28 of $n+1$. Sunset at the end of day 29 may not yet be far enough distant to make the crescent visible. Hence month $n+1$ will be 30 days long and the conjunction of month $n+2$ will have the date 28. In this way the dates of all consecutive syzygies can be established,[1] provided the visibility conditions are known on the basis of column P.

It is interesting to see how these rules were formulated in the procedure texts,[2] of course without any reference to a diagram. From $M(=M_1)$ in a given month we know date and large hour before sunset for the conjunction. From it, and other circumstances, one now decides whether this month is full or hollow. To obtain the next M $(=M_2)$ we have to add K to M_1. Then two possibilities exist: either the addition of K large hours gives a moment that still falls before the same sunset; or the total transgresses 6^H, i.e. the new conjunction falls after this sunset. Then the character of M_1 combined with such a "transgression" or no transgression of 6^H by M_1+K gives the following four cases for the next day-number:

M_1	transgression	no transgression
full:	date remains	date decreases
hollow:	date increases	date remains.

It is easy to see that these rules correspond exactly to the results of our graphical representation.

[1] Cf. the diagrams in ACT III, Pl. 140–150 which tell us much about the working of a real lunar calendar.

[2] Cf. ACT, p. 206f. (from Nos. 200 and 202).

2. First Visibility

We now come to the difficult question of how an evening after conjunction was selected to be the evening of the first visibility of the new crescent. The information we find in our sources is very unevenly distributed. For System A we have rules in procedure texts that allow us to follow the main steps that lead to column P but the ephemerides themselves suppress these intermediate steps between M and P. For System B the situation is opposite. The ephemerides contain the auxiliary columns between M and P but we have no general rules from procedure texts. This would not be too bad were it not for the fact that we only have a few badly damaged ephemerides which contain these final columns, that most numbers give only rounded values, and that internal consistency is in general deficient in System B. The result is that we have a fair understanding of the basic ideas in this section of Babylonian lunar theory but that the numerical details are not under our control to the same extent as the columns which lead up to M. In particular we will not be able to give definite numerical limits for the final visibility criteria.

The first step (column N in the ephemerides of System B) introduces the time interval between the given moment of the conjunction (M) and the moment of sunset in the evening which can be expected to be the evening of first visibility. It is, of course, possible that the following computation shows that an evening one day earlier or later should have been chosen. In this case the whole computation has to be repeated with the new value for N.

From column F for the lunar velocity one derives

$$F' = 1/6\,F \tag{1}$$

which gives the moon's progress in degrees per large hour. Assuming for the solar progress $1^{\circ/d}$ one finds in

$$1/6\,N(F-1) = N(F'-0;10) = O \tag{2}$$

the elongation at the evening in question.

The next step determines the setting time (in large hours, i.e. in time degrees or right ascension) of the elongation just found. This is done by multiplying O by given coefficients q which depend only on the longitude of the midpoint of the arc sun-moon. We shall discuss presently the numerical values of these coefficients. For the moment it suffices to have obtained numbers

$$Q = q \cdot O \tag{3}$$

which represent a time interval.

From column E we know the latitude of the moon at conjunction (now to be measured in degrees and applied with the proper sign). We assume that the lunar latitude changes 0;4° for every degree of motion in longitude, i.e. we assume that 90° of lunar motion from the node changes the latitude by $0;4 \cdot 1,30 = 6°$ which is the accepted value for the extremal latitude.[1] The lunar motion since conjunction

[1] Cf. above p. 515 (3), p. 520 (1a); p. 523.

is given by F′ · N, thus the lunar latitude at the critical evening is

$$\beta = \mathrm{E} \pm 0;4 \cdot \mathrm{F}' \cdot \mathrm{N} \tag{4}$$

depending on the ascending or descending trend of E.

The setting time Q, found in (3) from the elongation O, will also be modified by a certain correction

$$\mathrm{R} = r \cdot \beta \tag{5}$$

where r is a coefficient depending only on the longitude of the moon. Since we are concerned about the western horizon, the setting time of the moon at the latitude β will be greater than Q for $\beta > 0$, less than Q for $\beta < 0$ (cf. Fig. 77, p. 1348). Consequently

$$\mathrm{P} = \mathrm{Q} + \mathrm{R} \tag{6}$$

is the delay of moon-set over sun-set at the evening under investigation.

So far we have summarized the procedure for computing P as told in the procedure texts. Of the two ephemerides of System A which preserve a column P, one (ACT No. 18) shows fair agreement with the results one obtains by following the procedure described here. The other (ACT No. 5) ends in systematic deviations (of about −2;30°) which remain unexplained.

We still have to discuss the coefficients q and r in (3) and (5), listed in the procedure texts. Both form linear zigzag functions with their extrema at the equinoxes, i.e. at ♈ 10° and at ♎ 10° since we are dealing with System A.

The coefficients q for the consecutive "signs" (i.e. from 10° to 10°) are given as follows:

♈ 1;24	♊ 1; 8 ♒	♍ 0;44 ♏	
♉ 1;16 ♓	♋ 1 ♑	♎ 0;36	(7a)
	♌ 0;52 ♐		

It is easy to see that these numbers are equivalent to the "rising times" of System A. Since we are dealing in the present context with setting times we must associate the numbers in (7a) with the diametrically opposite signs in order to obtain data for risings. Thus we derive from (7a) the list

♈ 0;36 ♉ 0;44 ♊ 0;52 etc.[2] (7b)

For the rising time of "Aries" (i.e. from ♈ 10 to ♉ 10) we have to take the coefficient that belongs to the midpoint of this arc, i.e. according to (7b) the coefficient 0;40. Hence the whole sign has the rising time

$$0;40 \cdot 30 = 20°.$$

Similarly Taurus: 0;48 · 30 = 24°, etc., i.e. a zigzag function with the difference $d = 4°$. Such are exactly the rising times in System A as we know them from the scheme for the length of daylight.[3]

The coefficients r for latitudes are given in two versions. One (in ACT No. 200, Sect. 15)

♈ 0	♊ 0;20 ♒	♍ 0;50 ♏	
♉ 0;10 ♓	♋ 0;30 ♑	♎ 1	(8)
	♌ 0;40 ♐		

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[2] Explicitly given in ACT No. 201, Sect. 5 (p. 239 f.).

[3] Above p. 368 (1).

the other (in No. 201 a, Sect. 2[4])

♈0;18	♊0;26♒	♍0;38♏	
♉0;22♓	♋0;30♑	♎0;42	(9)
	♌0;34♐		

both for first visibility, thus concerning sunset and moonset.

Obviously the coefficients r should be directly related to the coefficients q. Let us assume plane triangles for the small arcs in question near the western horizon (cf. Fig. 77, p. 1348). Since the triangles MM′ W′ and LWS are similar we should have

$$q=\frac{\mathrm{Q}}{\mathrm{O}}=\frac{r\beta}{\mathrm{MW'}}=\frac{r\beta}{\beta\cot\alpha}=r\tan\alpha$$

thus

$$r=q\cdot\cot\alpha. \tag{10}$$

For Babylon with $\varphi\approx 32°$ the angle α between ecliptic and horizon varies between about 34° (for ♎) and 82° (for ♈). For the ratio q one finds about 0;40 and 1;9 as the extrema, thus

α	$\cot\alpha$	q	$q\cot\alpha$	
82°	0; 8	1; 9	0;9	(11)
34	1;30	0;40	1	

which agrees fairly well with (8). If we take for q the values from (7a) one finds

$\cot\alpha$	q	$q\cot\alpha$	
0; 8	1;24	0;11	(12)
1;30	0;36	0;54	

which also comes a little nearer to (9). At any rate the two lists (8) and (9) are clearly not derived from (7a) by a correct mathematical process but they nevertheless represent reasonable latitudinal corrections.

The computation of column P in System B proceeds in the same fashion as in System A, except for the definition of P which is no longer given by (6) but by

$$\mathrm{P}=\mathrm{O}+\mathrm{Q}+\mathrm{R}. \tag{13}$$

This implies that Q is no longer simply $q\cdot\mathrm{O}$ but only a correction of O. In other words if we again define Q by $q'\cdot\mathrm{O}$ then the new coefficients q' should be related by $q'\approx q-1$ to the coefficients q in System A. The analysis of the numbers preserved in the ephemerides confirms by and large this expectation.[5] As for the column O itself the preserved numbers indicate that the definition (2) was amplified by a correction such that

$$\mathrm{O}\approx\mathrm{N}(\mathrm{F}'-0;10)-1;30° \tag{14}$$

or similar. Schaumberger suggested[6] as explanation of the new term a correction for twilight.

[4] The inverse order (for full moons) is found in ACT No. 201, Sect. 6.

[5] Cf. for details ACT, p. 83.

[6] Schaumberger, Erg., p. 388f.

3. Last Visibility and Full Moons

The situation at the morning of the last visibility of the moon is only the mirror image of the configuration at first visibility but without calendaric interest. To compute and to tabulate the data for the waning moon is not much more than a mathematical exercise.

The case of the full moons is quite different, particularly because of its importance for the prediction or exclusion of lunar eclipses. Since the moment of opposition will rarely coincide exactly with sunrise or sunset one must determine the delay or advance of one luminary with respect to the other in the crossing of the horizon. This led the Babylonian astronomers to distinguish four cases

$$\left.\begin{array}{ll}\text{šú}: & \lambda_☾ < \lambda_\odot + 180^\circ \\ \text{na}: & \lambda_☾ > \lambda_\odot + 180\end{array}\right\} \text{near sunrise} \qquad \left.\begin{array}{ll}\text{me}: & \lambda_☾ < \lambda_\odot + 180 \\ \text{ge}_6: & \lambda_☾ > \lambda_\odot + 180\end{array}\right\} \text{near sunset.} \tag{1}$$

In the first two cases the sun rises before, respectively after, the moment of opposition; in the last two cases sunset precedes, respectively follows, opposition. These four cases give rise to four time intervals to predict which numerically is the goal of ephemerides for full moons:

šú: from moonset to sunrise
na: from sunrise to moonset[1]
me: from moonrise to sunset
ge_6: from sunset to moonrise.

The determination of these intervals follows the pattern established for new moons, i.e. it is based on the same sequence of steps N, O, Q, and R. A careful explanation of the proper combination of the additive and subtractive components is given in the procedure texts.[2] The values in Q are again derived in the form $Q = q \cdot O$. The numerical values are the same as before but now referred to the position of the moon, 180° distant from the sun. Consequently the procedure texts for System A give an arrangement between zodiacal signs and coefficients q and r in the inverse order to (7a) and (9).[3]

A peculiar rule for the determination of the time interval between opposition and sunrise has been adopted in the column N_2 of an ephemeris of System B (ACT No. 102). When the opposition falls into daytime the time until the subsequent sunrise is taken (corresponding to the case of na). For an opposition at night, however, N_2 is determined by the preceding sunrise (case of šú). Hence in both cases N_2 exceeds 3^H, perhaps the reason for this norm. In the case of new moons the initial choice for N seems to be restricted to the interval from 5^H to 14^H, for full moons from 4^H to 11^H.

[1] The term na is also used for new moons where it means the time between sunset and moonset; cf. below p. 552.
[2] Cf. for the details ACT No. 201, p. 234–236 and Figs. 48–50.
[3] ACT No. 201, Sect. 1–4 (p. 223) and Sect. 6 (p. 240), respectively. We have no full moon version for the coefficients (8).

From the time intervals the elongations are derived. We have evidence for abbreviated computations which ignore the solar motion, making

$$O = N \cdot F'$$

both for oppositions and for last visibility. On the other hand we also have examples of the type (14), p. 537 with a correction near $-0;30°$ instead of $1;30°$.

As mentioned before[4] column Q in System B is only the correction of the elongation with respect to right ascension (i.e. in Fig. 77, p. 1348, proportional to the difference LW − LS), hence it can obtain negative values. Some ephemerides give only the negative values for Q_1 (new moons) and only positive values for Q_3 (last visibility), obviously because only a negative correction would be of interest if O_1 by itself would indicate visibility; similarly only a positive correction could change invisibility based on O_3 into visibility.

In the computation of ephemerides the coordinates of "sun" and "moon" must mean the midpoints of the respective disks. For positions "in" the horizon, however, this still leaves a semicircle visible; we have, of course, no evidence for any recognition of refraction. Hence it seems natural to modify the previously discussed intervals by the time it takes to remove the disks of sun or moon completely from sight. Such a correction (which can be positive, e.g. for me, or negative, e.g. for ge_6) would be the sum of a solar and a lunar radius, i.e. of the order of magnitude of a lunar diameter. In procedure texts we indeed find references to this situation expressed as "add (or subtract) the 2 of the disk".[5] Since all quantities so far were measured in time degrees it would be natural to assume the same for "the 2 of the disk". The assumption of a lunar diameter of 2° is, however, difficult to reconcile with correct distances of the moon from Normal-Stars in Diaries[6] and with the generally good agreement of eclipse magnitudes with modern values.[7]

4. Visibility Conditions

One can hardly doubt that procedure texts existed which gave criteria for first and last visibility, based on the values of column P and perhaps other parameters. Unfortunately no such text has come to light and therefore it is only from the individual cases appearing in ephemerides that we can estimate limits required for visibility.

In System A where M gives the time before sunset for the syzygies the cases of special interest are those in which the time interval N was chosen as low as M or as high as $M + 2^d$. Similarly one finds for the last visibility alternatives between $N_3 = C - M + 6^H$ and a morning 2 days earlier. Investigation of such alternative situations in the ephemerides suggest that it was not the value in P

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[4] Above p. 537.

[5] The term ḫab-rat for "disk" is also used for eclipse magnitudes; cf. below p. 550 and ACT, p. 197f. and p. 237f., Figs. 51 and 52.

[6] Cf. below p. 546.

[7] Cf. above p. 525, p. 527.

alone that was decisive but probably a criterion of the form

$$\mathrm{O}+\mathrm{P}>c \qquad (1)$$

with $c \approx 21°$ for first visibility and a number between 18° and 22° for last visibility. The inclusion of the elongation O in a formula like (1) can be motivated by the fact that the apparent width of the crescent is proportional to the elongation.[1]

System B does not exclude the existence of a criterion of the form (1). Two ephemerides would agree with $c=23°$ at last visibility and others suggest for first visibility $c=23°$, or 20°, or 17°. We also have some lists of values for P at first visibility covering the new moons for several years[2] that seem to suggest, however, that P alone had been used for selecting the evening of first visibility. In these lists we find 11;20°, 11;40° and 11;50° as lowest values (singled out by a note "ina pi" of unknown meaning) and one alternative solution 20;30° in case of a full month against 10;30° if hollow ("ina 30-šú"). For S.E. 122 I a value as high as 25;10° is given without an alternative. An ephemeris preserved for the same year accepts instead the low value of 12° for P. This changes, of course, the character of the preceding month from full to hollow. For the hollow month before that the ephemeris gives P = 17;30° as compared with 17;10° in the list. This shows again that the computation of data following column M is not of the same strictness as in the columns before it.

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[1] Cf. ACT, p. 67, Fig. 31.

[2] ACT No. 180 for S.E. 120 to 125 (−191 to −185).

C. Early Babylonian Astronomy

The development of a mathematical astronomy, some time in the Persian period, was made possible only by the existence of highly advanced mathematical techniques, the most important of which was the sexagesimal place value notation. We know today that Babylonian mathematics preceded astronomy by more than a millenium but it is impossible to say why its application to astronomy occurred so much later or why and how it occurred at all. It is evident, however, that without flexible numerical methods no real astronomy would have been possible. Purely geometrical ingenuity leads nowhere, as we see in the example of Eudoxus' homocentric spheres. The decisive role played by the computational procedures in the development of astronomy was badly obscured by the fascination of historians with the geometric aspects of planetary theory. It is correct to say that geometrical models are essential for the Hipparchian-Ptolemaic astronomy, but without the numerical techniques inherited from Babylonia a work like the Almagest could never have been written.

In the following pages I shall make no attempt to outline a consistent history of the steps which led to the astronomy of the Seleucid period. The material available for such a task is still much too fragmentary and every new text destroys some aspect of our reconstructions. In particular I consider all attempts to determine a definite date of invention for any one of the "Systems" of the ACT level not only unconvincing but seriously misleading. If we have learned anything from "atypical" texts, which somehow lead us back into the Persian period or even earlier, then it is the fact that the sharp distinction between the Systems becomes more and more obliterated and that we must assume a period of transition during which many attempts were made to construct mathematical procedures capable of predicting astronomical phenomena far into the future. I would even be inclined to suggest that only System A of the lunar theory represents a uniform "System" in the proper sense of the word, whereas "B" reflects the stage of continued experimentation. Similar things can be said about the planetary theories. I have no doubt that the discovery of any new archive of astronomical tablets would greatly disturb our simplistic historical pictures.

§ 1. Calendaric Data, Celestial Coordinates

Sexagesimal division of time and the 19-year cycle testify to the influence of Babylonian astronomy up to the present. As we have mentioned before, the origin of the rigidly regulated intercalation pattern of the 19-year cycle cannot be traced farther back than the middle of the fifth century B.C.[1] Since any reasonably

[1] Above p. 354f.

accurate sequence of intercalations must average 7 intercalary months in 19 years it is impossible either to prove or to disprove, on the basis of historically attested intercalations, that the strict 19-year cycle was preceded by a period in which one was fully aware of the significance of a 235-month interval, without adopting definite rules as to the position of the intercalary months and their character (XII_2 or VI_2).

In fact the general quality of a 19-year cycle can easily be derived from the famous 18-year eclipse cycle, the "Saros" of 223 months, which was discovered much earlier and was incorporated in the "Goal-Year texts" for the prediction of eclipses and of the lunar phases near the syzygies.[2] Since it is obvious that 223 months are 11 days longer than 18 solar years[3] it is tempting to utilize the 11-day epact of the lunar year to eliminate this excess. Thus

$$223+12=235=19\cdot 12+7 \tag{1}$$

months can readily be expected to be the equivalent of 19 solar years.

This estimate can be sharpened by using the parameters of "System A" of the luni-solar theory.[4] Since in this model

$$1\ \text{year}=12;22,8\ \text{months}=46,23/3,45=\Pi/Z$$

one has

$$223Z\equiv 1,21 \mod \Pi^{5} \quad\text{and}\quad 12Z=45,0\equiv -1,23 \mod \Pi. \tag{2}$$

That is to say $223+12=235$ months differ only by $1,21-1,23=-2$ basic intervals from an exact return, i.e. only 0;16° on the fast arc, 0;15° on the slow arc[6] are missing from 19 revolutions of the sun.

The solstice-equinox scheme and its companion for Sirius antedates by far the introduction of the 19-year cycle.[7] Unfortunately its arithmetical pattern is not rigid enough to make it an instrument for the dating of pre-Seleucid texts. It is only through historically attested sequences of intercalations that a text of this type can be dated.

This is the case with the earliest solstice-equinox-Sirius text which we have.[8] From its pattern of month names the intercalary years can be determined, and comparison with attested intercalations refers the extant text to the years from Nabopolassar 10 to Nebuchadnezzar II 17 (i.e. -615 to -587; actually the text might have covered about 60 years). The first column, for the summer solstices, is completely broken away. The next three columns concern autumnal equinoxes, winter solstices, and vernal equinoxes. The last column is a list of pairs of Sirius settings (Ω) and risings (Γ). These Sirius dates follow the familiar pattern of the later period[9]: the time of invisibility of the star amounts to

$$\Omega\rightarrow\Gamma=2\ \text{months}\ 5\ \text{days}. \tag{3}$$

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[2] Cf. Sachs [1948], p. 283, p. 281.

[3] Cf. above p. 502 (3).

[4] This remark is due to Aaboe-Henderson [1975].

[5] Cf. above p. 503 (5).

[6] Above p. 502 (4).

[7] Cf. above II Intr. 3, 2 and 3, 3.

[8] Published Neugebauer-Sachs [1967], p. 183–190.

[9] Cf. Sachs [1952, 1].

The dates given in the text increase regularly by 11 days from one year to the next, with the occasional insertion of a 12-day interval to account for the accumulated excess of the epact over 12 days.

Of special interest in this text is the fact that it gives, for the cardinal points of the year, not only the name of the month and the numbers of the day but also "hours", i.e. time-degrees before or after sunset or sunrise; e.g. for the winter solstices in four consecutive years:

$$\begin{array}{rl} 27^\circ & \text{after sunrise} \\ 21 & \text{before sunset} \\ 1{,}15 & \text{after sunset} \\ 45 & \text{before sunrise.} \end{array} \tag{4a}$$

At the winter solstice the night has its greatest length. If we assume that this maximum is $M = 3{,}36°$, which is the norm in the later astronomical texts,[10] then we can replace (4a) by

$$\begin{array}{rl} 3{,}36+27=4{,}3^\circ & \text{after sunset} \\ 6{,}0-21=5{,}39 & \text{after sunset} \\ 1{,}15=1{,}15 & \text{after sunset} \\ 3{,}36-45=2{,}51 & \text{after sunset.} \end{array} \tag{4b}$$

This justifies our introducing 3,36° for the longest night because (4b) reveals that (4a) is based on a constant increment

$$\delta = 1{,}36°. \tag{5a}$$

Interpreting the "days" and their fractions as tithis[11], the usual norm in these schemes, the yearly increment would be

$$\delta = 1{,}36° = 0;16^{\mathrm{r}} \approx 0;15{,}45^{\mathrm{d}} \tag{5b}$$

and hence the length of one year

$$12^{\mathrm{m}} + 11;15{,}45^{\mathrm{d}} = 365;15{,}45^{\mathrm{d}}, \tag{6}$$

a result mentioned before.[12]

No hours are mentioned, of course, for the phases of Sirius, because Ω always occurs shortly after sunset, and Γ shortly before sunrise. We have evidence that Sirius is not the only fixed star whose phases were put into a definite relation to the solstice-equinox scheme.[13] A small fragment from Babylon[14] contains in two parallel columns[15] the dates for Γ and Ω of the Pleiades during the years

[10] Cf. above p. 366.
[11] Above p. 358.
[12] Above p. 529.
[13] This is, incidentally, further evidence for the fact that the Babylonian astronomers were not aware of the existence of precession.
[14] Published in Neugebauer-Sachs [1967], p. 190.
[15] The alignment of the text is not very carefully observed by the scribe. The original arrangement obviously was to have in the same line first Γ of year N, then Ω in $N+1$. The text as written ends with Γ in S.E. 86, followed by Ω in 88.

S.E. 64 to 88 (-247 to -223), related as follows to the vernal equinoxes:

$$\Gamma = \text{V.E.} + 45^\tau, \qquad \Omega = \text{V.E.} + 16^\tau. \tag{7}$$

Hence these stars are invisible exactly one month.

It can hardly be doubted that phases for other fixed stars, occasionally mentioned in "Almanacs"[16] are also not observed but taken from arithmetical schemes related to the solstices.

The determination of the cardinal points of the year must ultimately rest on some observational methods, however influenced by schematization they may be. One of these methods consists in the recording of shadow lengths, preserved in tabular form in the second tablet of the series $^{\text{mul}}$Apin[17], thus antedating the origin of mathematical astronomy by centuries. The imprint of a much older mathematical tradition is particularly evident in this early attempt of numerical description of astronomical data.

The contents of this table of shadow lengths can be adequately described in the following fashion. Let s be the length of the shadow (measured in cubits[18]), t the time after sunrise, counted in time degrees ($1^d = 24^h = 6{,}0°$) then we have the following relations:

A		Equinox		B	
s	t	s	t	s	t
1	1,0°	1	1,15°	1	1,30°
2	30	2	37;30	2	45
3	20	3	25	3	30
4	15			4	22;30
5	12			5	18
6	10			6	15
8	7;30			8	11;15
9	6;40			9	10
10	6			10	9

The columns A and B refer to the solstices. The time that elapses after sunrise until the shortest shadow is reached is greater in B than in A. Consequently B must be the summer solstice, contrary to the text which has summer and winter interchanged.

The arithmetical structure of this table reflects the arrangement of the Old Babylonian tables of reciprocals[19] – note the omission of the "irregular" number $s=7$. All numbers t can be obtained from

$$t = c/s \tag{8}$$

[16] Sachs [1948], p. 280, No. 26.

[17] Cf. for this text below p. 598. The shadow tables were published by Weidner [1924], repeated by van der Waerden [1951, 1] and AA, p. 80, BA, p. 84.

[18] Weidner (and hence van der Waerden) misinterpreted the text by taking the phrase "*ina* 1 kùš" to refer to a gnomon of 1 cubit length. In fact this expression only means "reckoning in cubits" [Sachs].

[19] Cf. Neugebauer, MKT I, p. 9.

with $c=1{,}0$ in A, 1,15 at the equinoxes, 1,30 in B. If we substitute $s=0;50$ in (8) we find

$$\begin{array}{ll} \text{winter solstice (A):} & t=1{,}12^\circ=4{,}48^h \\ \text{equinox:} & t=1{,}30^\circ=6^h \\ \text{summer solstice (B):} & t=1{,}48^\circ=7;12^h \end{array}$$

i.e. for the length of daylight at the solstices

$$M:m=3{,}36^\circ:2{,}24^\circ, \tag{9}$$

again the standard ratio for Babylon.

The satisfaction with this result is spoiled by the implication that the noon shadow is always 5/6 cubits long, independent of the seasons. I do not see a plausible model of a sundial that would explain such a norm.

The inaccuracy of time measurement, be it based on sundials or water clocks, or on the length of daylight schemes, probably had very little influence on the progress of Babylonian astronomy. The mere fact that a large proportion of its efforts are directed toward visibility problems, which concern the time near sunset or sunrise, makes it possible to express the critical values in time degrees, i.e. in relatively small angular distances. Of much greater interest than the time coordinates are the positional parameters which play such a prominent role in all predictions.

Looking at what we may superficially call coordinate systems in Babylonian astronomy we notice two greatly different approaches which exist side by side during the whole Seleucid period. On the one hand we have a straightforward ecliptical coordinate system in the ephemerides with a sidereally fixed vernal point, called ♈ 10° in System A, ♈ 8° in System B. On the other hand the "Diaries," the "Goal-Year texts," and the "Normal-Star Almanacs" describe the positions of the planets and of the moon with respect to a set of 31 reference stars, called "Normal Stars," in Epping's terminology.[20] The ecliptic with its division into 12 signs is present in these texts but the positions of the celestial bodies are related to the Normal Stars not by coordinates counted in degrees but by distances measured in "cubits" and "fingers."

Fig. 78 gives a schematic diagram for the positions of the Normal Stars[21] which all belong to a zone between $+10^\circ$ and $-7;30^\circ$ of latitude.[22] The modern identification of these 31 stars is quite secure, in part because they are well-known bright stars like Regulus, Spica, the Gemini, etc., and in part through their connection with lunar and planetary positions which are known from modern tables.[23] The principle which led to the selection of just these 31 stars, is less obvious. Some are so near to each other that they barely characterize different positions in the path of planet. On the other hand the last four signs of the ecliptic are only very poorly represented. It is true that this area does not contain very bright stars near the ecliptic but it does contain many stars of magnitude 3 or 4, not rejected for Normal Stars in the first half of the ecliptic.

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[20] Epping AB, p. 115.

[21] Longitudes are drawn to scale, latitudes are exaggerated, but to scale relative to each other.

[22] A catalogue of stars in the Handy Tables is similarly concerned only with latitudes between $\pm 10^\circ$; cf. below p. 1050.

[23] For a list of the Normal Stars with their coordinates for -600, -300, and 0 cf. Sachs [1974], p. 46.

Thanks to a fragment of a catalogue of Normal Stars, discovered by Sachs,[24] we know how the positions of these stars were defined (at least in this one catalogue). The extant list comprises the six stars from θ Leonis to β Librae; with each star is mentioned a longitude, rounded to integer degrees, but no latitude. Sachs tentatively dated the text to the end of the Persian period. To investigate its norm we therefore compare its longitudes with the actual longitudes of these stars for -300. Each century earlier would increase the differences by about 1.4°. For -300 we find

	λ	Catal.	Cat.$-\lambda$
θ Leo	♌ 11.5	♌ 20	8.5°
β Vir	♌ 24.5	♍ 1	6.5
γ	♍ 8.5	♍ 16	7.5
α	♍ 22	♍ 28	6
α Lib	♎ 13	♎ 20	7
β	♎ 17	♎ 25	8

The differences indicate that the longitudinal norm underlying this catalogue is compatible with a vernal point located at ♈ 8° or ♈ 10°. Ignoring precession this norm for the Normal Stars remains valid into the period of the ephemerides. This seems to establish at least some connection between the Normal Stars and the longitudes in the ephemerides and their norms for the daylight schemes.

Normal Stars are known to us from the Diaries which go back to the seventh century[25] and, for the Seleucid period, from the Goal-Year texts and from the Normal-Star-Almanacs. In all these texts, positions of planets at given dates are recorded in relation to a nearby Normal Star. In the Diaries the same also holds for the moon.[26] Although much of these data is not directly observed but computed, we have here an enormous collection of ultimately empirical data.

The terminology used by these texts to describe the motion of a planet or of the moon with respect to the stars is rather loose. We know that the expression "in front of" a Normal Star means that the planet is located to the west of the star. This implies that the "forward" motion is the direction of the daily rotation in the same sense in which Greek astronomy takes the terms "leading" and "following."[27]

Serious difficulties are caused by the expressions "above" and "below" a Normal Star, always combined with an integer number of cubits, or occasionally with simple fractions of cubits, at greater distances, fingers at smaller intervals. Since modern tables provide us with the ecliptic coordinates of star and planet one can graphically determine the direction from the star to the planet at the date in question. If "above" and "below" referred to a sidereally fixed direction (e.g. perpendicular to the ecliptic or to the equator) the directions from a given star to any planet above or below it should always coincide. In fact, however,

[24] Sachs [1952, 2]; cf. also Huber [1958], p. 205f.

[25] The sequence of the still extant Diaries becomes relatively dense in the middle of the fourth century; cf. the graph in Sachs [1974], p. 47.

[26] Cf. Sachs [1948] for the topics recorded in each class of these texts; also Sachs [1952, 2], p. 149.

[27] Cf., e.g., below p. 807.

this is not the case.[28] For example Mars is said to be below θ Leonis in S.E. 114 VII 22 = −197 Oct. 29. The star was at $\lambda = 132.86$ and $\beta = 9.65$, the planet at $\lambda = 135.4$ and $\beta = 1.8$. The line connecting these two points meets the ecliptic at a longitude about 3° greater than the orthogonal from the star. Similarly Mars in S.E. 81 VI 28 = −230 Oct. 8 (at $\lambda = 221.4$, $\beta = -0.6$) is "above" α Scorpii (at $\lambda = 218.74$, $\beta = -4.28$). The line from the star to the planet again points to a longitude about 3° greater than the star's longitude.

This, and many similar cases, rule out any system of spherical coordinates in fixed relation to the ecliptic or the equator. What remains as a possibility seems to be orthogonality to the horizon but this would involve rules as to the time of the night at which the observation had to be made.

To summarize the situation: I think one has to admit that the texts which operate with Normal Stars are farther removed from spherical geometry than even the arithmetical patterns used in the ephemerides for the oblique ascensions and their applications. This being the case it must have been difficult to make full use of the empirical data accumulated in the Normal Star texts. For example a "conjunction" of a planet and a star on an altitude circle cannot be easily used as information about ecliptic coordinates. The Normal Star texts and the ephemerides appear to be two unrelated approaches to planetary theory.

§ 2. The Moon

As we have noted before[1] a simple estimate leads to the conclusion that the length of the anomalistic month is about 0;56 synodic months, a ratio which corresponds to a period $P = 14$ for the anomalistic month. There exists indeed a text[2] which is based on this crude period, due to the following parameters which we give in comparison with column F in System A:

$$\begin{array}{rll} m = 11;3^{\circ/d} & \text{col. F: } 11;4^{\circ/d} & \\ M = 15;57 & \text{same} & \\ \mu = 13;30 & 13;30,30 & \qquad (1) \\ d = 0;42 & \text{same} & \\ P = 14 & 13;57,8,\ldots & \end{array}$$

The same tablet also contains dates concerning Mars, its phases and its entries into the zodiacal signs[3] in the year Philip Arrhidaeus 5 (= −318/317). Exactly the same year was used in an example for the computation of lunar latitudes, following System A, in the important procedure text ACT No. 200.[4] Hence

[28] I investigated about a hundred cases, involving all Normal Stars, from a variety of texts put at my disposal by A. Sachs.

[1] Above p. 476.

[2] Published Neugebauer-Sachs [1969], p. 92f. For F in System A cf. above p. 479 (7).

[3] Cf. above p. 456, n. 10. For the years 4 to 7 of Philip we also have an uncanonical lunar ephemeris; cf. above p. 506 n. 5.

[4] This was discovered by A. Aaboe in No. 200, Sect. 6, Obv. I, 24 and I, 31; cf. Aaboe-Henderson [1975].

what seems a rather primitive scheme for the lunar velocity is chronologically associated with a sophisticated theory of the lunar latitude. This is part of a general experience. Simultaneously with a groping and more primitive lunar theory in the fourth or fifth century we find evidence for the extended use of the function Φ, related perhaps even to the 14-month intervals known from (1).[5] Such facts are not easily fitted into a simple evolutionary picture.

Techniques designed to produce new functions from Φ or F, in particular G for the length of the synodic month,[6] are attested in texts which clearly belong to the pre-ACT level. For example[7] we know of a function F′, truncated at[8]

$$m' = 11;25 \qquad M' = 15;26 \tag{2a}$$

and with

$$d = 0;42 \qquad \Delta = 4;52,48 \qquad \text{thus } P = 8,8/35 = 13;56,34,\ldots. \tag{2b}$$

The accurate extrema are not known.

Related to this function F′ is a function G with the following parameters

$$\begin{array}{ll} m = 1;49,40^{\mathrm{H}} & d = 0;23,20 \\ M = 4;32,20 & \Delta = 2;42,40 \\ \mu = 3;11 & \text{thus } P = 8,8/35 = 13;56,34,\ldots. \end{array} \tag{3}$$

Of particular interest in (3) is the mean value because it implies a mean synodic month of

$$29^{\mathrm{d}} + 3;11^{\mathrm{H}} = 29;31,50^{\mathrm{H}}. \tag{4}$$

Thus we have here a function G with a very good mean value. In fact (4) is the parameter on which the Islamic lunar calendar is based, and very near to the accurate value of the mean synodic month in System B.[9]

The differences of G and F′ are, of course, proportional:

$$d_{\mathrm{G}} = 0;33,20\, d_{\mathrm{F'}} = 0;33,20 \cdot 0;42 = 0;23,20. \tag{5}$$

This is reminiscent of the connection between Φ and G in System A, a relationship which is indeed discussed in the next section of the same text.[10] In the ACT material G is computed from Φ, and Φ is related to F[11] by

$$d_{\mathrm{G}} = 9;20\, d_{\Phi} \qquad d_{\mathrm{F}} = 15;11,15\, d_{\Phi}$$

such that one would expect

$$d_{\mathrm{G}} = 9;20/15;11,15\, d_{\mathrm{F'}} \approx 0;36,52\, d_{\mathrm{F'}}$$

instead of (5). Consequently neither G nor F are identical with the corresponding functions in System A. Obviously the variety of functions F and G is much greater than the ephemerides from the Seleucid period let us suspect.

The periodic oscillations of the moon to the north and to the south of the ecliptic are easily perceived, e.g. by observing the changes of the rising and

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[5] Neugebauer-Sachs [1967], p. 199; also below p. 553.
[6] Cf. above II B 4, 3 C 1.
[7] From Neugebauer-Sachs [1969], Text K, Sect. 6 (p. 108–110). Cf. also below p. 552.
[8] Compare also the truncation of F in System A, above p. 501.
[9] Cf. above II B 3, 1.
[10] Neugebauer-Sachs [1969], p. 110f. Cf. also above p. 511.
[11] Cf. above p. 487 (6) and p. 501 (11).

setting amplitudes of the moon in relation to the points of sunrise and sunset. This, however, is by no means equivalent to introducing coordinates perpendicular to the ecliptic and to the determination of values for the extremal latitudes reached by the moon. Perhaps one may assume that it was the observation of the moon in relation to the Normal-Stars [12] that numerical data for the lunar latitude were established. Of decisive importance must have been the insight that eclipses are restricted to a narrow zone near the ecliptic, an insight which directed all efforts on a theory of nodal motion and lunar velocity and relegated the problem of extremal latitudes to a second rank. Thus it could happen that the same text mentions both 5° and 6° as greatest lunar latitude.[13] As we have seen [14] 6° remained in use in the ephemerides, column E, of System A.

From the same text we also know about a schematic description of the nodal motion as 1;40° per month. One should not take this as a crude observational result, preparatory, e.g., to the better value $k = 1;33,55,30°$ in System A.[15] In fact this 1;40° is the numerical equivalent of the statement that the nodes make one revolution in 18 years, i.e. in one "Saros." Indeed, if we schematically equate one "year" with 12 "months" we have

$$1;40^{\circ/m} \cdot 12^{m} \cdot 18 = 6,0^{\circ} \tag{6}$$

which is a simple relation for the proper order of magnitude, easy to remember.

A similar convenient rounding is the statement that 3 draconitic months amount to 82 days.[16] Hence

$$1 \text{ drac.m.} = 27;20^{d} \tag{7}$$

is a short range estimate, comparable to the $29;30^{d}$ for the lunar month, or the 1° for the motion of the sun per day.

Ptolemy's famous statement that practically complete eclipse records from Babylon existed, beginning with the mid-eighth century B.C., is undoubtedly correct.[17] Thus at an early period the material was available from which one could establish the periodicity of lunar eclipses.[18] Hence we may assume that the "Saros" belongs to the earliest phase of mathematical astronomy, making the prediction of eclipses possible. On the basis of the period relations represented in the Saros [19] one can establish definite rules for the empirical fact that lunar eclipses are generally 6 months apart with the occasional insertion of 5-month, or rather 11-month intervals. This leads to the tabulation of eclipse data in 18-year columns with groups for the 6-month intervals, properly separated by 5-month intervals. A table of this form is, e.g., the "Saros-Canon" which was

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[12] Regularly reported in the "Diaries"; in a schematic form also in the "atypical" text discussed here (cf. Neugebauer-Sachs [1967], p. 203).

[13] Neugebauer-Sachs [1967], p. 203.

[14] Cf. above p. 535, n. 1.

[15] Above p. 515 (8 b).

[16] Neugebauer-Sachs [1967], p. 205.

[17] Cf. above p. 352, n. 3.

[18] The periodicity of solar eclipses cannot be detected in empirical data from only one locality. Hence it is not surprising when a text concerned with lunar motion mentions only the periodicity of lunar eclipses; cf. Neugebauer-Sachs [1967], p. 205.

[19] Cf. above p. 502 (1).

compiled in the third century but included eclipses as far back as the 5th century.[20] Nothing more than the dates are given in most of these compilations. We know, however, that lists with computational details existed,[21] again written at some later time, but extended back into the Persian period. We have no means of knowing whether such tables were compiled from records actually written in an earlier period or whether one computed that far back, adding greater detail (e.g. the extended function Φ) for the sake of comparison of the later theory with whatever original records existed from earlier centuries. Our knowledge concerning general theoretical concepts and techniques related to eclipses is very defective. For example we are on very insecure grounds in our attempts to reconcile different data for the apparent diameters of the luminaries and of the "shadow" and we know nothing about the prediction of duration or phases of an eclipse.[22]

Nor do we know the Babylonian astronomers "explained" eclipses. The problem is perhaps simpler for solar eclipses. The theory of solar and lunar motion indicates that at a solar eclipse the positions of sun and moon coincide, or nearly coincide, both in longitude and latitude. This readily suggests that it is the "disk" of the moon that obscures the "disk" of the sun. To explain what causes a lunar eclipse is another matter. To introduce the concept of the earth's circular shadow is paramount to postulating the sphericity of the earth, a concept which otherwise is completely lacking in Babylonian astronomy. One could invent the existence of a special object, a dark "disk" that obscures the moon, moving always at 180° elongation from the sun. The mathematics of the ephemerides would allow for this interpretation, but we have no trace of evidence for such a cosmological hypothesis, comparable to the Indian "Head" and "Tail" of the Dragon that move in computable fashion as members of the planetary family or to the Greek "Counter-Earth".

We do not know of a Babylonian explanation even for the lunar phases. We have a fragmentary statement[23] that the "disk" of the moon is halved on the 16th day of the month, [1/4] on the 8th, last visible on the 28th. The computations of the visibility intervals near the syzygies do not imply a specific geometric model, nor do they exclude it, but they give us no right to assume considerations of the type we find with Aristarchus. The nonsense about a translucent lunar hemisphere,[24] presented by Berosus (around 300 B.C.) to a Greek audience might well be a concession to the taste of Greek philosophical speculations.

The term ḫab-rat rendered in the foregoing as "disk," undoubtedly covers this general meaning.[25] In a technical context ḫab-rat (or ḫab) stands for "eclipse magnitude," computed in column Ψ of the ephemerides. We have, e.g., the sentence[26] "opposite 1 finger partial eclipse magnitude (literal: fractional ḫab-rat) you put 1,38,24," meaning that $\Psi = 1^f$ is the eclipse magnitude for a lunar latitude

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[20] Cf. my discussion in [1937, 3], p. 248–253.

[21] Cf. the tables discussed p. 525ff.

[22] It is probably in connection with the determination of the duration of an eclipse that the colophon of a table of lunar eclipses (System B, ACT No. 220) gives a small table of solar velocities as function of the single zodiacal signs; cf. above p. 530.

[23] ACT, p. 197, E (from TU 14).

[24] Vitruvius, Arch. IX, II (Budé, p. 16f.); Schnabel, Ber., p. 258, Frg. 22.

[25] Cf. ACT, p. 198.

[26] ACT No. 204, Sect. 4 (p. 247/8); cf. also ACT No. 200, Sect. 9 (p. 196).

$E = 1{,}38;24$ še. From this and similar statements one can construct a table of eclipse magnitudes and lunar latitudes in agreement with the ephemerides of System A[27]:

$$\begin{array}{ll} \Psi = 0 & E = 1{,}44;24\text{ še} = 17;24^{f} = 1;27^{\circ} \\ 1^{f} & 1{,}38;22\text{ še} = 16;24^{f} = 1;22^{\circ} \\ \cdot & \qquad\cdot \\ 12^{f} & \qquad 32;24\text{ še} = 5;24^{f} = 0;27^{\circ} \\ \cdot & \qquad\cdot \\ 17;24^{f} = c & \qquad 0 \end{array} \tag{8}$$

It follows from this table that the sum of the radii of the moon and of the shadow amounts to

$$r_{☾} + u = 17;24^{f} = 1;27^{\circ}. \tag{9}$$

In System B we have $c = 18^{f}$ for the greatest eclipse,[28] suggesting $\Psi = 0$ for $E = 18^{f}$, hence

$$r_{☾} + u = 18^{f} = 1;30^{\circ}. \tag{10}$$

So far we are on fairly secure grounds. Both (9) and (10) imply a total which is too large; actually $u \approx 3\, r_{☾}$ and $r_{☾} \approx 0;15$ hence the sum of the radii amounts to only about 1°.

All further conclusions depend on evidence for $r_{☾}$ or u alone, or for the ratio $u/r_{☾}$. In a text, which probably reflects a pre-ACT level,[29] we are told that the lunar diameter measures 1°:

$$d_{☾} = 12^{f}. \tag{11}$$

Hence it follows from (10) or (9) that

$$u \approx 2\, r_{☾}. \tag{12}$$

This ratio is also attested elsewhere in antiquity: we find it, e.g., with Aristarchus[30] and in Geminus' Isagoge, a work well-known for its information about Babylonian lunar theory.[31] The relation (11) also strongly suggests that the Greek division into 12 "digits" of the apparent diameters of sun and moon is inherited from Babylonian astronomy.

If we accept the conclusion that the lunar diameter was assumed to be 1° we have to admit that we have no satisfactory explanation for the origin of this parameter. Not only is it difficult to understand how direct observation could result in so gross an error; neither should its consequences for the theory of eclipses have escaped astronomers who developed the most sophisticated methods for the computation of lunar ephemerides. On the basis of (10) a central lunar eclipse would result in a duration of about 6^{h}, i.e. about 2^{h} more than ever observed. There is little consolation in remembering that Ptolemy silently ignored the doubling of the moon's apparent diameter in consequence of his cinematic model for the second inequality.[32]

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[27] Above p. 522.
[28] Above II B 6, 2 and p. 527.
[29] Published Neugebauer-Sachs [1967], Text E (p. 203). This is the same text which gives 5° as well as 6° for the extremal latitude of the moon (cf. above p. 549).
[30] Cf. below p. 667.
[31] Cf. below p. 593.
[32] Cf. above p. 88.

Our difficulties do not end here. We have seen that the term ḫab-rat also occurs in connection with the visibility phenomena at new and full moon,[33] apparently as correction leading from the midpoints of the luminaries to the rims. If this interpretation is correct then the correction in question represents the sum of the radii. Since its amount is repeatedly given as 2° it would imply another doubling of the apparent size, in flagrant contradiction to our previous conclusion. I see no way of solving this puzzle.

Texts dealing with visibility problems always present us with great difficulties of understanding. This is particularly true for a level that precedes the methods described in the procedure texts of the ACT period where we are able to illustrate the essential steps by consistent mathematical models which have their counterpart in simple configurations on the celestial sphere. At an earlier level we are deprived of the help of spherical geometry because we are dealing with an attempt to compute the critical time intervals at first and last visibility without having yet reached a full insight into the elements which influence the problem. It suffices to mention the fact that the lunar latitude is not recognized as an essential parameter. Symmetries which exist for parameters associated with rising- and setting-times are not yet detected.

The text I have in mind from which we know about an early approach to visibility problems contains so many unresolved difficulties[34] that I can only describe the basic idea of the procedure. Let us assume we know for a given month the time interval (called "na") between sunset and moonset at the evening of first visibility (and similarly the "kur" of last visibility). One wishes to find the na (and the kur) for the next month. The answer to this problem is given by a list of numerical coefficients which should be added, or occasionally subtracted, to yield the desired na (or kur). The additive numbers (we call them t) assume that the months are full months (30 days long). Since the conjunction always recedes from month to month the new na must be greater than the preceding one. This increment is given by t, tabulated for the beginning points of the 12 zodiacal signs and found by interpolation for other longitudes. This process of adding positive numbers t will lead to a value of the na so large that it is obvious that the first crescent was visible one day earlier. In this case the value of the na must be reduced because of the elimination of one whole day. The amount of this reduction is a quantity s, again tabulated for all signs.

As an example I give here the list of the t and s for the na, unfortunately only partially preserved

	♈	♉	♊	♋	♌	♍	♎	♏	♐	♑	♒	♓
t:	14	12	10	5	3	3	5	8	[·	·]	11	11
s:	22	20	18	16	14	12	10	14	[·	·	·]	18

The units in this table are time degrees.

A similar table for the kur, and two more analogous tables for full moons, show clearly that the numbers are not the result of combining data for rising

[33] Above p. 539.

[34] For details cf. Neugebauer-Sachs [1969], Text K (p. 96–111).

times as should be the case. Apparently we have here a combination of some empirical data and of arithmetical approximations (stretchwise linear but not zigzag functions[35]) which would, however, after a short while become valueless because the lunar motion would destroy all fixed relations to the zodiacal signs assumed in these tables.

In order to apply the above described procedure one must know how much the true conjunctions recede in longitude from month to month with respect to the evening of first visibility. Consequently one must know the length of the true synodic month and indeed the examples computed in the text give values of the function G (which have to be added to 29^d). These values are associated, as in the ephemerides, with the function Φ (though not connectible with the ephemerides in the canonical fashion). Not only are values of Φ introduced in the examples but the truncation of a velocity function and a related function G is described.[36] Thus we find methods which belong to the most advanced lunar theory associated with a primitive attempt to deal with visibility problems.

§ 3. The Planets

In our sources, aside from the ephemerides, the planets are comparatively well represented. In the "Diaries", which go back to the middle of the seventh century, the planets are always taken into consideration. They are at the center of the "Goal-Year texts", of the "Almanacs", and of the "Normal-Star Almanacs", all texts that follow rigidly defined patterns. There remains a third class of texts which we called, for the lack of a better name, "atypical".[1] These texts do not conform to any definite rule for the type and arrangement of their content. They are, in fact, related to both canonical classes, e.g. by describing the planetary motions with respect to Normal Stars, by containing solstice-equinox-Sirius schemes, together with sections from the lunar function Φ, or System A planetary longitudes. Chronologically this group seems to belong, by and large, to the early Seleucid period, or to the end of the Persian period.

As an example an "atypical" text may be described which concerns Mars, Venus, and Φ.[2] This tablet was copied from an older one since the first five lines of column I say "broken", followed by only the last sign of each line of the original. The first column concerns the phases Ω, Γ, Φ, Ψ for Mars, but no opposition. The dates comprise only year numbers, presumably regnal years, and months, but no day numbers. The longitudes are computed, to three digits, with a System A method, which, however, does not always agree with the norm of the ephemerides, the most striking deviation being velocity zones not always exactly two signs in length. From the sequence of the month names one can deduce the intercalation pattern and compare it with the historically known sequences. In this way only two dates seem possible which agree fairly well with the given longitudes: Arta-

[35] Cf. the graphs in Neugebauer-Sachs [1969], p. 103.
[36] Cf. for this above p. 506ff.
[1] Neugebauer-Sachs [1967], [1969]; Aaboe-Henderson *et al.* [1975].
[2] Neugebauer-Sachs [1967], Text C.

xerxes II 46 to Artaxerxes III 17 (i.e. -359 to -341) or, perhaps, S.E. -1 to 17 (-311 to -294).

The next column concerns one synodic period of Venus, probably for the years $-423/2$ or $-432/0$ and the year following.[3] The dates and longitudes are based on an essentially linear scheme: for 4 months after the station as morning star (Φ) the planet moves with the solar velocity (30° per month), and similarly before Ψ. The transition from morning star to evening star takes place with 36° per month, for about 10 months. No definite rule seems to cover the retrogradation from Ψ to Φ, probably in order to utilize this interval for whatever remains from the synodic period.

The text ends with a list of ten numbers which are values of the function Φ for full moons and 14 additional numbers some of which are differences of the the function G.[4] The addition to the first of these numbers of the name of Kandalanu, year (?) 36 (which would be the year $-401/0$) is a complete mystery.[5]

An entirely unexpected revelation came from an atypical text[6] by providing evidence for the existence of a theory of planetary latitudes, completely absent in the ephemerides. The Diaries and the related texts seemed to confirm the conclusion that one did not operate with distances from the ecliptic but only with respect to the Normal Stars. To this rule we now have an exception in the form of a simple linear scheme. As extremal latitudes we find $\pm 1;20°$, valid from the middle of Virgo to Scorpius (maximum) and from Pisces to Taurus (minimum). Both ascent and descent between these extrema is assumed to be linear. The location of the extrema suggests Saturn, though the latitude of this planet can reach almost $\pm 3°$.

A remarkable group of texts are the "Goal-Year texts"[7], based on the periodicity of the planetary phenomena. A Goal-Year text, for the five planets and for the moon, concerns one given year N. Coordinated with N is a collection of data which occurred in the year $N-P$ where P is a period characteristic for each of the six celestial bodies:

Jupiter:	71 and 83 years	
Venus:	8 years	
Mercury:	46 years	
Saturn:	59 years	(1)
Mars:	47 and 79 years	
Moon:	18 years	

One recognizes in this list the synodic periods for the planets, well known in linear combinations with cruder or with larger periods.[8] The ambivalence of the periods of Jupiter and Mars has been explained by A. Sachs.[9] An example of the recurrence

[3] Thus Artaxerxes I/Darius II.
[4] Column ΔG in Table 14, above p. 509.
[5] Kandalanu ruled only 22 years, thus 36 could only mean a regnal year if the text had been written during Kandalanu's lifetime.
[6] Neugebauer-Sachs [1967], Text F.
[7] Terminology introduced by Sachs [1948], p. 282.
[8] Cf., e.g., above II A 2, p. 391 or II A 6, 1 C, p. 426.
[9] LBAT, p. XXV.

of the phases of Jupiter is shown for the years S.E. 1 + 71 and 1 + 83:[10]

S.E. 1	IV	14	♋24.9	S.E. 72	IV	14	♋20.5	S.E. 84	IV	2	♋25.0	
	V	12	♌ 0.9		V	13	26.5		IV	28	♌ 1.0	
	IX	14	17.5		IX	15	♌13.5		IX	1	17.9	(2)
	XI	15	12.5		XI	15	8.5		XI	3	12.9	
	XII$_2$	18	7.5		XII$_2$	18	3.5	S.E. 85	I	5	7.9	

The 71-year period restores the dates remarkably well whereas the 83-year interval repeats closely the longitudes. Similar, but not as obvious, are the returns to longitudes and dates for Mars after 47 and 79 years. In one of the atypical texts[11] the 79-year period is explained as the combination of a 32-year period, which falls 5 days short of the completion of sidereal years, with the 47-year period, which results in an excess of 4 days. All this is said under the heading "the passings by of Mars at the Normal Stars" which underlines the sidereal character of the reference system.

It has long been recognized that the planetary data in the Almanacs and in the Normal-Stars Almanacs do not reflect observations but predictions.[12] The ephemerides can be excluded as the source of such predictions because they do not contain elements which could readily provide data for the "passing by" above or below the Normal Stars. Such information is provided by the Goal-Year texts, a fact which strongly suggests that the use for the Almanacs is a motive for compiling Goal-Year texts. For the ultimate observational source one is led to the Diaries which contain a huge amount of observational data, though embedded among many other items, e.g. meteorological conditions, economical and political events, etc. To sift out the relevant data from a huge mass of miscellaneous records, extending over several decades, is surely not an easy way of obtaining observational data, particularly in the order needed for a Goal-Year text. Perhaps excerpts for consecutive years from Diaries concerning individual planets are a preliminary step for preparing the Goal-Year texts.[13]

The existence of ephemerides seems to be ignored in the Diaries and their descendants,[14] the bulk of which belongs to the same two centuries from which we have ephemerides. The accurate prediction of the time and place of the planetary phases is not very interesting for astrological purposes. The ephemerides are equally useless for the Almanacs and the Normal-Star Almanacs, which in turn also contain much useless information. We were able to penetrate quite far into the mathematical aspects of Babylonian astronomy but most of its character as an historical phenomenon remains unknown.

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[10] Computed by A. Sachs.
[11] Neugebauer-Sachs [1967], p. 206 (10).
[12] Cf. Sachs [1948], p. 287.
[13] Cf. Sachs [1974], p. 49.
[14] The Goal-Year texts are attested from about −250 on; cf. Sachs [1974], p. 49.